A discrete spherical Laplacian

Ivan Izmestiev

TU Wien

Workshop on the geometry of circle packings

Fields Institute, April 12–16

Motivation

Circle packings/patterns	$\to$	Polyhedra in $\mathbb{H}^3 \cup \mathrm{d}\mathbb{S}^3$
Combinatorics $+$ angles	$\to$	(Dual) boundary metrics

Circles $\to$ faces of $P \subset \mathbb{H}^3 \to$ vertices of $P^* \subset \mathrm{d}\mathbb{S}^3$.

Angles between faces $\to$ distances between vertices.

Alexandrov theorem, discrete analog of the Weyl problem: a sphere with a metric of positive curvature embeds isometrically into $\mathbb{R}^3$. In the polyhedral case the boundary metric determines the combinatorics.

Rivin–Hodgson'93: a generalization of Andreev's theorem.

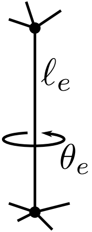
Fillastre–I.'11: a similar result for the torus, variational proof.

Variational methods: convexity/nondegeneracy of the functional.

The discrete Hilbert-Einstein functional

A Euclidean simplicial complex has cone angles around the edges.

The discrete Hilbert-Einstein functional:



$$F(\ell) = \sum \ell_e(2\pi - \theta_e)$$

$$\frac{\partial F}{\partial \ell_e} = 2\pi - \theta_e \quad \text{because of } \sum \ell_e d\theta_e = 0$$

Critical points $\leftrightarrow$ metrics without singularities.

There are versions for hyperbolic and spherical metrics:

$$F(\ell) = \pm 2\,\mathrm{vol}(M) + \sum \ell_e(2\pi - \theta_e).$$

If there are interior vertices, then F is neither convex nor concave.

Discrete Laplacians

The smooth Hilbert-Einstein functional tends to be convex on conformal deformations and concave on anticonformal ones.
Hidden behind this: Laplacian of the conformal factor and Laplacian of a tensor field, respectively.

We need discrete Laplacians and discrete conformal deformations, also in higher dimensions.

In this talk:

- Review of discrete Euclidean Laplacians.
- Definition of a discrete spherical Laplacian.

Discrete Euclidean Laplacians

The graph Laplacian

Let (V, E) be an abstract graph. Consider a function $f\colon V \to \mathbb{R}$.
Notation: $f(i) =: f_i$. Define a new function $\Delta f\colon V \to \mathbb{R}$ by

$$(\Delta f)_i = \sum_{\{i,j\} \in E} (f_i - f_j).$$

- f harmonic $\Leftrightarrow$ f_i is the average of its neighbors.
- The linear operator $\Delta\colon \mathbb{R}^V \to \mathbb{R}^V$ is self-adjoint and positive semidefinite: $\langle \Delta f, f \rangle = \sum_{ij} (f_i - f_j)^2$.

The spectrum of Δ reflects combinatorial properties of the graph:

- The number of spanning trees is $\frac{1}{n} \prod_{\lambda_i > 0} \lambda_i$.
- λ_1 is related to the expansion properties of the graph (Cheeger's inequality).

The cotangent Laplacian

Our graphs come from geometric objects: they are graphs of geometric triangulations. Geometry should be reflected in

$$(\Delta f)_i = \sum_{\{i,j\}\in E} w_{ij}(f_i - f_j).$$

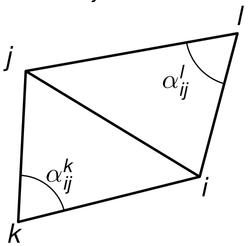
If $w_{ij} \geq 0$, then harmonic functions satisfy the maximum principle, and Δ is positive semidefinite: $\langle \Delta f, f \rangle = \sum_{ij} w_{ij}(f_i - f_j)^2$.

For a Euclidean triangulation put $w_{ij} = \cot \alpha_{ij}^k + \cot \alpha_{ij}^l$.

Because of $\cot\alpha + \cot\beta = \frac{\sin(\alpha+\beta)}{\sin\alpha \sin\beta}$,

$w_{ij} \geq 0 \Leftrightarrow \alpha_{ij}^k + \alpha_{ij}^l \leq \pi$,

the triangulation is Delaunay.



Why these weights?

A function f on the vertices of a triangle extends to a piecewise linear function $\tilde{f}$ on the triangle. $\Delta\tilde{f}$ does not make much sense, but $\nabla\tilde{f}$ does. We may recast the formula

$$\int_M (\Delta f) h \,\mathrm{dvol} = \int_M \langle \nabla f, \nabla h\rangle \,\mathrm{dvol} \qquad (f, M \text{ smooth})$$

into a definition

$$\langle \Delta f, h\rangle := \int_M \langle \nabla \tilde{f}, \nabla \tilde{h}\rangle \,\mathrm{dvol} \qquad (M \text{ polyhedral})$$

This turns out to be $\sum_{ij} w_{ij}(f_i - f_j)(h_i - h_j)$ with our cotangent weights.

