Branched Circle Packings as Discrete Rational Maps

Edward Crane

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Overview

This talk is about *tangency* packings of *triangulations* of the 2-sphere, in which *branching* is allowed. This means that the chain of neighbours of a disc in the packing may wind around it more than once before closing up.

The generalized branching near the end of the talk was joint work with James Ashe and Ken Stephenson:
Circle packing with generalized branching, Journal of Analysis, (2016).

I have not worked actively on this for a few years. But this talk is an opportunity for me to advertise my favourite open problem in circle packing, and to share some approaches to it. I would be very pleased if anyone here can build on these ideas to solve it, or solve it in a completely different way!

Metric circle packings

Consider a simplicial complex K homeomorphic to a closed connected oriented differentiable surface S. K has vertex set V, edge set E and face set F. Let ρ be a Riemannian metric on S.

A metric circle packing of K with respect to ρ is a collection of closed balls of ρ, called *discs*, such that

- there is one disc D_v for each vertex $v \in V$, with radius $r_v > 0$ and centre c_v,
- the interiors of the discs are disjoint, and
- for each $(v, w) \in E$, the discs D_v and D_w have a unique common boundary point.

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- for each $(v, w) \in E$, the discs D_v and D_w have a unique common boundary point.

The graph $G = (V, E)$ is called the nerve of the packing.

We *do not* require that r_v must be less than the injectivity radius of ρ at c_v. So the discs D_v need not have smooth boundary and need not be topological discs.

Schramm's metric packing theorem (1991)

Let ρ be a Riemannian metric on the 2-sphere S^2. Let K be a simplicial complex homeomorphic to the 2-sphere and let a, b, c be the vertices of one face of K.

Fix D_1, D_2, D_3, three closed balls of ρ with positive radii and disjoint interiors, each pair meeting in a unique common boundary point. Let $\mathcal{I}$ be an interstice, i.e. a connected component of $S^2 \setminus (D_1 \cup D_2 \cup D_2)$ whose boundary meets all three of D_1, D_2, and D_3.

Then there exists a unique metric packing of K with respect to ρ which extends the assignment $D_a = D_1$, $D_b = D_2$, $D_c = D_3$, so that for all vertices $v \in V \setminus \{a, b, c\}$, $c_v \in \mathcal{I}$.

Koebe's theorems

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Corollary (Fáry's theorem):
Any planar graph can be drawn in the plane with straight edges.

In fact you can draw the planar dual with straight edges at the same time, according to a theorem of Brightwell and Scheinermann.

Koebe's theorems

Koebe's circle domain theorem (Kreisnormierung)

Any finitely-connected plane domain U is conformally equivalent to a *circle domain*, unique up to Möbius maps.

Koebe proved his circle packing theorem by considering the *contact limit* of a sequence of such domains.

Circle packing in the hyperbolic plane

Thurston studied circle patterns in which neighbouring vertices in G are represented by circles on the Riemann sphere which meet at a specified angle in $[0, \pi/2]$. One angle is specified for each edge of G. These circle patterns correspond to three-dimensional hyperbolic polyhedra with specified dihedral angles.

Thurston and Andreev independently proved an existence and uniqueness theorem for these patterns. There is a necessary and sufficient condition in terms of obstructing cycles.

Thurston also gave a practical iterative algorithm for computing such patterns. Its convergence was proven by Colin de Verdière.

Circle packing in the hyperbolic plane

We explain a version of Thurston's algorithm in the tangency case (overlap angle 0).

Pick one vertex v_0, to be represented by the complement of the open unit disc $\mathbb{D}$. The vertices not adjacent to v_0 will correspond to closed discs contained in $\mathbb{D}$; these are metric balls of finite radius in the hyperbolic metric on $\mathbb{D}$.
The vertices adjacent to v_0 will be represented by horodiscs (discs internally tangent to the unit circle). Horodiscs are limits of sequences of hyperbolic balls whose radii tend to infinity.

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Thurston's method uses an iterative approximation scheme to compute the hyperbolic *radii* of all the discs, without saying anything about the locations of their centres. This is quite natural because the hyperbolic radii are invariant under Möbius transformations fixing the unit disc.

Local conditions

Given any three radii $r_1, r_2, r_3 \in (0, \infty]$, there is a configuration of mutually tangent hyperbolic discs with these radii (horodiscs for $r = \infty$). It is unique up to isometry. The geodesic triangle joining the hyperbolic centres (or ideal boundary points) makes an angle θ_1 at the centre of the circle of radius r_1.

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A *packing label* is a putative set of radii $r : V \to (0, \infty]$. From a packing label we can compute the implied angles in each face at each vertex, and add them for each vertex. If the sum at a vertex v exceeds 2π then it can be reduced to 2π by making the radius $r(v)$ larger. If the angle sum is less than 2π then we can fix it by decreasing $r(v)$.

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Monotonicity: If we decrease the radius for just one vertex v, and keep all other radii fixed, then the angle sum of v increases, but angle sum of each of its neighbours decreases. Also the area of each hyperbolic triangles involving v decreases.

Upper Perron method (Phil Bowers)

Start with a packing label for which all interior angle sums are all at most 2π. Call this a *supersolution* because each radius is too large given its neighbours' radii.

Now cycle repeatedly through the vertices. For each vertex in turn, adjust its radius so that its angle sum becomes exactly 2π. This is always a radius decrease, and by the monotonicity properties, the new packing label is again a supersolution.

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The limit label is the vertexwise infimum of all supersolutions. For each interior vertex with positive limit label, the angle sum is exactly 2π.

A Gauss-Bonnet argument using hyperbolic areas and Euler characteristics of subcomplexes shows that the limit is nowhere zero, so in fact all interior angle sums are 2π.

Layout

Once we have a packing label where all interior vertices have angle sum 2π, then we can place the discs, starting wherever we like with two neighbours, then laying out discs in turn like dominoes.

A *monodromy theorem* shows that the eventual placement does not depend on the order in which we lay out the discs, because the underlying complex is simply-connected.

Why doesn't Perron work in spherical geometry?

One obvious problem with trying to compute circle packings on the sphere by computing the spherical radii directly is that the group of Möbius maps does not act by similarities of the spherical metric.

So if there is any solution to the equations expressing the angle sum conditions in terms of the spherical radii, then there is a non-compact 6–parameter family of solutions. We need an algorithm that will pick out one solution, so it must break the symmetry somehow and avoid escaping to infinity.

Why doesn't Perron work in spherical geometry?

Monotonicity still holds, so long as we restrict to packing labels with $r(v) + r(w) \leq \pi/2$ for all $(v, w) \in E$.
This is easy to see by differentiating the spherical trig formula

$$\cot^2(\theta_1/2) = \sin^2 r_1(\cot r_2 \cot r_3 - 1) + \cos r_1 \sin r_1(\cot r_2 + \cot r_3)\,.$$

But we know no version of Thurston's algorithm that works in spherical geometry. In fact, if we start with the spherical radii of a packing and scale down all the radii by the same factor then all the angle sums get smaller, so we get a supersolution. That is bad news, because the steps of the iterative method all make the radii smaller still, taking us further away from the correct solution.

So the *positive curvature* is a problem.

What about other methods?

Several proofs of the Koebe-Andreev-Thurston theorem and its relatives rely on the global minimization of some functional.
(Leibon, Colin de Verdière, Brägger, Rivin, Bobenko and Springborn.)
These all work in hyperbolic geometry or in Euclidean geometry, but not in spherical geometry.

