13 April 2021

Fields Institute : The Geometry of Circle Packings

Circle Packings on Complex Projective Surfaces

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(joint work with S. Ballas, A. Casella, L. Ruffoni)

Part 1

1 • Geometric Structures on Surfaces – 3 flavors: complex, hyperbolic, and complex projective.

2 • Circles on Complex Projective Surfaces

3 • The Koebe-Andre'ev-Thurston Theorem (KAT Theorem)

4 • Deformation Spaces of Geometric Surfaces (Compact Case)

5 • Packable Surfaces and the Kojima-Mizushima-Tan Theorem (KMT Theorem)

6 • Packable Surfaces and the Kojima-Mizushima-Tan Conjecture (KMT Conjecture)

Part 2

1 • Non-compact Surfaces: similarities and differences with the compact case

2 • The Conformally Thrice Punctured Sphere (CTPS) and Its Geometric Structures

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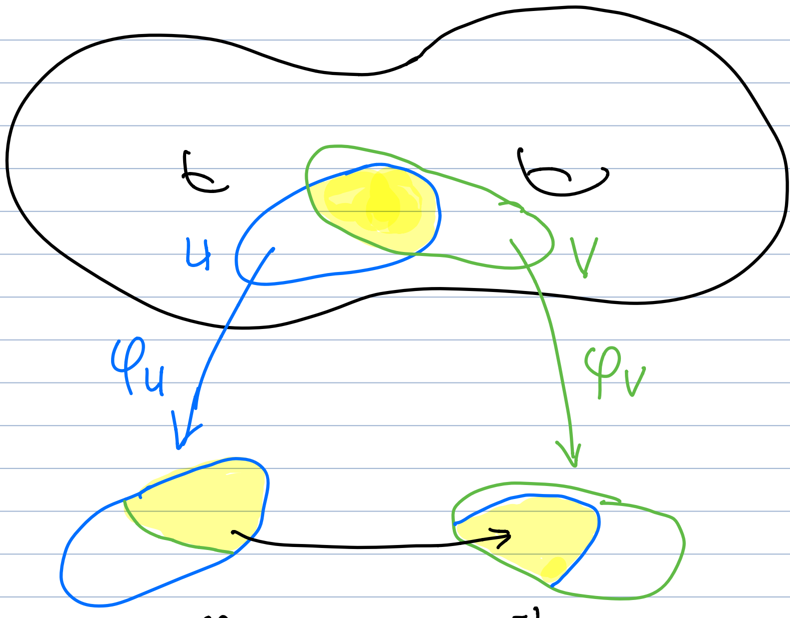
6 • Combinatorial Tool for Detecting Punctures - Vertex Extremal Length

Part 1

1. Geometric Structures on Surfaces - Three Flavors



$$\varphi_{UV} = \varphi_V \circ \varphi_U^{-1}$$

S topological surface

atlas of charts

* $\mathcal{A} = \{(U, \varphi_U)\}$
* $\mathcal{A}$ maximal w.r.t ...

$(S, \mathcal{A})$ - geometric surface

$\mathbb{C}P^1$ complex projective line = Riemann sphere

Flavor 1 Riemann Surfaces $S_{\mathbb{C}}$

Conformal (or Riemann) Structure Target Space $\mathbb{C}$

Here the transition maps φ_{UV} are <u>conformal homeomorphisms</u> (complex analytic with non-zero derivatives) $\varphi_{UV}' \neq 0$

Flavor 2 Hyperbolic Surfaces $S_{\mathbb{H}^2}$

Hyperbolic Structure Target Space $\mathbb{D} \subset \mathbb{C}$

The target space is any disk $\mathbb{D}$ equipped with its Poincaré metric and φ_{UV} are <u>hyperbolic isometries</u>

$\varphi_{UV} \in \mathrm{Aut}\,\mathbb{D} = \text{Möb}(\mathbb{D}) \cong PSL(2, \mathbb{R}) \cong \mathrm{Isom}^+(\mathbb{H}^2)$

Flavor 3 Complex Projective Surfaces $S_{\mathbb{C}P^1}$

Möbius or $\mathbb{C}P^1$ Structure Target Space $\mathbb{C}P^1$

The transition maps φ_{UV} are <u>Möbius transformations</u>

$\varphi_{UV} \in \text{Möb}(\mathbb{C}P^1) \cong PSL(2, \mathbb{C})$

IMPORTANT OBSERVATION

$$S_{\mathbb{H}^2} \rightsquigarrow S_{\mathbb{CP}^1} \rightsquigarrow S_{\mathbb{C}}$$

$$\mathcal{A}_{\mathbb{H}^2} \subset \mathcal{A}_{\mathbb{CP}^1} \subset \mathcal{A}_{\mathbb{C}}$$

$\rightsquigarrow$ means "uniquely determines up to isomorphism"

$$S_{\mathbb{C}} \rightsquigarrow S_{\mathbb{H}^2}$$

Uniformization Theorem (U.T.)

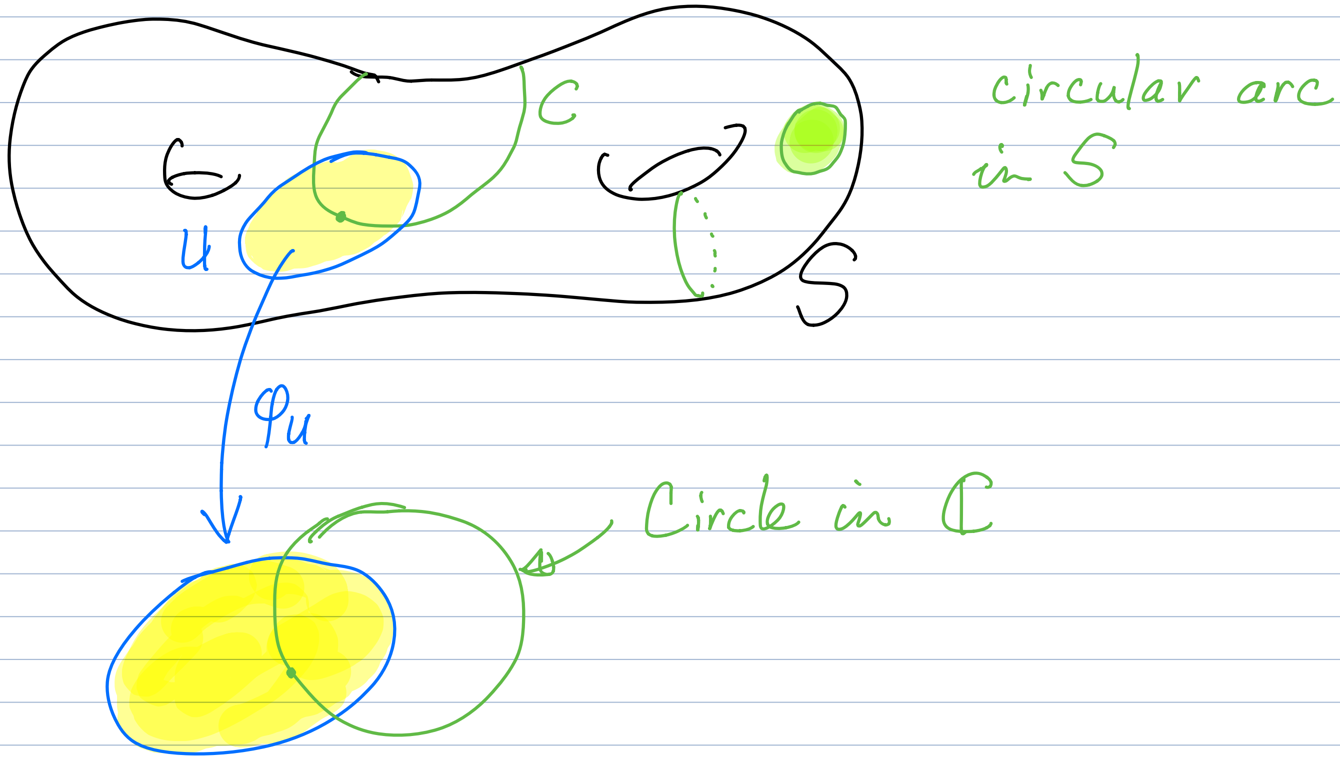
$$\begin{tikzcd}
\widetilde{S}_{\mathbb{C}} \arrow[r, phantom, "\overset{\text{U.T.}}{\cong}"] \arrow[d] & \mathbb{H}^2 \arrow[d] \\
S_{\mathbb{C}} \arrow[r, phantom, "\cong"] & \mathbb{H}^2/\pi_1(S)
\end{tikzcd}$$