What is $\langle \Delta f, h\rangle$? It may but need not be $\sum_{i\in V}(\Delta f)_i h_i$.

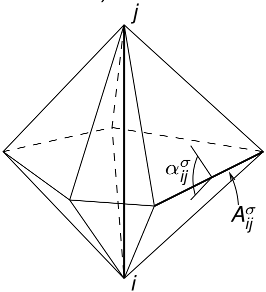
Duffin'59, Pinkall-Polthier'93

Higher dimensions, Dodziuk-Whitney Laplacians

The same approach in higher dimensions leads to

$$\langle \Delta f, h \rangle = \frac{1}{n(n-1)} \sum_{\{i,j\}} \left(\sum_{\sigma \supset \{i,j\}} A_{ij}^{\sigma} \cot \alpha_{ij}^{\sigma} \right) (f_i - f_j)(h_i - h_j).$$

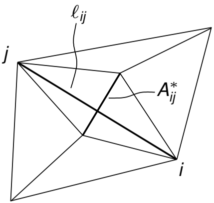
Here α_{ij}^{σ} is a dihedral angle opposite to the edge ij, and A_{ij}^{σ} is the volume of the corresponding codimension-2 face.



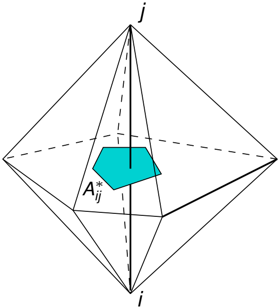
The approach also leads to the discrete Hodge Laplacian on simplicial cochains: simplex $\sigma \mapsto$ Whitney form W_{σ}.
Eckmann'45, Dodziuk'76, Forman'03

The Discrete Exterior Calculus Laplacian

Let together with a triangulation a dual decomposition be given (e. g. Voronoi diagram or power diagram).
Denote by A^*_{ij} the volume of the cell dual to the edge ij. DEC weights:



$$w_{ij} = \frac{A^*_{ij}}{\ell_{ij}}$$



(Weighted) Delaunay + (Power) Voronoi $\Rightarrow w_{ij} \geq 0$.

Dodziuk-Whitney and DEC Laplacians coincide in dimension 2 but are different in higher dimensions.

Do we need positive weights?

On a Delaunay triangulation, the weights of DEC Laplacian are ≥ 0, but the weights of the Dodziuk-Whitney Laplacian may be negative (in higher dimensions).

Still, the Dodziuk-Whitney Laplacian is positive semidefinite, for any triangulation:

$$\langle \Delta f, f \rangle = \int_M \|\nabla \tilde{f}\|^2 \,\mathrm{dvol} \geq 0.$$

If one wants to keep the maximum principle, one has to choose triangulations carefully.

We now move to a spherical discrete Laplacian

Two old theorems and two new ones

Theorem

For every infinitesimal isometric deformation (IID) of the paraboloid the vertical component of the deformation is a harmonic function.

Theorem

The vertical components of IIDs of convex polyhedra inscribed into the paraboloid are discrete harmonic functions.

Theorem

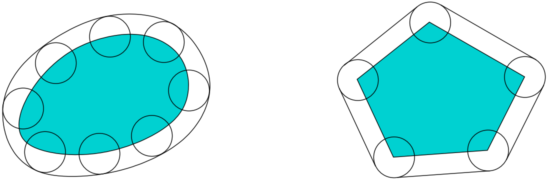
For every IID of a subset of the sphere the radial component of the deformation is an eigenfunction of $\Delta^{\mathbb{S}^2}$ to the eigenvalue 2.

Theorem

The vertical components of IIDs of convex inscribed polyhedra are eigenfunctions of the discrete spherical Laplacian to the eigenvalue 2.

The Steiner formula: smooth and polyhedral

M: body with smooth boundary, P: convex polyhedron.
Compute the volumes of their t-neighborhoods.



$$\mathrm{vol}(M_t) = \mathrm{vol}(M) + t \cdot \mathrm{area}(\partial M) + t^2 \int_{\partial M} \frac{\kappa_1 + \kappa_2}{2}\, dx + t^3 \int_{\partial M} \frac{\kappa_1 \kappa_2}{3}\, dx$$

$$\mathrm{vol}(P_t) = \mathrm{vol}(P) + t \cdot \mathrm{area}(\partial P) + t^2 \sum_{\text{edges}} \ell_e \frac{\pi - \theta_e}{2} + t^3 \sum_{\text{vertices}} \frac{\varphi_v}{3}$$

The total mean curvature is related to the sum of lengths times angles.

The Steiner formula: general

The coefficients in the Steiner formula depend continuously on the body (wrt the Hausdorff metric).

For an arbitrary convex body K one has

$$\mathrm{vol}(K_t) = W_0(K) + 3tW_1(K) + 3t^2 W_2(K) + t^3 W_3(K).$$

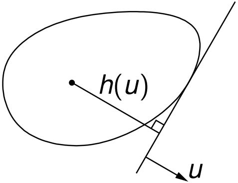
$$W_0(K) = \mathrm{vol}(K), \quad W_1(K) = \frac{1}{3}\,\mathrm{area}(\partial K), \quad W_2(K) = ?, \quad W_3(K) = \frac{4\pi}{3}$$

W_i are called i-th *quermassintegral*, it is proportional to the average projection volume to $(3-i)$-dimensional subspaces.