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For example, Bobenko and Springborn's Euclidean functional is a convex function of the logs of the radii. It becomes strictly convex when restricted to a hyperplane, to fix the scale. Thurston's algorithm corresponds to minimizing the functional one variable at a time. Much better numerical methods exist for minimizing convex functions of many variables!

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Boris Springborn wrote down a functional of the spherical radii such that the critical points correspond to solutions of the angle sum conditions in spherical geometry. But *the critical points are not minima* so it is harder to find them computationally. Experiment shows that we can find a critical point of this functional by Newton's method, if we start close enough to one.

Discrete conformal maps

A combinatorial closed disc is a simplicial complex homeomorphic to a closed disc.

There is considerable flexibility in circle packing such a complex.

Thurston had the idea that circle packings provide a discrete analogue of conformal structure.

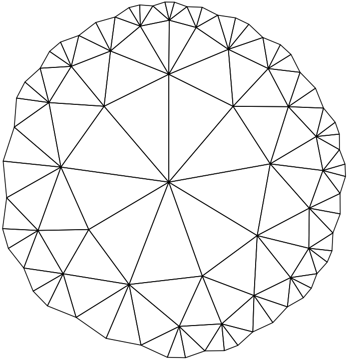
So if we have two circle packings for the same complex, we can think of the correspondence as a *discrete conformal mapping*.

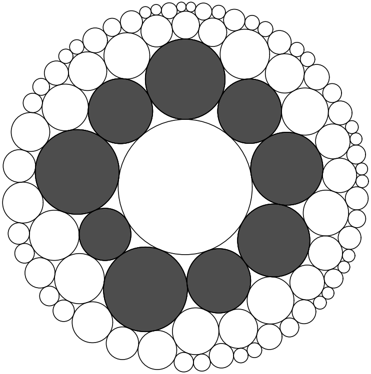
If the two packings are both in Euclidean geometry, we can get a genuine mapping by piecewise linear interpolation of the mapping between vertex locations in the two packings.

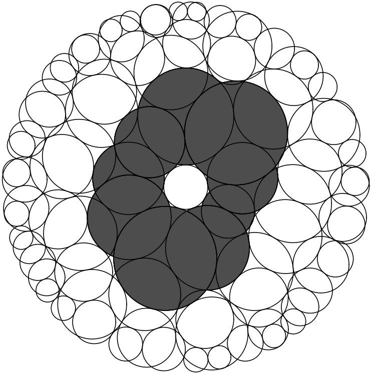
Branching

To throw more light on the failure of Thurston's algorithm in the spherical metric, we will look at branched circle packings.

In a branched circle packing, we allow a circle to have a chain of neighbours that winds around it more than once before joining up.







Branched packings as metric packings

We can think of a branched packing as a collection of metric balls with disjoint interiors in a branched covering surface spread over the Riemann sphere.

The metric is the path metric associated to the lift of the spherical metric to the covering surface.

This is a singular metric, not a Riemannian metric. It has constant positive curvature except at some cone points of angle $4\pi, 6\pi$, etc.

Example: discrete polynomials

In Thurston's algorithm, we can aim for certain interior vertices to attain angle sums that are multiples of 2π, say $2\pi k_v$. The upper Perron method still works, (in hyperbolic geometry), but the limit radii can be zero.

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Bowers (and Garrett, independently) gave a necessary and sufficient condition for the limit to be positive. The condition is that for every edge-connected set V of interior vertices,

$$F_V > 2 \sum_{v \in V} k_v \,,$$

where F_V is the number of faces incident on V.
In particular, for any simple closed cycle of edges, the length of the cycle must be more than twice the total amount of branching enclosed by the cycle, which is the sum $\sum (k_v - 1)$ over interior vertices of this disc.

Example: discrete polynomials

When we layout the maximal packing, we get a pattern of tangent circles inside the disc such that for any vertex v the neighbouring circles to v wind around it k_v times before closing up. The boundary horodiscs go around the unit circle $\sum(k_v - 1)$ times before closing up.

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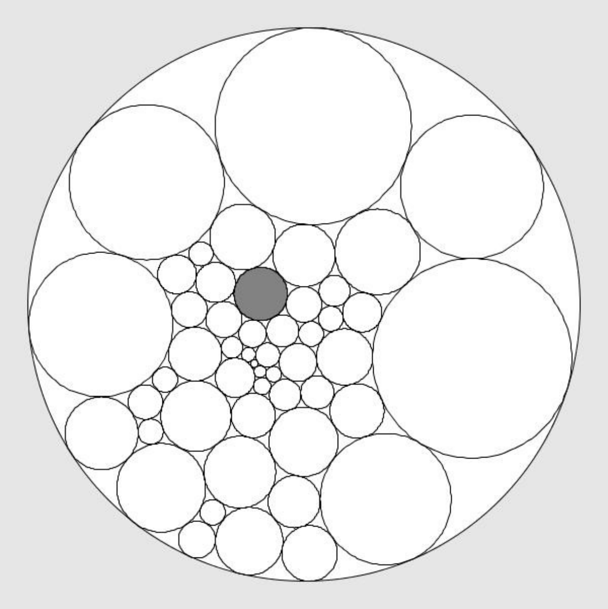
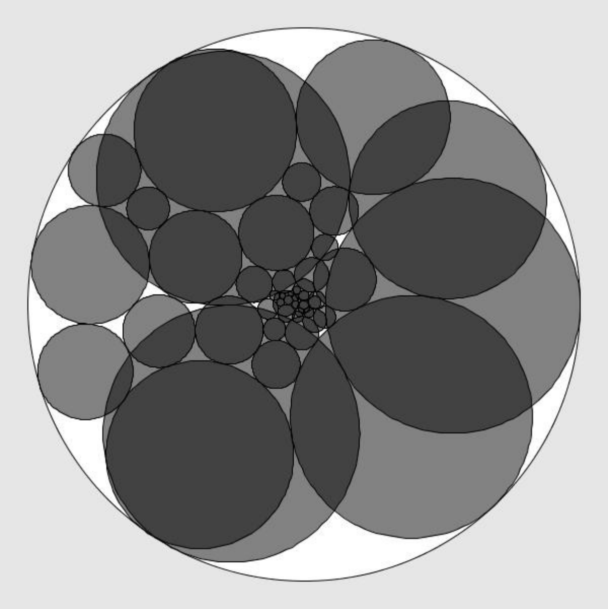
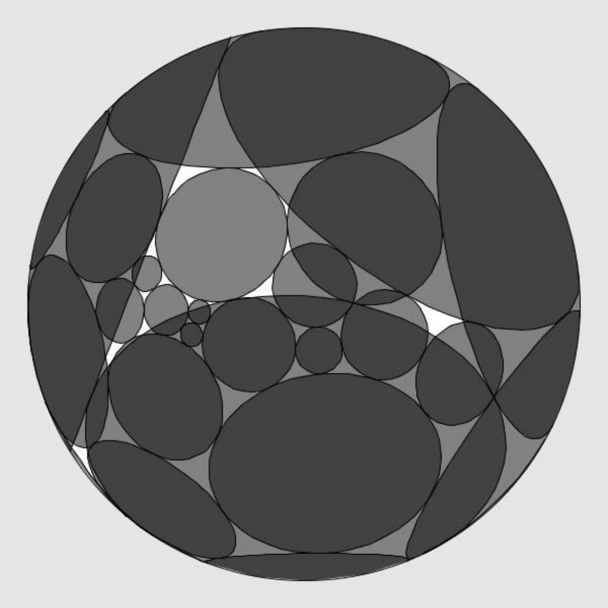
We also have an unbranched maximal sphere packing of the same complex. Comparing them gives us a discrete polynomial mapping, with half of its total branching at one circle (the one centered at infinity).







