Circle patterns on surfaces with complex projective structures: Cotangent Laplacian

Wai Yeung Lam

Beijing Institute of Mathematical Sciences and Applications
Tsinghua University

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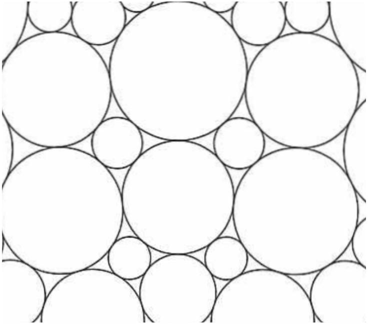
Circle patterns

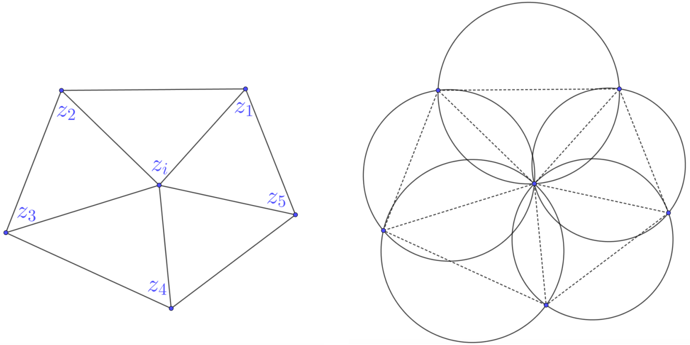
- Circle pattern is
  1. **realization of a planar graph in $\mathbb{C} \cup \infty$ such that the vertices of each face lie on a circle**
  2. **neighbouring circles intersect with prescribed intersection angle** $\Theta : E \to [0, 2\pi)$
- Circle packing $+$ dual packing
  $\leftrightarrow$ Circle pattern with intersection angles $\Theta_{ij} \in \{0, \pi/2\}$



Cross ratios of 4 points $z_1, z_2, z_3, z_4 \in \mathbb{C}$:

$$X(z_1, z_2, z_3, z_4) := -\frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)} \in \mathbb{C}$$

Subdividing mesh into triangulation, cross ratio for every interior edge $X : E \to \mathbb{C}$



Around each interior vertex i

$$1 = \Pi_{j=1}^{n} X_{ij} \quad (1)$$

$$0 = (X_{i1}) + (X_{i1}X_{i2}) + \cdots + (X_{i1}X_{i2}\ldots X_{in}) \quad (2)$$

Delaunay cross ratio system

Definition

Given $M = (V, E, F)$ a triangulation of a closed surface, a cross ratio system is a map $X : E \to \mathbb{C}$ such that for every vertex i

$$1 = \Pi_{j=1}^{n} X_{ij}$$
$$0 = (X_{i1}) + (X_{i1}X_{i2}) + \cdots + (X_{i1}X_{i2}\ldots X_{in})$$

Definition

A Delaunay angle structure is an assignment $\Theta : E \to [0, \pi)$ satisfying

1. For every vertex i, $\sum_j \Theta_{ij} = 2\pi$.
2. $\sum_{i=1}^{n} \Theta_{ij} > 2\pi$ for any closed loop on the dual graph bounding more than one face.

$P(\Theta)$ space of all cross ratio systems X with $\mathrm{Arg}\, X \equiv \Theta = \mathrm{Im} \log X$.
$\leftrightarrow$ space of circle patterns with prescribed angle Θ on complex projective surfaces.

$P(M)$ space of complex projective structures.
$\mathcal{T}(M)$ Teichmüller space.

Theorem (A)

Fixing any triangulation and Delaunay angle structure Θ on a torus,

1. *$P(\Theta)$ is a real analytic surface homeomorphic to $\mathbb{R}^2$.*
2. *$f : P(\Theta) \to P(M)$ is embedding*
3. *The holonomy map is embedding*

$$\mathrm{hol} : P(\Theta) \to \mathrm{Hom}(\pi_1(M), PSL(2,\mathbb{C})) /\!\!/ PSL(2,\mathbb{C})$$

Theorem (B)

The projection $\pi \circ f : P(\Theta) \to \mathcal{T}(M)$ is a homeomorphism.