$\pi_1(S)$ acts on $\widetilde{S}_{\mathbb{C}}$ by automorphisms

$\therefore$ on $\mathbb{H}^2$ by isometries

$\cong$ conformal isomorphisms

2 • Circles on Complex Projective Surfaces



$\varphi_U(C \cap U)$ is a circular arc in $\mathbb{C}$

- For hyperbolic surfaces, circles that bound disks are metric circles.
- In general, there is no metric on a complex projective surface that defines the circles.

A couple of notes

Hyperbolic surface : Riemannian metric of constant curvature -1

: $\mathbb{H}^2/\pi_1(S)$ where $\pi_1(S) \curvearrowright \mathrm{Isom}^+(\mathbb{H}^2)$

$\mathbb{C}P^1$ surface : described by a developing pair (dev, ρ)

developing map

where $\mathrm{dev} : \tilde{S}_{\mathbb{C}P^1} \to \mathbb{C}P^1$, the analytic continuation of a chart map φ_{U}, is equivariant w.r.t. $\rho : \pi_1(S) \to PSL(2,\mathbb{C})$ so that

holonomy representation

$$\mathrm{dev}(\gamma \cdot x) = \rho(\gamma) \cdot \mathrm{dev}(x) \qquad x \in \tilde{S}_{\mathbb{C}P^1},\ \gamma \in \pi_1(S).$$

: In general, no metric of constant curvature, and none that defines the circles

: dev is a local homeomorphism, not necessarily proper so not a covering map

: a circle in $S_{\mathbb{C}P^1}$ lifts to one in $\tilde{S}_{\mathbb{C}P^1}$ that develops via dev to a circle in $\mathbb{C}P^1$.

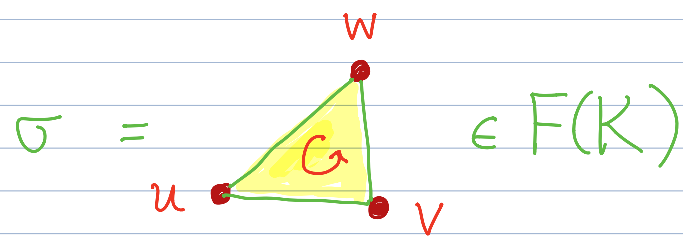
3. The Koebe–Andre'ev–Thurston Theorem (KAT)

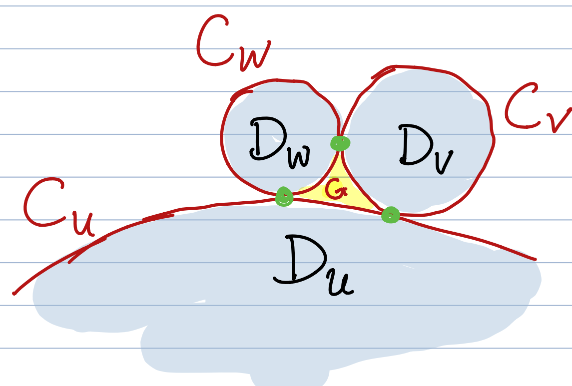
Tangency Version

Let S be a surface and K a simplicial triangulation of S. Then K determines an essentially unique, complete metric of constant curvature ρ and a univalent circle packing

$$\mathcal{C}(K) = \{ C_v : v \in V(K) \}$$

in S_ρ with the combinatorics of K. Moreover, $\mathcal{C}(K)$ is unique up to isometries.





$$\mathcal{C} = \{ C_v : v \in V(K) \}$$

- each C_v is a metric circle in S_ρ that bounds a disk D_v.
- $uv \in E(K) \Longrightarrow C_u \cap C_v = \{ \bullet \}$
- $uvw \in F(K) \Longrightarrow C_u, C_v, C_w$ determines an interstice (orientation matters)
- univalent means the disk interiors $\mathring{D}_v$ are pairwise disjoint

4 • Deformation Spaces of Geometric Surfaces (Compact Case)

Fix a compact topological surface S of genus $g \geq 2$. Then S may be endowed with complex, hyperbolic, and complex projective structures.

$\mathcal{M}^g_{\mathbb{C}}$ - moduli space of marked complex structures on S

(Teichmüller Space) - $6g-6$ dim'l Euclidean space

$\mathcal{M}^g_{\mathbb{H}^2}$ - moduli space of marked hyperbolic structures on S

- $6g-6$ dim'l Euclidean space

$\mathcal{M}^g_{\mathbb{CP}^1}$ - moduli space of marked complex projective structures on S

- $12g-12 = 2(6g-6)$ dim'l Euclidean space

Forgetful Map f

$$
\begin{tikzcd}
\mathcal{M}_{\mathbb{H}^2} \arrow[r, hook, "i"] & \mathcal{M}_{\mathbb{CP}^1} \arrow[d, "f"] & \supset f^{-1}(X) \\
 & \mathcal{M}_{\mathbb{C}} \arrow[ul, "h"] & \ni X
\end{tikzcd}
$$

continuous section

$$
\begin{tikzcd}
S_{\mathbb{H}^2} \arrow[rr, squiggly, "\text{enlarge atlas}", "i"'] & & S_{\mathbb{CP}^1} \arrow[dl, squiggly, "f"] \\
 & S_{\mathbb{C}} \arrow[ul, squiggly, "h"] &
\end{tikzcd}
$$