Discrete rational maps

Cut-and-paste construction:

Take a univalent circle packing on the Riemann sphere, with its geodesically embedded nerve.

Fix a branched covering map $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, defined by a classical rational map R such that every critical value of R is a circle centre of our packing.

Lift every vertex, every edge and every face to copies of themselves on every sheet of the covering surface. Also lift the discs. Something special happens for discs centred on critical values of R!

We get a complex on the covering surface, which is the domain of R, a copy of the Riemann sphere. This complex is represented by a branched circle packing, where overlapping discs on different sheets actually coincide. This is an example of a *discrete rational map*.

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We can use this construction to approximate a genuine rational map with given critical values and *monodromy*. Circle packing followed by Newton's method is the best way I know to solve this uniformization problem.

The most beautiful discrete rational map?

Take 12 spherical discs, centred at the vertices of a regular icosahedron, each with spherical radius $\arctan\left(\frac{1+\sqrt{5}}{2}\right)$. This gives a branched circle packing of the standard icosahedral triangulation of the sphere, in which every vertex is a simple branch point. That is, every disc has five neighbours, which wind around it twice.

The discrete critical points and the critical values that they map to (i.e. disc centres in the unbranched and branched packings of the same triangulation) can be arranged to agree exactly with the critical points and critical values of the rational map

$$z \mapsto \frac{z^2(z^5 - 7)}{7z^5 + 1} .$$

The mapping of the critical points to critical values is a permutation that sends neighbouring vertices of the icosahedron to vertices at combinatorial distance 2 on the icosahedron.

Counting classical rational maps I

Theorem (Hurwitz / Crescimanno and Taylor, 1995)

Let $d \geq 3$. For a generic choice of $2d-2$ distinct points in $\widehat{\mathbb{C}}$, there are exactly $(2d-2)!\, d^{d-3}/d!$ rational maps of degree d having those $2d-2$ points as critical values, up to equivalence by pre-composition with a Möbius map.

This number arises from counting the number of ways to write the identity permutation as the product of $2d-2$ transpositions which generate the symmetric group S_d, which turns out to be $(2d-2)!d^{d-3}$. (This count isn't easy!)

For $d = 3$ this gives 4 rational maps. For $d = 4$ it gives 120 already.

This theorem tells us how many different discrete rational maps we can create from a given triangulation of the sphere with $2d-2$ marked vertices as branch values, by the cut-and-paste construction.

Main open problem on discrete rational maps

Suppose we are given a triangulation of the sphere, and specified branch orders k_v at each vertex, such that $\sum(k_v - 1)$ is even and strictly less than half of this sum comes from any one vertex? Under what further conditions does there exist a circle packing on the sphere with this complex and this branching?

If it exists, how can we compute it?

This is a discrete analogue of specifying $2d - 2$ points in the Riemann sphere and asking whether they are the critical points of some rational map of degree d.

Counting classical rational maps II

Theorem (Goldberg, 1991)

Let $d \geq 2$. For a generic choice of $2d-2$ distinct points in $\widehat{\mathbb{C}}$, there are exactly $C_d = \frac{1}{d}\binom{2d-2}{d-1}$ rational maps of degree d having those $2d-2$ points as critical points, up to equivalence by post-composition with a Möbius map.

These are Catalan numbers. For $d = 2$ there's just one rational map. For $d = 3$ there are generically two. The sequence grows exponentially: $C_d \sim 4^d/(d^{3/2}\sqrt{\pi})$ as $d \to \infty$.

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Remark: Eremenko and Gabrielov showed that if all the $2d-2$ points are real then each equivalence class has a representative that is a rational map with real coefficients. (Catalan numbers count non-crossing matchings of the critical points by arcs of the real locus in the upper half-plane.)

A refined problem

Show that for any triangulation K of the sphere with any $2d - 2$ distinct vertices marked, there are at most C_d branched circle packings of K with simple branching at these $2d - 2$ marked vertices and no others, up to equivalence by post-composing with Möbius maps.

Show that for each $d \geq 2$ there exists a triangulation of the sphere with $2d - 2$ marked vertices for which this bound is achieved.

Non-uniqueness in degree $d = 3$

Here is a construction showing that more than one Möbius equivalence class of solutions may exist with the same combinatorial and branching data.

Non-uniqueness

Take your favourite triangulation of the sphere, G, say. If you can, pick three vertices, v_1, v_2, and w, no two adjacent, such that v_1 and v_2 have degree at least 5 and w has degree at least 7.

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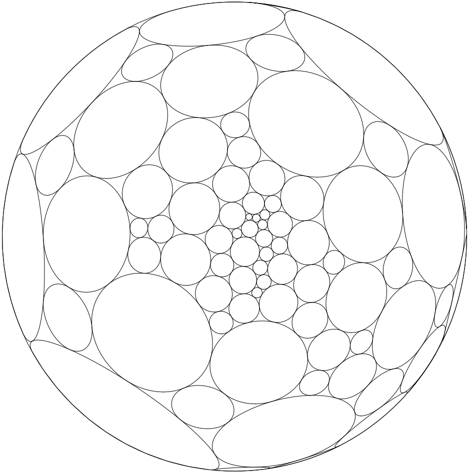
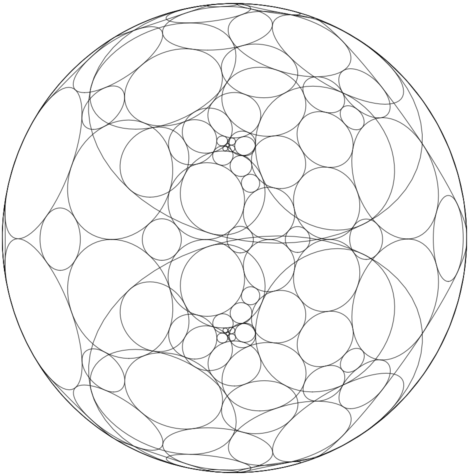
Now we construct the domain packing that our two discrete cubic maps will have in common. Circle pack G in the Riemann sphere so that the circle C_w is the extended real line. Reflect all the other circles in C_w, delete C_w, and insert 'ball bearing' circles of degree 4 in the resulting 4-sided interstices.



Non-uniqueness

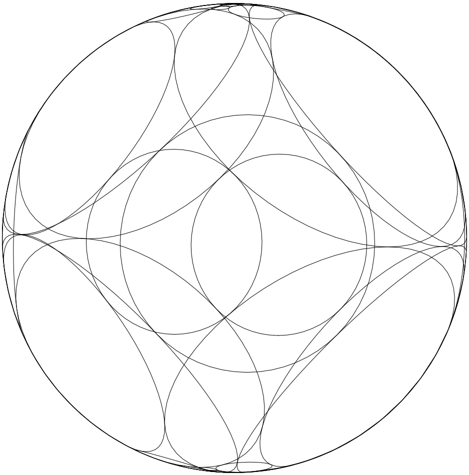
Now we make the first of our range packings. Start by making a discrete quadratic polynomial with nerve G and branching of order 2 at v_1 and v_2. Apply a Möbius map to move C_w to the real line. Reflect all the other circles in C_w, delete C_w and insert circles into the resulting 4-sided gaps.