It proved Kojima-Mizushima-Tan conjecture for torus (g=1).
Discrete conformality far from Classical conformality.
Remain conjecture for $g > 1$

OUTLINE

- Main result: Kojima-Mizushima-Tan conjecture for torus
- $P(\Theta)$ **real analytic surface**
- $P(\Theta)$ homeomorphic to $\mathbb{R}^2$, Affine structure
- $\pi \circ f : P(\Theta) \to \mathcal{T}(M)$ proper map
- $\pi \circ f : P(\Theta) \to \mathcal{T}(M)$ local homeomorphism, Cotangent Laplacian
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$P(\Theta)$ REAL ANALYTIC SURFACE

X_t family of cross ratios, $\operatorname{Im} \log X_t = \Theta \implies q := \frac{d}{dt} \log X_t \big|_{t=0} = \frac{\dot{X}}{X} \in \mathbb{R}$

LEMMA

We have $q : E \to \mathbb{R}$ and for every vertex i

$$0 = \sum_j q_{ij} \tag{3}$$

$$0 = q_{i1} X_{i1} + (q_{i1} + q_{i2}) X_{i1} X_{i2} + \cdots + (q_{i1} + \cdots + q_{im}) X_{i1} X_{i2} \ldots X_{im} \tag{4}$$

Note: q is in the kernel of the Jacobian of the algebraic system $(1)(2)$.

Remark: q ia called a discrete quadratic differential, which is a self-stress in the setting of circle patterns.

Discrete harmonic function $\implies \dim\{q : E \to \mathbb{R} | (3)(4)\} = 2$

Constant rank theorem $\implies P(\Theta)$ real analytic surface

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- $P(\Theta)$ real analytic surface
- $P(\Theta)$ **homeomorphic to $\mathbb{R}^2$, Affine structure**
- $\pi \circ f : P(\Theta) \to \mathcal{T}(M)$ proper map
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Complex affine structures on tori

Definition

An affine structure is a maximal atlas of charts to $\mathbb{C}$ such that the transition functions are affine maps $z \mapsto az + b$ for some $a, b \in \mathbb{C}$.

Euclidean torus: $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ where $\tau \in \mathbb{C}$, $\operatorname{Im} \tau > 0$
Developing map of affine torus $d : z \mapsto e^{cz}$ for some $c \in \mathbb{C} - \{0\}$
Affine structures on a torus are parameterized by (τ, c).
(Note: $c = 0 \leftrightarrow$ Euclidean torus.)

Write γ_1, γ_2 generators of $\pi_1(M)$.
Affine holonomy:

$$d(z \circ \gamma_1) = d(z+1) = e^c d(z) = \rho_1 \circ d(z)$$
$$d(z \circ \gamma_2) = d(z+\tau) = e^{c\tau} d(z) = \rho_2 \circ d(z)$$

REDUCE $\mathbb{C}P^1$-STRUCTURE TO AFFINE STRUCTURE

$\pi_1(M)$ is abeliean: $\gamma_1 \circ \gamma_2 = \gamma_2 \circ \gamma_1$
$\implies$ Holonomy of $\mathbb{C}P^1$-structures $\tilde{\rho}_1, \tilde{\rho}_2$ share same eigenvectors
$\implies$ $\tilde{\rho}_1, \tilde{\rho}_2$ share 1 fixed point or 2 fixed points
$\implies$ Euclidean structure or Non-Euclidean affine structure

PROPOSITION (GUNNING)

Every complex projective structure on a torus can be reduced to an affine structure.

Complex projective structure $\iff$ Affine structures $(\tau, \pm c)$

Theorem (B) in terms of affine structures $\implies$ 2-1 projection to Teichmüller space.