uniformization

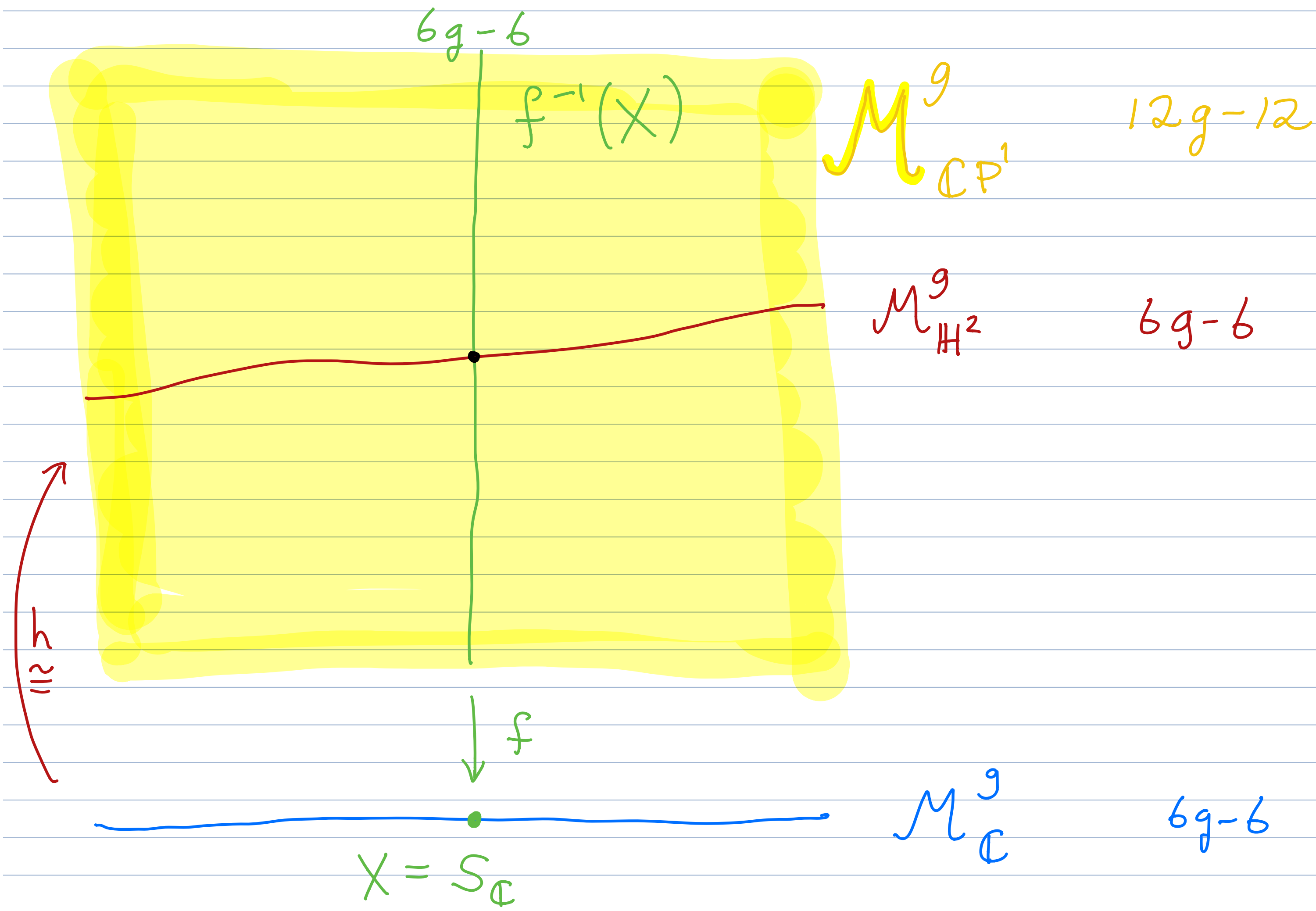
enlarge atlas

This is an analytic homeomorphism.

The fiber $f^{-1}(X)$ parameterizes the (marked) complex projective structures that are pairwise conformally equivalent and hence determine the same complex structure.

$$\dim f^{-1}(X) = 6g-6.$$

Cartoon Picture of the Moduli Spaces



5 • Packable Surfaces and the Kojima-Mizushima-Tan Theorem (KMT Theorem)

K - simplicial triangulation of surface S

- KAT $\Rightarrow \exists!$ hyperbolic metric ρ on S such that K may be realized as a circle packing $\mathcal{C} = \{C_v : v \in V(K)\}$. Call ρ the Thurston metric for S.

  Gives a geodisic triangulation of S_ρ by connecting circle centers with geodesic segments

Definition The hyperbolic structure determined by the Thurston metric ρ is the KAT solution.

Question Are there complex projective structures other than that of ρ that have a circle packing in the pattern of K?

Fix K and let $\text{Pack}(K)$ denote the collection of (marked) K-packable $\mathbb{C}P^1$-surfaces with the subspace topology:

$$\text{Pack}(K) \subset \mathcal{M}^g_{\mathbb{C}P^1}$$

Can one deform the complex projective structure of S_ρ away from that of the Thurston metric and retain packability in the pattern of K?

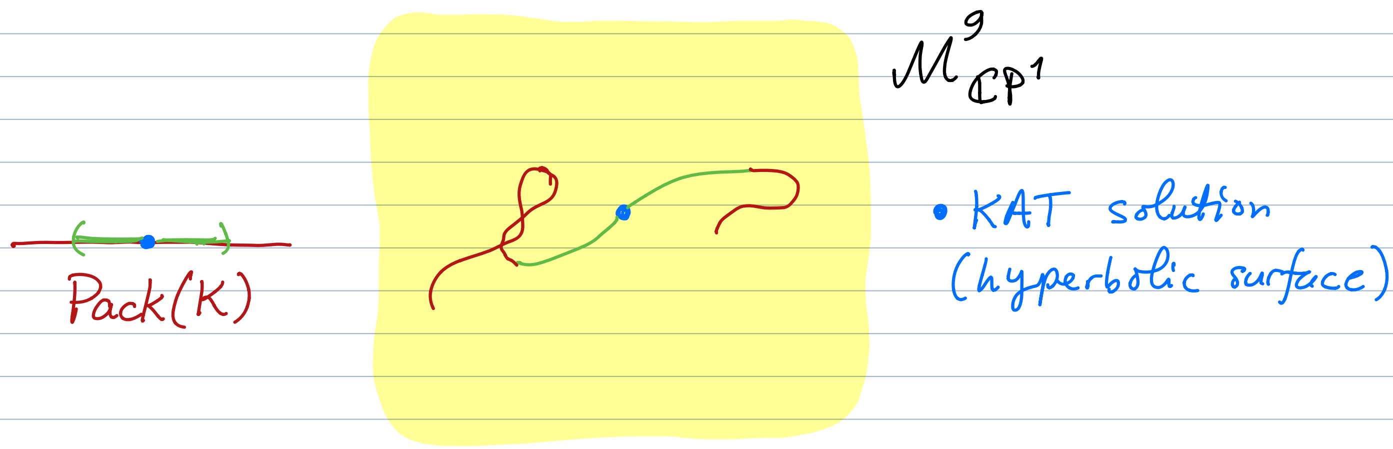
Kojima, Mizushima, and Tan in 2003:

Circle Packings on Surfaces with Projective Structures
J. Diff. Geo. 63 (2003) 349-397

answered this affirmatively.

KMT Theorem (2003)

(1) There is a nbhd $\mathcal{U}$ of the KAT solution in Pack(K) homeomorphic with $\mathbb{R}^{6g-6}$ and that embeds in $\mathcal{M}^g_{\mathbb{CP}^1}$.



(2) If K has exactly one vertex and $\tilde{K}$ is simplicial then Pack(K) is homeomorphic with $\mathbb{R}^{6g-6}$.

6 • Packable Surfaces and the Kojima–Mizushima–Tan Conjecture (KMT Conjecture) + Refinements!

KMT Conjecture: For any simplicial triangulation K of S,
$\mathrm{Pack}(K) \cong \mathbb{R}^{6g-6} \hookrightarrow \mathcal{M}^g_{\mathbb{C}P^1}$.