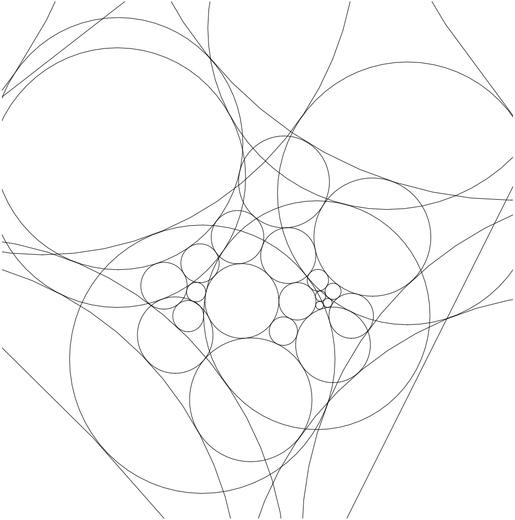
The result is a discrete rational map of degree 3 with branch vertices $v_1, v_2, \widehat{v_1}, \widehat{v_2}$.



Non-uniqueness

To construct the second range packing, take the maximal branched hyperbolic packing of $G \setminus \{w\}$ with angle sum 4π at v_1 and v_2 and 2π everywhere else. Do the same trick to double it across its boundary by inserting 'ball bearings'. We have constructed a discrete Blaschke product of degree 3.





Non-uniqueness

The two discrete rational maps that we have constructed are certainly different: in the second case the circles coming from $G \setminus \{w\}$ do not overlap those coming from the other half of $\tilde{G}$, but in the first case some of them do.

Non-uniqueness

What does non-uniqueness have to say about the nature of any existence proof and about algorithms for computing discrete rational maps given the combinatorics and the branching data?

No proof can work by giving a globally convex functional of the spherical radii such that mimima correspond to solutions, unless it works in a smaller space that only contains one solution. But it is difficult to see how to encapsulate the topological difference between our two discrete rational maps of degree 3 purely in terms of conditions on the spherical radii.

Iterative algorithms are likely to be faced with the problem of chaotic orbits separating the basins of attraction of different solutions, just as for Newton's method for finding roots of polynomials.

Morse theory approach

It is tempting to try to count the critical points of a functional such as Springborn's spherical functional using Morse theory. (First we need to impose a normalization, say by fixing the spherical radii of three adjacent discs.)

I do not know how to do this, but a first step would be to relate the index of each critical point of the functional to the geometry or the topology of the corresponding discrete rational map.

One would also have to understand the topology of the very high and very low level sets of the functional.

An approach via Schramm's metric packing theorem

Branched circle packings on the sphere are a special case of metric packings, where the metric is the spherical path metric on the covering surface.

These metrics are not Riemannian metrics, but we can make them Riemannian by making a local mollification in the neighbourhood of each critical point. If we do this in a radially symmetric way, then a disc centred on the critical point remains a disc in the mollified metric.

We can allow metric packings with respect to the spherical path metric in which some discs contain singular points in their interior or on their boundary. We call this *generalized branching*.

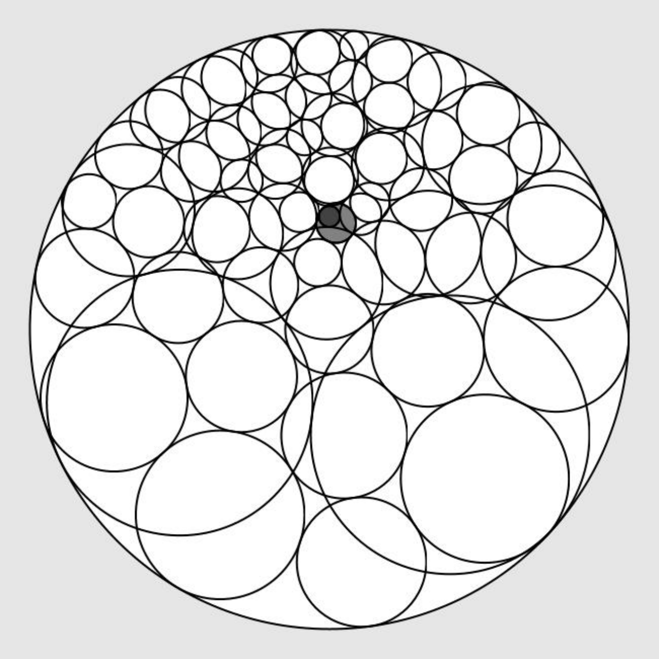
Parametrized generalized branching

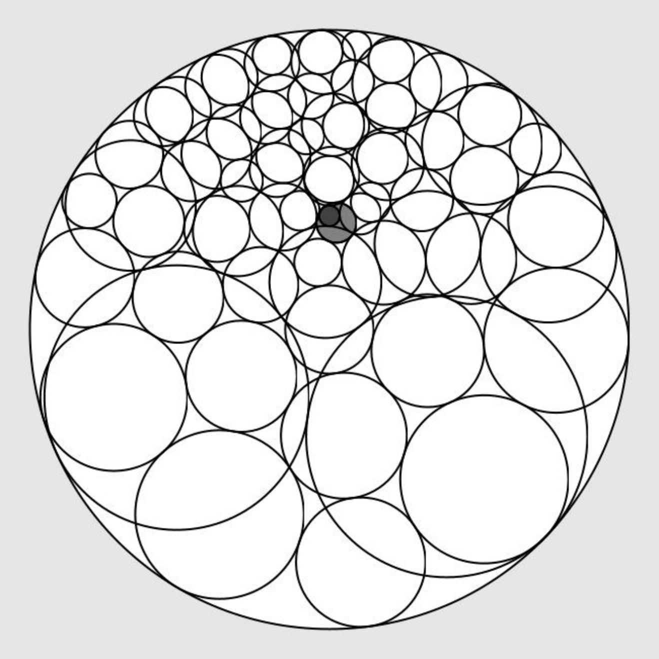
For example, at a simple critical point of the covering map R, three discs with disjoint interiors may meet. Their projections under R may overlap, but where this happens, they lie on different sheets of the covering. There are two real parameters describing the angles between their tangents at the singular point of the metric. We call this situation *singular branching*.

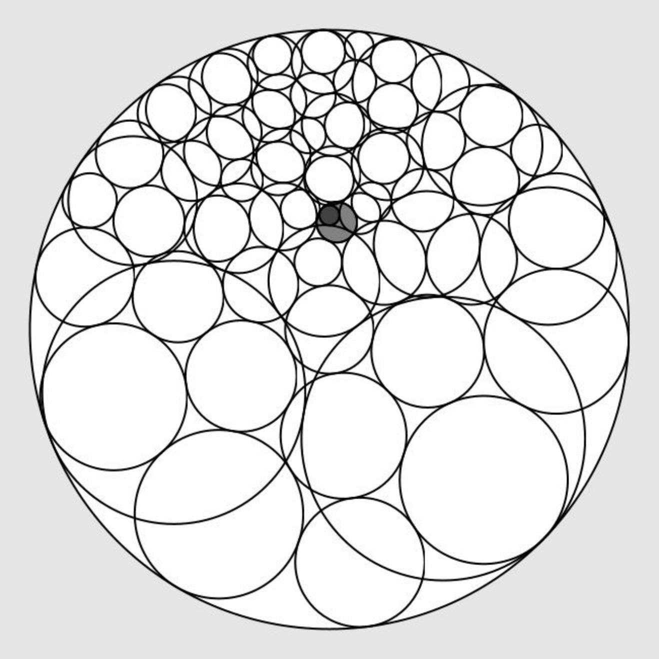
A disc with center z may contain a single simple critical point w in its interior, in which case its boundary winds twice around w, using both sheets of the cover. It has two different radii of curvature, one on each sheet. The projection of this disc under R looks like a big disc centred on z with a smaller disc centred on w internally tangent to it that is covered twice. We call this case *shifted branching*. It may also be parametrized by two real parameters together with some combinatorial information: between which pairs of neighbours does the radius of curvature change?