Circle patterns on affine tori are parametrized by the scaling part of the holonomy.

Proposition (Rivin 1994)

Let Θ be Delaunay angle and $A_1, A_2 \in \mathbb{R}$. Then there exists unique affine structure with holonomy $\rho_r(z) = \alpha_r z + \beta_r$ such that

- *scaling part of the holonomy* $\operatorname{Re}\log\alpha_r = A_r$
- *exist circle pattern X with* $\operatorname{Arg} X \equiv \Theta$

Moreover, $A_1 = A_2 = 0$ gives the Euclidean torus.

We can assume $\beta_r = 0$ for non-Euclidean torus.

It proves the rest of Theorem (A):

1. $P(\Theta)$ is homeomorphic to $\mathbb{R}^2$
2. $f : P(\Theta) \to P(M)$ is embedding
3. The holonomy map is embedding

$$\text{hol} : P(\Theta) \to \operatorname{Hom}(\pi_1(M), PSL(2,\mathbb{C})) /\!\!/ PSL(2,\mathbb{C})$$

OUTLINE

- Main result: Kojima-Mizushima-Tan conjecture for torus
- $P(\Theta)$ real analytic surface
- $P(\Theta)$ homeomorphic to $\mathbb{R}^2$, Affine structure
- $\pi \circ f : P(\Theta) \to \mathcal{T}(M)$ **proper map**
- $\pi \circ f : P(\Theta) \to \mathcal{T}(M)$ local homeomorphism, Cotangent Laplacian
- $\pi \circ f : P(\Theta) \to \mathcal{T}(M)$ is homeomorphism

$\pi \circ f : P(\Theta) \to \mathcal{T}(M)$ IS PROPER

Rotation part of holonomy $\operatorname{Im}(\log \alpha_i)$ is determined by walking along γ_i.

It is bounded by a constant depending on the triangulation only.

LEMMA

$$\operatorname{Im}(\log \alpha_i) \leq |\gamma_i|\pi$$

where $|\gamma_i|$ is the number of faces crossed by γ_i.

Recall:

$$\log \alpha_1 = c$$
$$\log \alpha_2 = \tau c = \tau \log \alpha_1$$
$$\operatorname{Im}\log \alpha_2 = (\operatorname{Im}\tau)(\operatorname{Re}\log \alpha_1) + (\operatorname{Re}\tau)(\operatorname{Im}\log \alpha_1)$$

τ in compact set $\implies$ $\operatorname{Re}\log \alpha_1, \operatorname{Re}\log \alpha_2$ in compact set.

$\implies$ $\pi \circ f : P(\Theta) \to \mathcal{T}(M)$ is proper.

Outline

- Main result: Kojima-Mizushima-Tan conjecture for torus
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- $\pi \circ f : P(\Theta) \to \mathcal{T}(M)$ proper map
- $\pi \circ f : P(\Theta) \to \mathcal{T}(M)$ **local homeomorphism, Cotangent Laplacian**
- $\pi \circ f : P(\Theta) \to \mathcal{T}(M)$ is homeomorphism

Cotangent Laplacian

Definition

Given $z : V \to \mathbb{C}$ a realization of a triangle mesh $M = (V, E, F)$, a function $u : V \to \mathbb{R}$ is harmonic if for every vertex i

$$\sum_j c_{ij}(u_j - u_i) = 0$$

where $c_{ij} = c_{ji} := \cot \angle jki + \cot \angle ilj$ is called the cotangent weight.

Proposition

If M simply connected, then a function $u : V \to \mathbb{R}$ is harmonic if and only if there exists $u^* : F \to \mathbb{R}$ such that for every edge $\{ij\}$

$$u^*_{ijk} - u^*_{jil} = \frac{1}{2}(\cot \angle jki + \cot \angle ilj)(u_j - u_i)$$

Here u^* is called the conjugate harmonic function and is unique up to an additive constant.