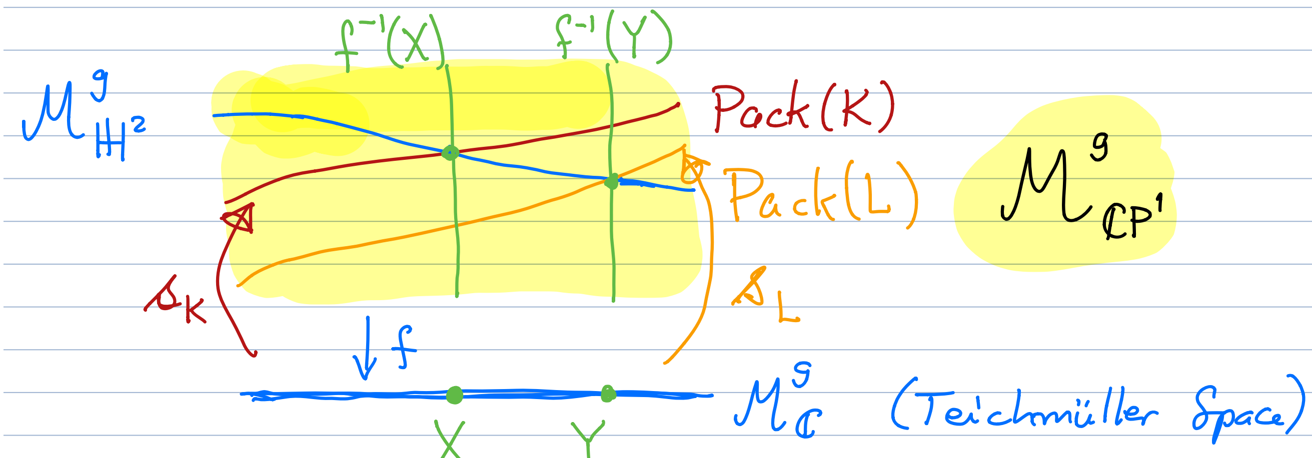
Refinement 1: For any simplicial triangulation K of S and for any complex structure X on S, there is a unique complex projective structure $\sigma \in f^{-1}(X) \cap \mathrm{Pack}(K)$.
i.e., Given the Riemann surface $S_{\mathbb{C}}$, $\exists!$ $\mathbb{C}P^1$-surface $S_{\mathbb{C}P^1}$ conformally equivalent to $S_{\mathbb{C}}$ that is packable in the pattern of K.

Refinement 2: For any simplicial triangulation K of S the forgetful map f restricts to a homeomorphism:

$$f : \mathrm{Pack}(K) \xrightarrow{\cong} \mathcal{M}^g_{\mathbb{C}}$$

Refinement 3: Every simplicial triangulation K of S determines a smooth section

$$\mathcal{M}^g_{\mathbb{C}} \xrightarrow[s_K]{} \!\!\!\!\twoheadrightarrow \mathrm{Pack}(K) \subset \mathcal{M}^g_{\mathbb{C}P^1}$$



The KAT solution for K is found from $\mathcal{M}^g_{\mathbb{H}^2} \cap \mathrm{Pack}(K)$.

Strategy for Proving KMT

(1) $\mathrm{Pack}(K)$ is a manifold of dimension $6g-6$;
(2) $f|_{\mathrm{Pack}(K)}$ is locally injective;
(3) $f|_{\mathrm{Pack}(K)}$ is proper;
(4) $\mathrm{Pack}(K)$ is connected.

(1)–(3) + invariance of domain $\Longrightarrow$ $f|_{\mathrm{Pack}(K)}$ is a covering map

+ (4) $\Longrightarrow$ KMT.

- 2018 J-M Schlenker & A. Yarmola verified step (3).

arXiv: 1806.05254 v1

Properness for circle packings and Delaunay circle patterns on complex projective structures

Shout out to Wai Yeung Lam – see his talk

Circle packings on surfaces with complex projective structures

on Friday at 9:30!

Part 2

1 • Non-compact Surfaces: similarities and differences with the compact case

The Teichmüller theory is well-behaved for complex structures as long as S is of <u>finite type</u>:

$$S = S_g \setminus \{x_1, \dots, x_n\},$$

the complement of n points in a surface of genus g, $n \geq 1$, $g \geq 0$ with $6g-6+3n > 0$. We still have

$$\mathcal{M}_{\mathbb{C}}^{g,n} \cong \mathcal{M}_{\mathbb{H}^2}^{g,n} \cong \mathbb{R}^{6g-6+3n}$$

complete hyperbolic metrics

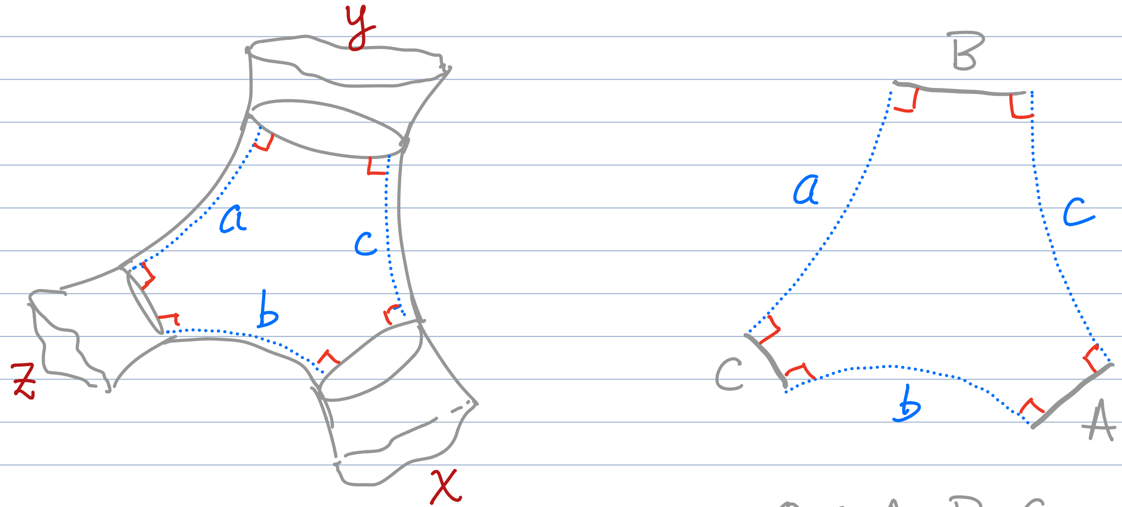
BUT $\mathcal{M}_{\mathbb{C}P^1}^{g,n}$ is infinite-dimensional!