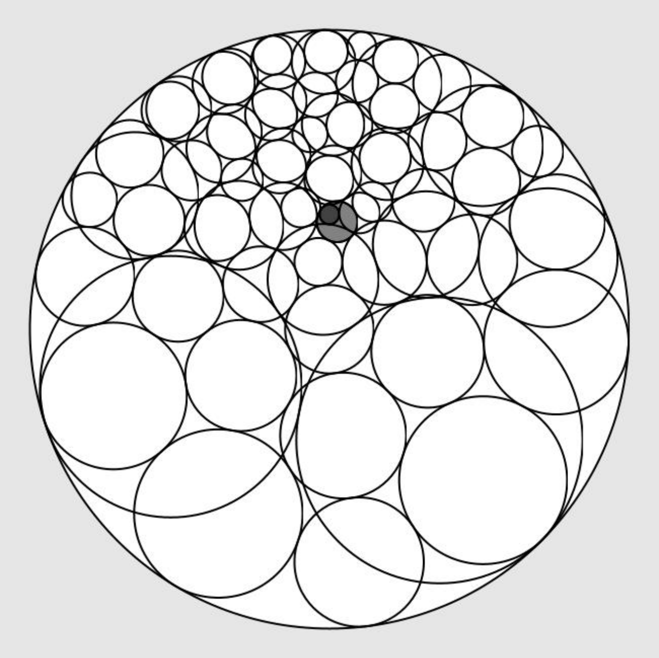
Generalized branching in `CirclePack`

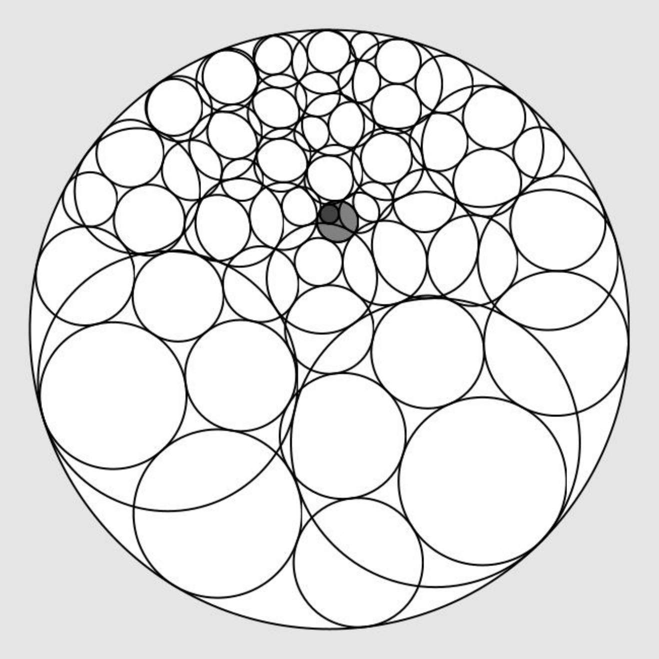
With James Ashe and Ken Stephenson, we figured out how to trick the `CirclePack` software into computing circle packings with generalized branching, in the case of packing a combinatorial disc into a finite branched cover of the hyperbolic plane. The trick was to modify the complex in the neighbourhood of each critical point, including *guide* circles whose overlap angles with the original circles are the variable parameters, and also some unfortunate circles whose radius is 0 by design in the final packing.
Using reflection methods we used this to produce some discrete rational maps with generalized branching.