For example, the surface area is twice the average projection area (Cauchy), and the total mean curvature is 2π times the average width.

The support function

A convex body is determined by its support function $h\colon \mathbb{S}^2 \to \mathbb{R}$.



Smooth boundary $\Rightarrow$ smooth h, and

$$W_2(M) = \frac{1}{3} \int_{\mathbb{S}^2} h \, \mathrm{darea}_{\mathbb{S}^2}$$

$$W_1(M) = \frac{1}{6} \int_{\mathbb{S}^2} h(2h - \Delta h) \, \mathrm{darea}_{\mathbb{S}^2}$$

Theorem

$W_2^2 \geq W_1 W_3$

Theorem

For the spherical Laplacian, $\lambda_1 = 2$.

Claim

These two theorems tell the same.

Proof of the claim

$$W_2^2 \geq W_1 W_3 \qquad \Leftrightarrow \qquad \lambda_1 = 2$$

$$W_3 = \mathrm{vol}(\mathbb{B}^3) = \frac{1}{3}\,\mathrm{area}(\mathbb{S}^2)$$

$$W_2 = \frac{1}{3}\int_{\mathbb{S}^2} h\ \mathrm{darea} = \frac{1}{3}\langle h, 1\rangle_{L^2} = \frac{\mathrm{area}(\mathbb{S}^2)}{3}\bar{h}$$

Put $f = h - \bar{h}$. Then $\langle f, 1\rangle_{L^2} = 0 \Rightarrow \langle \Delta f, f\rangle_{L^2} \geq 2\|f\|_{L^2}^2$ and

$$W_1 = \frac{1}{6}\langle h, 2h - \Delta h\rangle_{L^2} = \frac{1}{6}\left(2\,\mathrm{area}(\mathbb{S}^2)\bar{h}^2 + \langle f, 2f - \Delta f\rangle_{L^2}\right)$$

$$\leq \frac{1}{3}\,\mathrm{area}(\mathbb{S}^2)\bar{h}^2 = \frac{W_2^2}{W_3}$$

Polyhedra with fixed directions of normals

Let $u_1, \ldots, u_n$ be points on $\mathbb{S}^2$ such that every open hemisphere contains at least one of them. Then for every $h \in \mathbb{R}^n$ the polyhedron

$$P(h) = \{x \in \mathbb{R}^3 \mid \langle u_i, x \rangle \leq h_i\}$$

is bounded, and for some h it has exactly n faces with unit normals u_i.

Generically $P(h)$ is simple (three edges at every vertex), but its combinatorics usually depends on h.
A canonical choice: $h_i = 1$ for all i, circumscribed about the sphere.
The adjacency graph of the faces is then the Delaunay tesselation.

All possible combinatorics: weighted Delaunay/regular/coherent triangulations.
P. McMullen'73, Gelfand–Kapranov–Zelevinsky'94, Fillastre–I.'17

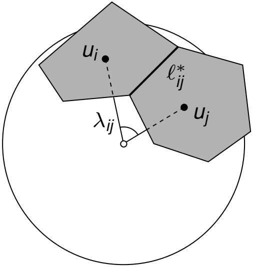
A discrete spherical Laplacian

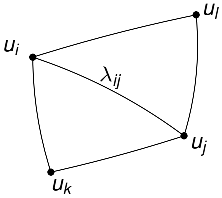
The discrete Laplacian associated with a non-negative spanning finite point set $U \subset \mathbb{S}^2$ is defined in a weak sense as

$$\langle 2f - \Delta f, f \rangle = 2\,\mathrm{Area}(f),$$

where $\mathrm{Area}(f)$ is the quadratic form computing the surface area of the polyhedron $P(f)$ under assumption of the circumscribed combinatorics.

$$\langle \Delta f - 2f, f \rangle = \sum_{ij} \frac{\ell^*_{ij}}{\tan \lambda_{ij}}((f_i^2 + f_j^2) - 2 \sec \lambda_{ij} f_i f_j)$$





Good things about this Laplacian

Theorem

If the sequence $U_n \subset \mathbb{S}^2$ fills the sphere, then the corresponding Laplacians converge to the spherical Laplacian in the weak sense.

Proof.

Continuity of the area of convex surfaces wrt the Hausdorff distance. □

Theorem

The spectral gap is equal to 2, as in the smooth case. That is, the quadratic form $\mathrm{Area}(f)$ has signature $(+, 0, 0, 0, -, \ldots, -)$.

Proof.

Follows from $W_2^2 \geq W_1 W_3$. □

Lorentzian signature... The "space of shapes" of convex polyhedra with fixed face normals carries a natural hyperbolic metric: Thurston'98, Bavard–Ghys'92, Fillastre-I.'17.