	Compact	Non-Compact
$\mathcal{M}_{\mathbb{C}} \cong \mathcal{M}_{\mathbb{H}^2}$	YES	YES
$\mathcal{M}_{\mathbb{C}P^1}$	twice the dimension of $\mathcal{M}_{\mathbb{C}}$	infinite-dimensional
KAT	K determines a ! hyperbolic metric	K determines a ! complete hyperbolic metric
$\text{Pack} = \bigcup_K \text{Pack}(K)$ (All packable surfaces)	$\text{Pack} \cap \mathcal{M}_{\mathbb{H}^2}$ is countable and dense (Packable hyperbolic)	$\text{Pack} \supset \mathcal{M}_{\mathbb{H}^2}$ (All complete hyperbolic surfaces are packable)
Equilateral Surfaces (Belyi surfaces)	Countable and dense in $\mathcal{M}_{\mathbb{C}}$	All surfaces of finite topological type are Belyi (Bishop-Rempe Mar. 2021)
KMT Conjecture	?	False

2 • The Conformally Thrice Punctured Sphere (CTPS) and Its Geometric Structures

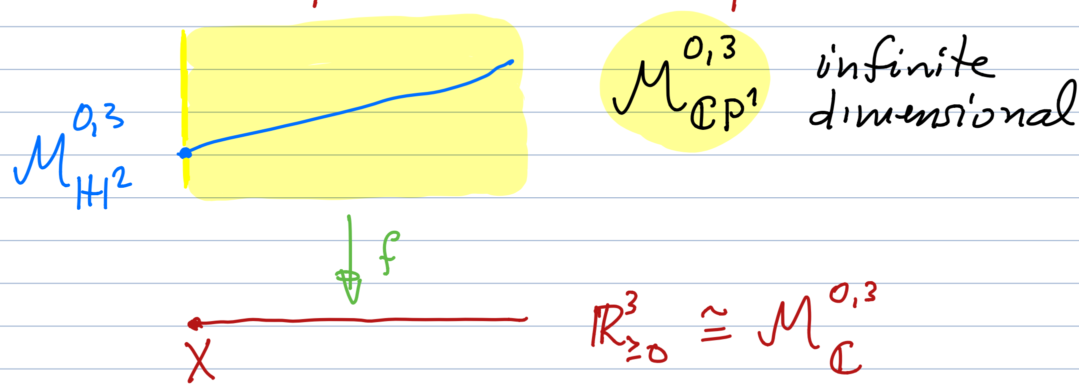
Let $S = S^{0,3} = \mathbb{S}^2 \setminus \{x, y, z\}$

$$\mathcal{M}^{0,3}_{\mathbb{C}} \cong \mathcal{M}^{0,3}_{\mathbb{H}^2} \cong \mathbb{R}^3_{\geq 0} = \{(A, B, C) : A, B, C \geq 0\}$$



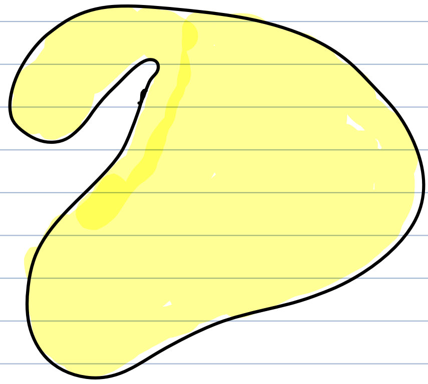
$0 \leq A, B, C < \infty$

When $A = 0$, x represents a cusp.

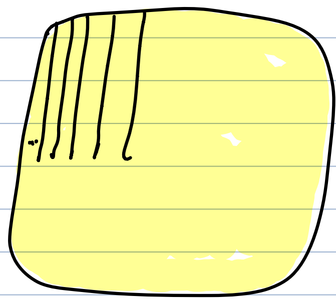


Let's concentrate on $A = 0, B = 0, C = 0$, the sphere with 3 cusps. There is one complex structure X, that of $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$, and there is a unique complete hyperbolic metric ρ, but $f^{-1}(X)$, the space of complex projective structures conformally equivalent to X, is infinite-dimensional.

$\mathcal{M}^{0,1}_{\mathbb{C}P^1}$ is infinite-dimensional



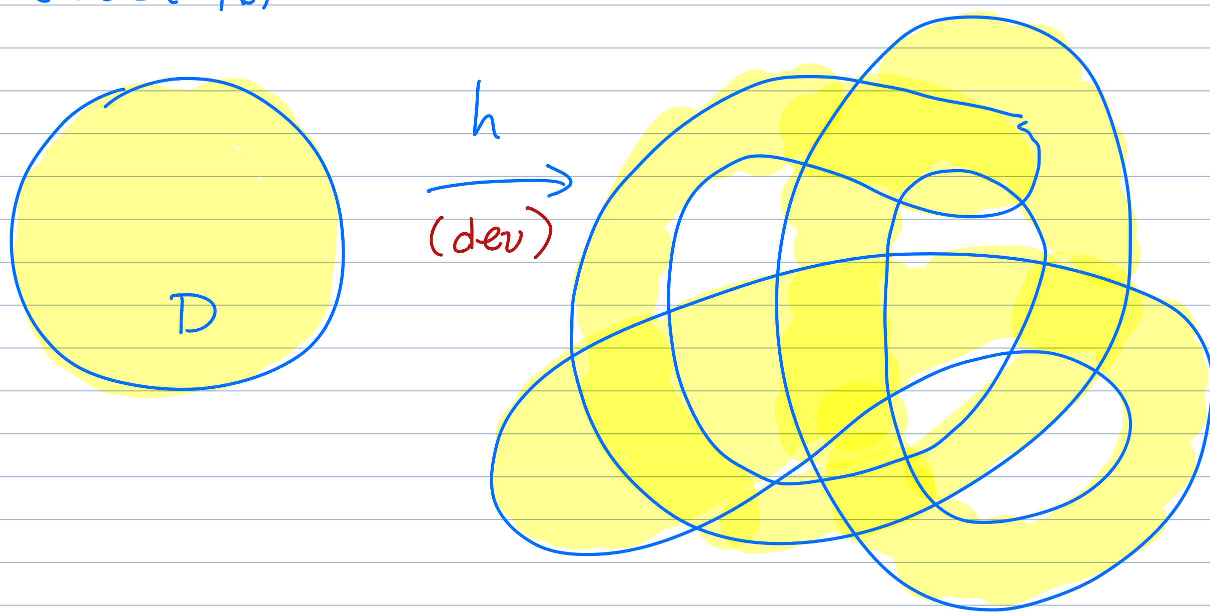
Any simply-connected domain in $\mathbb{C}$ defines a $\mathbb{C}P^1$-structure and two such D_1 and D_2 are isomorphic as complex projective surfaces iff $\exists\, \mu \in PSL(2,\mathbb{Z})$ s.t. $\mu(D_1) = D_2$.