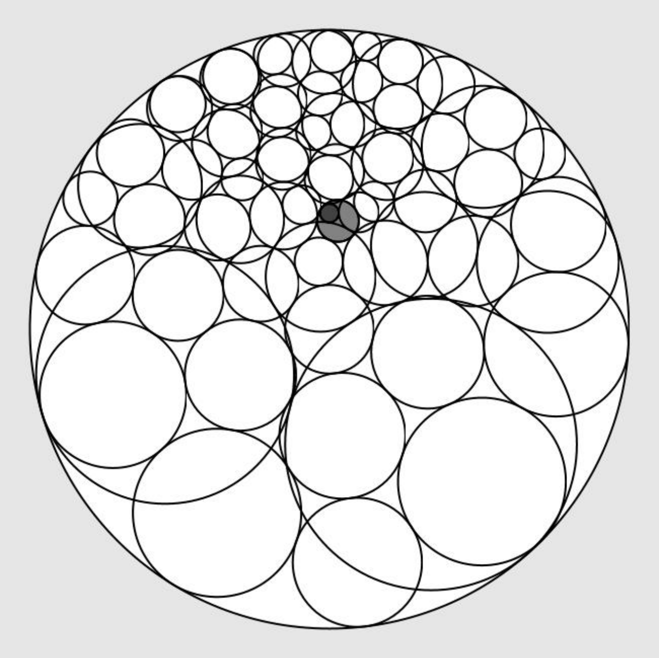


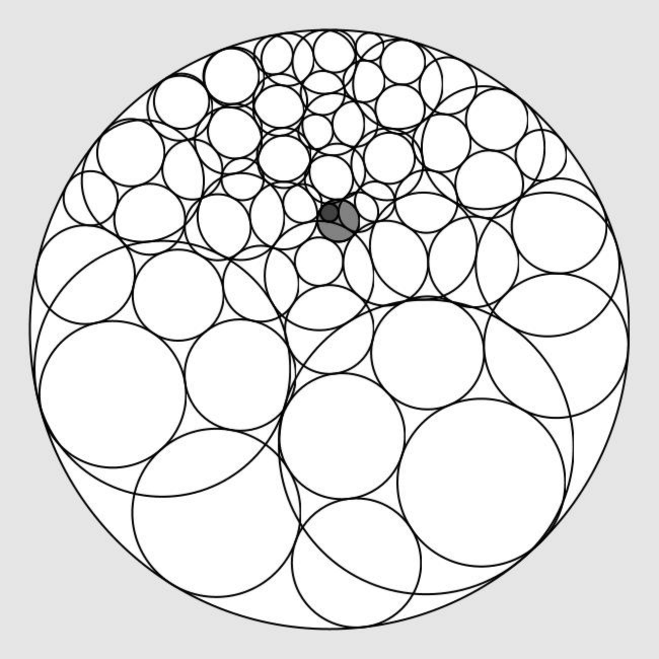


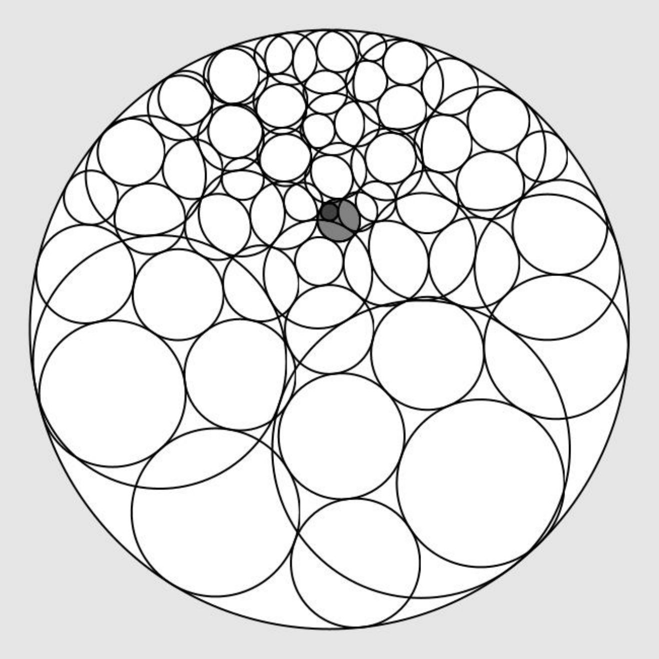


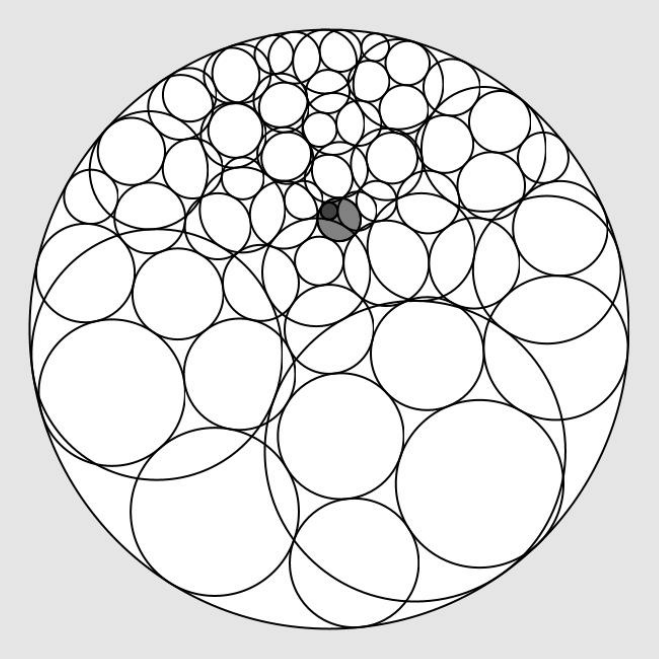


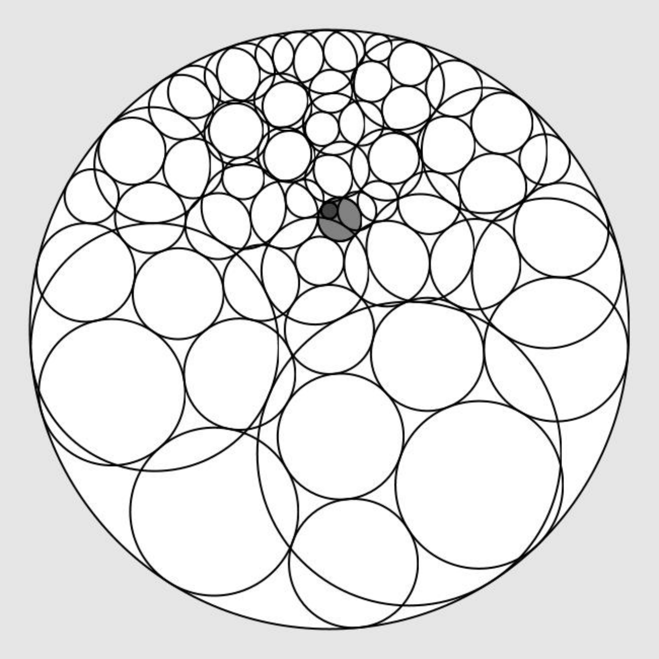


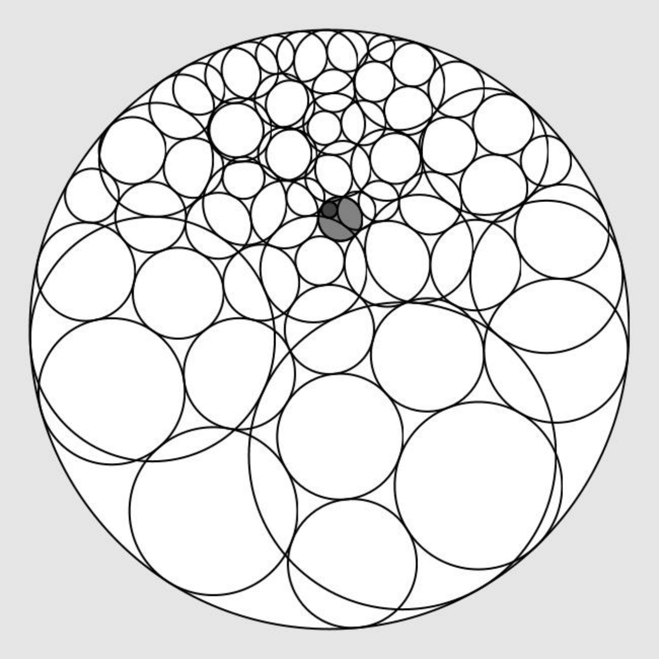


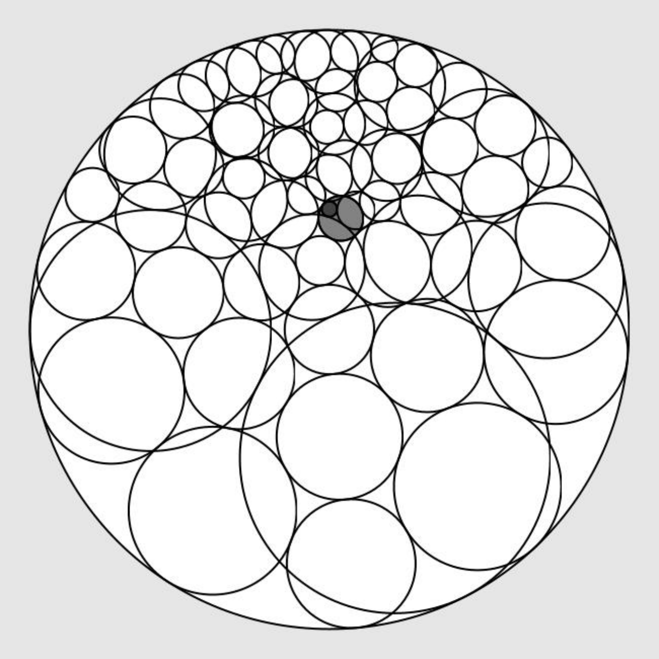


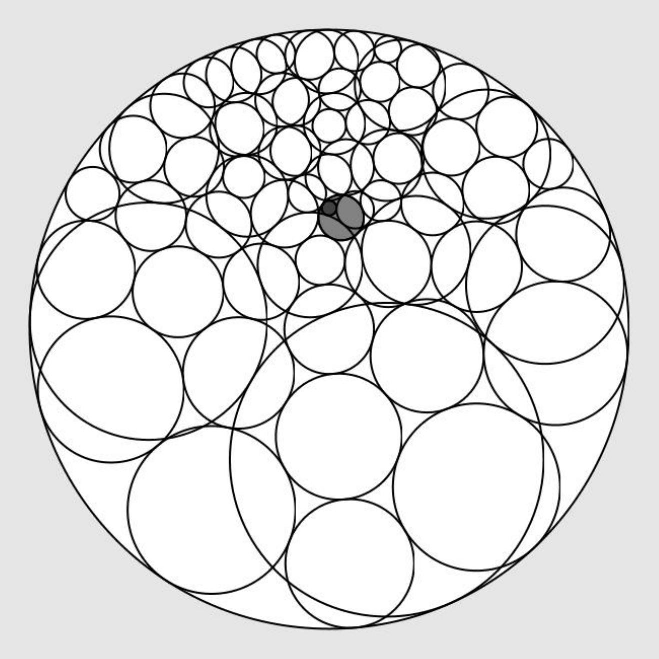


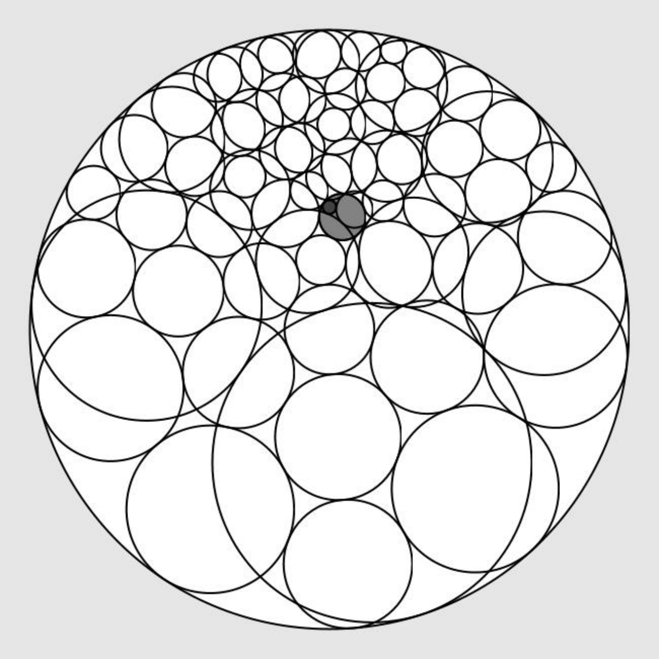


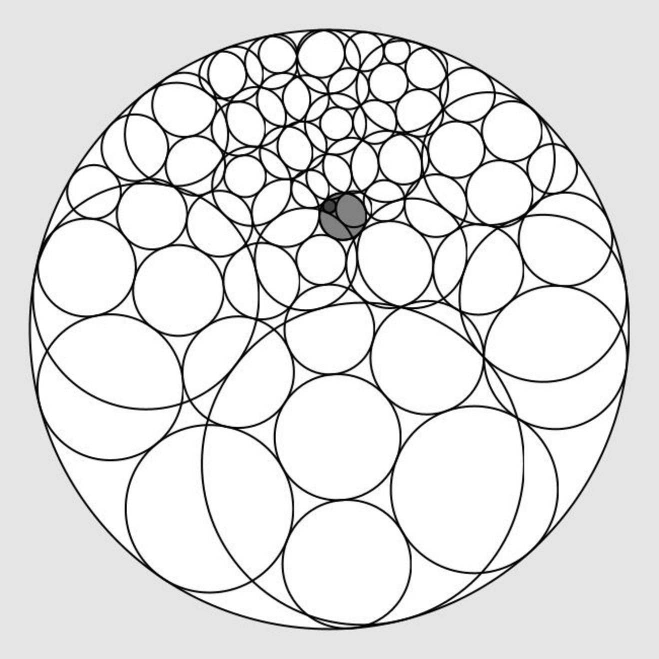


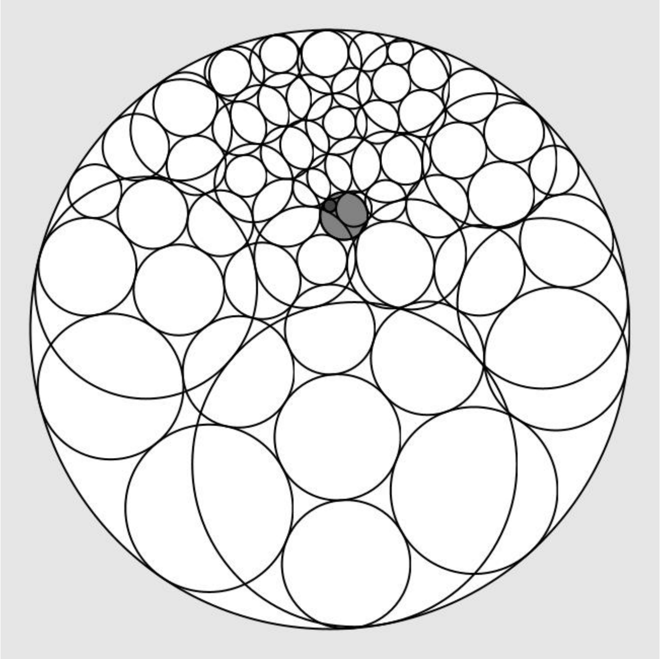


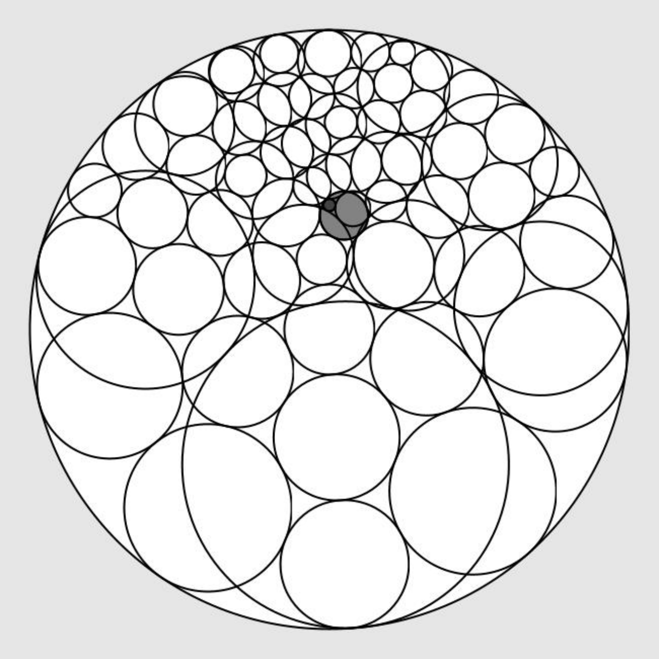


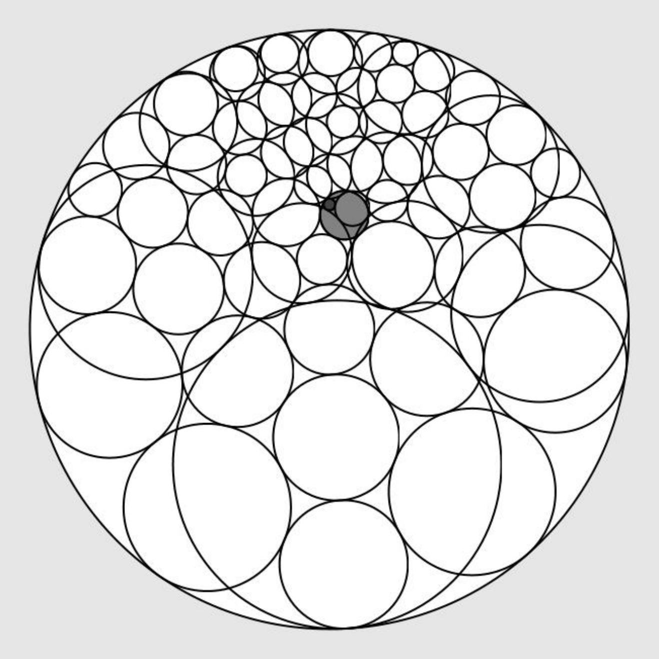


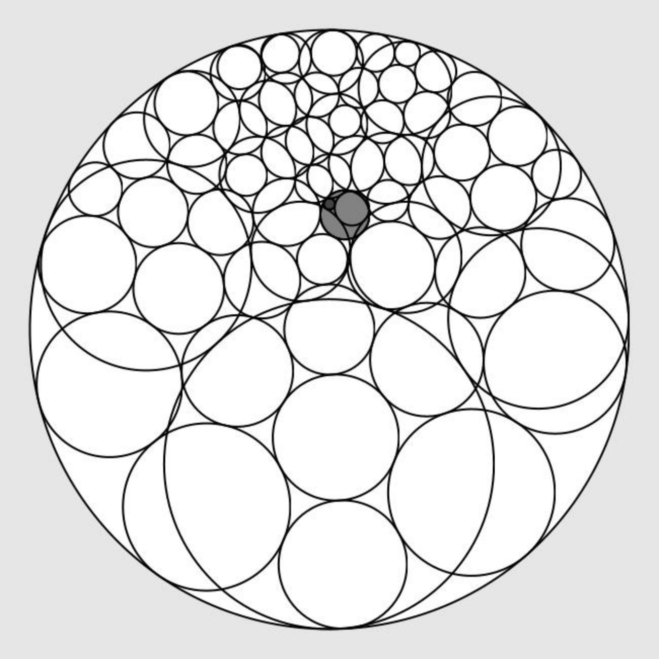


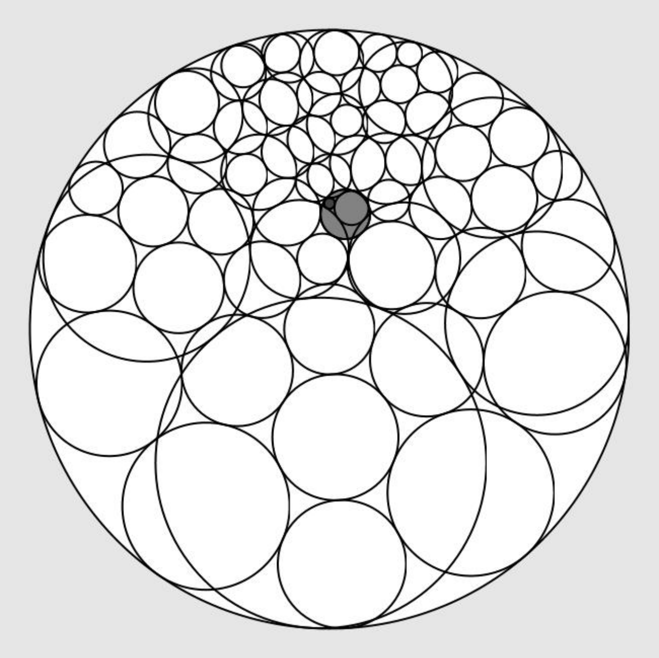


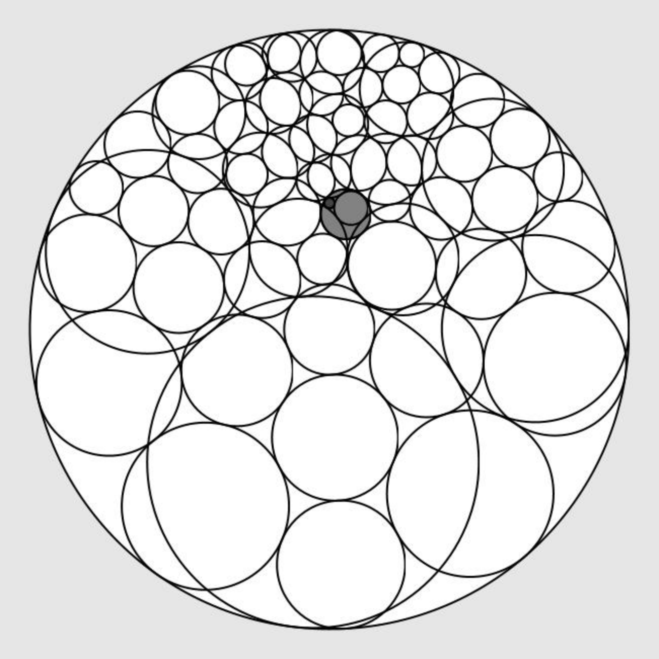


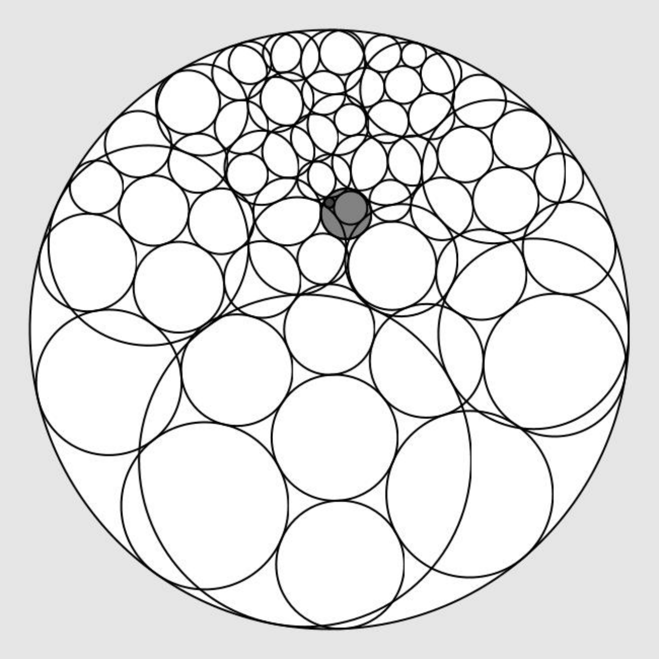












Limits of Schramm metric packings

Fix a triangulation K of the sphere. Fix a rational map R. Fix a face of K and assign to its vertices three mutually tangent discs in the spherical path metric on the Riemann surface of R. Suppose the geodesic triangle formed by the centers of these three discs contains no critical points of R.

Apply Schramm's metric packing theorem to pack K in a Riemannian mollifaction of the spherical path metric, using the given three centres and radii as the normalizing data.

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Do this for a sequence of mollifications whose support shrinks to the critical points. Take a subsequential limit of the packings. Under what conditions do all the radii remain positive?

If the limit radii are all positive then the limit is a metric packing for the singular path metric, i.e. a generalized branched packing.

Apply invariance of domain?

If we can solve that problem, and if the uniqueness in Schramm's metric packing theorem can be made to survive the limiting procedure, then we may be able to show in good cases that (at least locally) there is homeomorphism between the space of branched spherical packings of K with generalized branching and the $(4d-4)$-dimensional space of choices of three mutually tangent initial discs and $(2d-2)$ critical values, up to Möbius equivalence.

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Because there are two continuous parameters for each of the $2d-2$ simple generalized branch points, this local homeomorphism could follow by invariance of domain from uniqueness of the limiting Schramm packings, and their continuous dependence on the data.

Questions?

Thank you to the organisers!

I have enjoyed my virtual visit to Toronto so far. I may take a virtual tour of the CN tower later!