More generally, let D be a topological disk and

$$h : D \to \mathbb{C}P^1$$

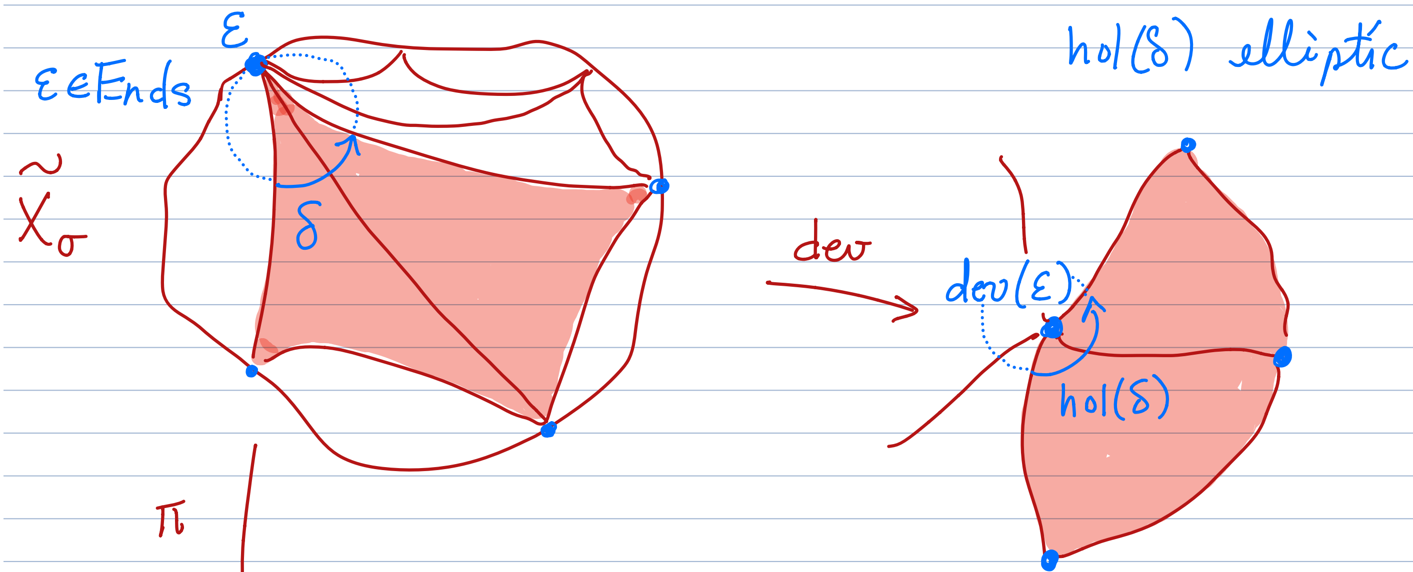
any locally injective map. Then h defines a complex projective structure. In fact $h = dev$ and $h_1 \cong h_2$ iff $h_2 = \mu \circ h_1$ for some $\mu \in PSL(2,\mathbb{C})$

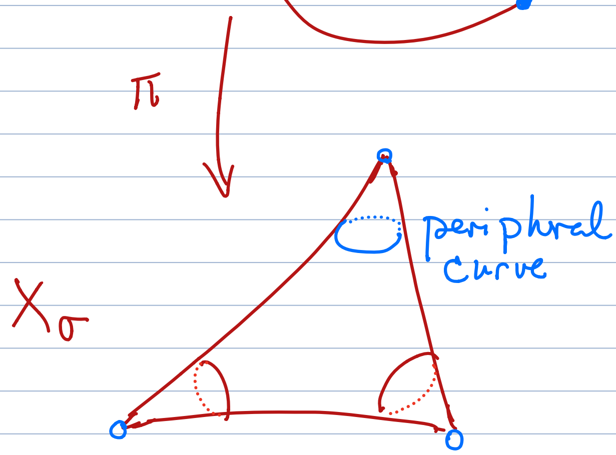


3 • Characterizing the Tame, Relatively Elliptic $\mathbb{CP}^1$-Structures on the CTPS

(Joint with S. Ballas, A. Carella, L. Ruffoni)

Let σ be a complex projective structure on X and denote X with structure σ as X_σ. $[f(X_\sigma)=X]$





$$\pi_1(X_\sigma) \xrightarrow{hol} PSL(2,\mathbb{C})$$

$$\text{Ends} \underset{\text{discrete}}{\subset} \overline{X_\sigma} = \tilde{X}_\sigma \cup \text{Ends}$$

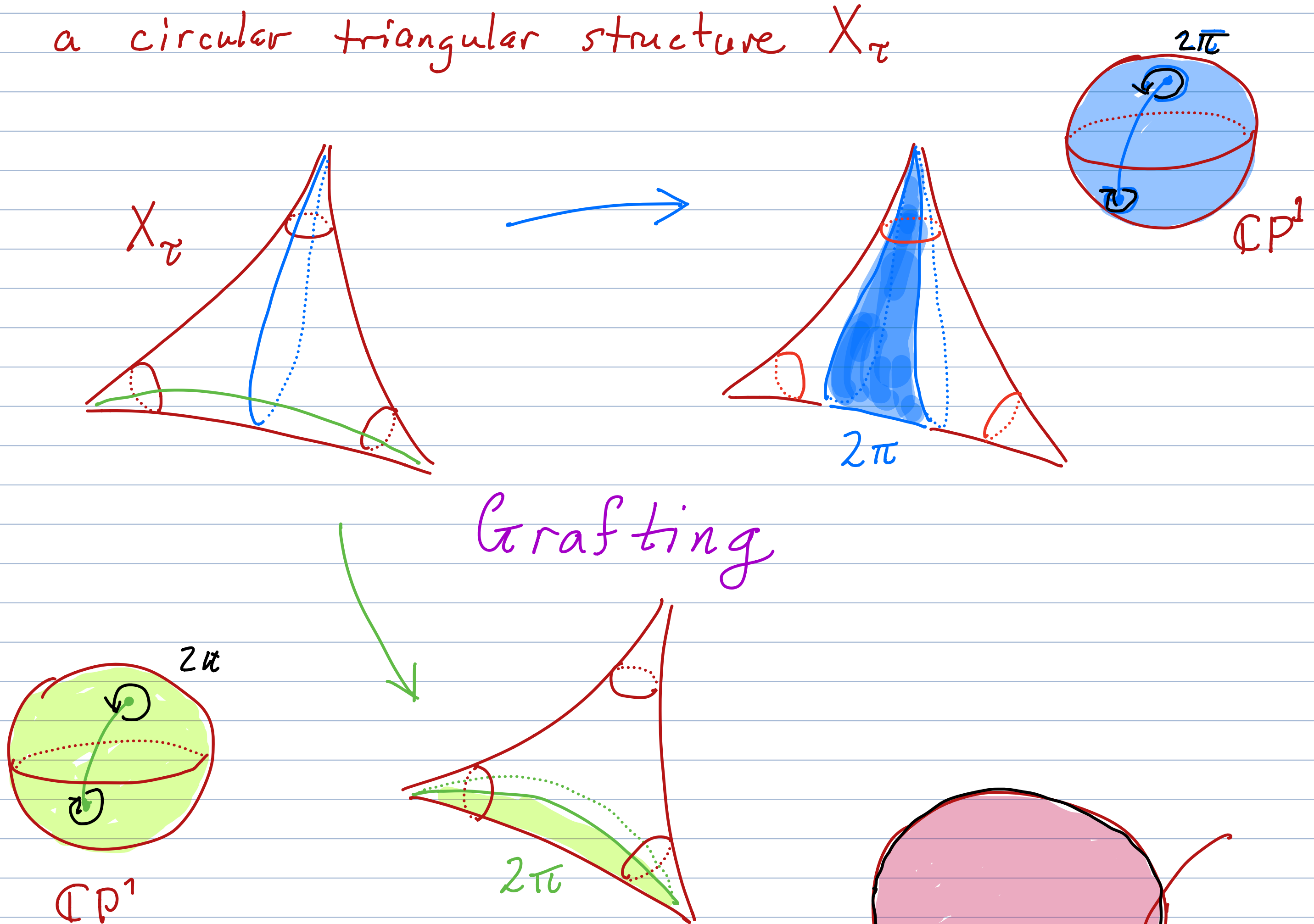
Tame – dev extends continuosly to Ends

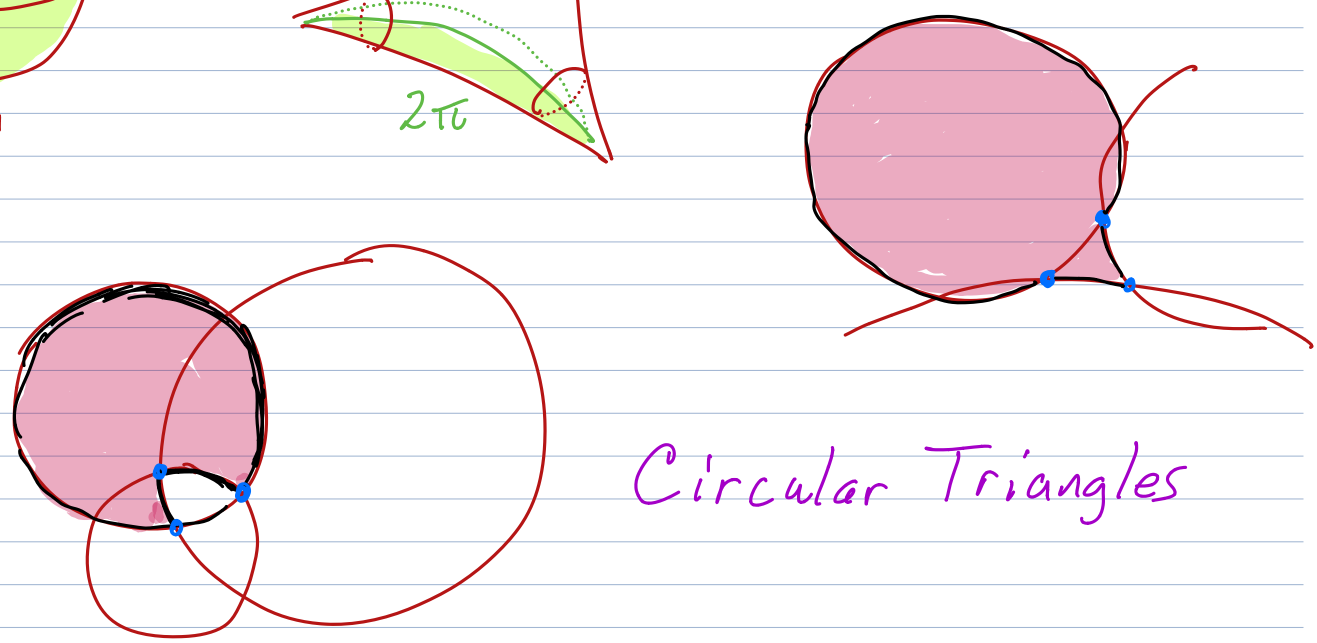
Relatively Elliptic – hol(δ) is an elliptic Möbius transformation for each peripheral deck transformation δ

Each X_σ is conformally equivalent to X, the conformally thrice punctured sphere.

Let $\mathcal{P}^\circ \subset \mathcal{M}^{0,3}_{\mathbb{CP}^1}$ be the collection of X_σ such that σ is tame and relatively elliptic.

<u>Thm</u> (Ballas, B, Casella, Ruffoni) $\mathcal{P}^\circ \cong \mathbb{R}^3$ and each X_σ may be obtained by a sequence of 2π-graftings along ideal arcs starting with a circular triangular structure X_τ



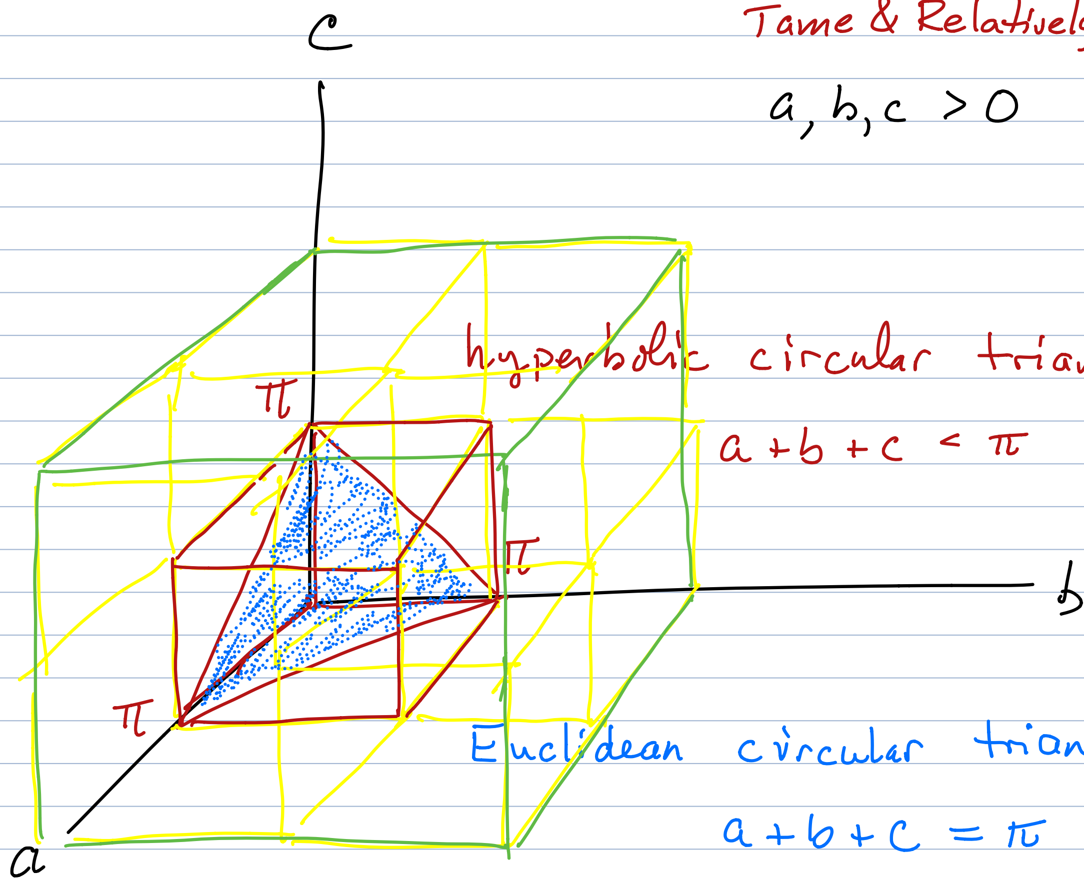


Circular Triangles

4 • The Moduli Space for $\mathbb{CP}^1$-Structures on the CTPS

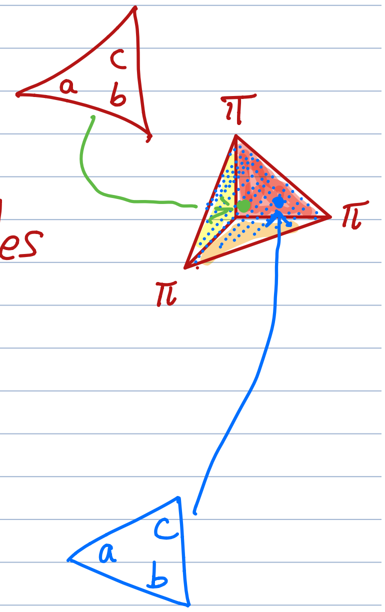
Tame & Relatively Elliptic

$a, b, c > 0$



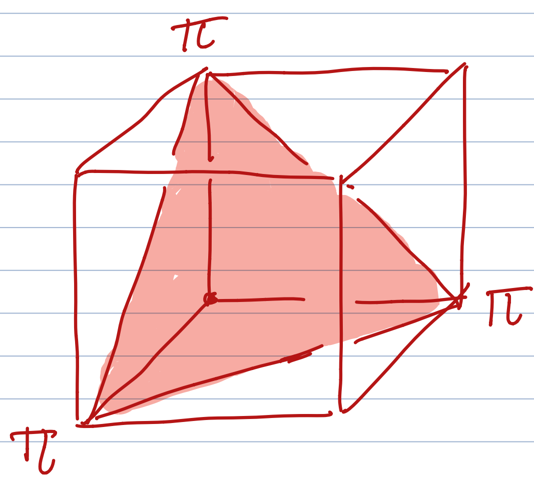
hyperbolic circular triangles

$a + b + c < \pi$

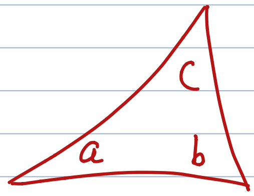


Euclidean circular triangles

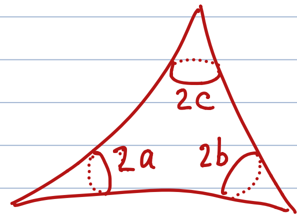
$a + b + c = \pi$



5 • Circle Packing Hyperbolic and Euclidean CTPS's



Hyperbolic Triangle



X_σ

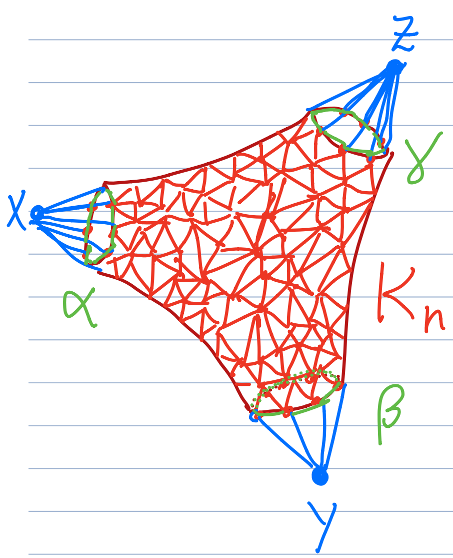
Joint with Lorenzo Ruffoni

Let K triangulate $S^{0,3}$

$$K = \bigcup_{n=1}^{\infty} K_n$$

K_n - finite core triangulation obtained by cutting off the three infinite ends along peripheral curves α, β, γ

$\overline{K}_n$ - add three vertices x, y, z and cone to the curves α, β, γ



$\overline{K}_n$

HOW?

$\mathcal{C}_n$ - circle packing of X_σ in the pattern of $\overline{K}_n$.

Take a limit as $n \to \infty$ to get a circle packing $\mathcal{C}$ of X_σ in the pattern of K

<u>But</u> What do the ends of $\mathcal{C}$ look like?

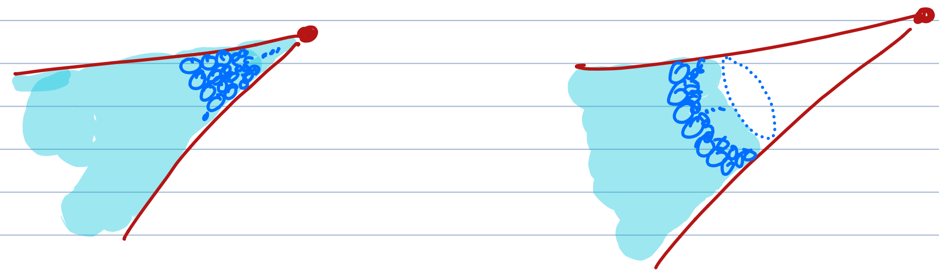
- Could be <u>punctures</u> or <u>holes</u>

conformally punctured disks — conformally annuli

6 • Combinatorial Tool for Detecting Punctures - Vertex Extremal Length

The ends need to be conformal punctures rather than conformal annuli:

Circles of $\mathcal{C}$ should accumulate at a pucture in X_σ, not along a peripheral curve of X_σ.



How do we ensure punctures rather than annular ends for $\mathcal{C}$?

Answer: Calculate the Vertex Extremal Length (VEL) of the curve family in K that accumulates at the end.

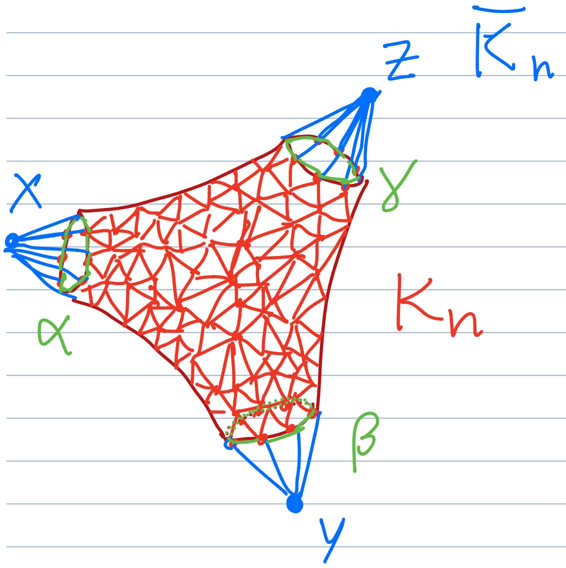
$$\text{VEL} = \infty \Rightarrow \text{puncture}$$

$$\text{VEL} < \infty \Rightarrow \text{annular}$$

$$\text{VEL}(\Gamma) = \sup_m \frac{\inf_{\gamma \in \Gamma} l_m(\gamma)^2}{\text{area}(m)}$$

m - metric on $V(K)$

$$l_m = \sum_{v \in \gamma} m(v)$$



HOW?

$\mathcal{C}_n$ - circle packing of X_σ in the pattern of K_n.

Theorem 4.1 of

B. The upper Perron method for labelled complexes with applications to circle packings, Math. Proc. Camb Phil. Soc. (1993), 114, 321-345.

implies the existence of $\mathcal{C}_n$.

One must check that $\forall$ edge-path connected set V of vertices of K_n, the invariant

$$F_V \pi - \sum_{v \in V} \theta(v)$$

is positive.

F_V = # faces of K_n that meet V

$\theta(v)$ = target angle sum at v

$$\theta(v) = \begin{cases} 2\pi & v \in V \cap K \\ 2a & v = x \\ 2b & v = y \\ 2c & v = z \end{cases}$$

Thank You!

References

(1) Ballas, Bowers, Casella, Ruffoni, Tame and relatively elliptic $\mathbb{C}P^1$-structures on the thrice-punctured sphere, preprint, 55 pages.

(2) Bishop and Rempe, Non-compact Riemann surfaces are equilaterally triangulable, arXiv: 2103.16702v1 30 Mar. 2021.

(3) Bowers, The upper Perron method for labelled complexes with applications to circle packings, Math. Proc. Camb. Phil. Soc. (1993) 114, 321-345.

(4) Bowers and Ruffoni, Circle packings on tame complex projective surfaces, in progress.

(5) Dumas, Complex projective structures, 10.1007/978-0-8176-4913-5_7 (Survey).

(6) Kojima, Mizushima, Tan, Circle packings on surfaces with projective structures, J. Diff. Geo. 63 (2003) 349-397.

(7) Schlenker and Yarmola, Properness for circle packings and Delaunay circle patterns on complex projective structures, arXiv: 1806.05254v1 13 Jun 2018.