Proposition

Discrete harmonic functions $\leftrightarrow$ infinitesimal deformation of circle patterns

1. *Change of radii*

$$u^* = \frac{\dot{R}}{R}.$$

2. *Change of vertex position*

$$\frac{\dot{z}_j - \dot{z}_i}{z_j - z_i} = \left(\frac{u^*_{ijk} \cot \angle ilj + u^*_{ilj} \cot \angle jki}{\cot \angle ilj + \cot \angle jki} + \mathbf{i}\frac{u_i + u_j}{2}\right)$$

3. *Change of cross ratio*

$$q_{ij} = \frac{\dot{X}_{ij}}{X_{ij}} = \frac{\dot{z}_i - \dot{z}_k}{z_i - z_k} - \frac{\dot{z}_l - \dot{z}_i}{z_l - z_i} + \frac{\dot{z}_j - \dot{z}_l}{z_j - z_l} - \frac{\dot{z}_k - \dot{z}_j}{z_k - z_j}$$

Cotangent Laplacian on affine tori

Proposition

Suppose $z : \tilde{V} \to \mathbb{C}$ *developing map of affine torus. Then*

1. *cotangent weights are invariant under deck transformations, i.e.* $c_{\gamma(ij)} = c_{ij}$ *for any deck transformation* γ
2. $c_{ij} \geq 0$ *(Delaunay condition)*
3. *the maximum principle holds: a discrete harmonic function* $u : \tilde{V} \to \mathbb{R}$ *achieving a local minimum or maximum at an interior vertex must be constant.*

Infinitesimal deformation of circle patterns $\implies$ change in holonomy

$$\rho_1(z) = e^c z, \quad \rho_2(z) = e^{c\tau} z$$
$$\dot{\rho}_1(z) = \dot{c} e^c z, \quad \dot{\rho}_2(z) = (\dot{c}\tau + c\dot{\tau}) e^{c\tau} z$$
$$\frac{\dot{\rho}_1}{\rho_1} = \dot{c}, \quad \frac{\dot{\rho}_2}{\rho_2} = \dot{c}\tau + c\dot{\tau}$$

Proposition

The corresponding discrete harmonic function $u : \tilde{V} \to \mathbb{R}$ satisfies

$$(u \circ \gamma_1) = u + \operatorname{Im}(\dot{c})$$
$$(u \circ \gamma_2) = u + \operatorname{Im}(\dot{c}\tau + c\dot{\tau}).$$

while the conguate $u^* : F \to \mathbb{R}$ satisfies

$$(u^* \circ \gamma_1) = u^* - \operatorname{Re}(\dot{c})$$
$$(u^* \circ \gamma_2) = u^* - \operatorname{Re}(\dot{c}\tau + c\dot{\tau}).$$

u integral of harmonic 1-form on torus,

Dirichlet energy

Given $u : \tilde{V} \to \mathbb{R}$

$\implies$ Piecewise linear extension over faces $u : \tilde{M} \to \mathbb{R}$

$\implies$ $\operatorname{grad} u : \tilde{F} \to \mathbb{C}$ piecewise constant

$\implies$ Dirichlet energy over a fundamental domain

$$\mathcal{E}(u) := \iint_M |\operatorname{grad} u|^2 dA = \frac{1}{2} \sum_{ij \in E} (\cot \angle jkl + \cot \angle jil)(u_j - u_i)^2$$

Note: $u_j - u_i$ is well defined on torus

$\implies$ $\mathcal{E}(u)$ independent of fundamental domain chosen.

If u is harmonic with conjugate u^*, then

$$\mathcal{E}(u) := \sum_{ij \in E} (u^*_{ijk} - u^*_{jil})(u_j - u_i) = -\operatorname{Im}(\dot{c}(\overline{\dot{c}\tau + c\dot{\tau}}))$$

Local homeomorphism

Suppose there exists an infinitesimal deformation of circle pattern that preserves conformal structure $\dot{\tau} = 0$

$\iff$ There exists harmonic

$$(u \circ \gamma_1) = u + \mathrm{Im}(\dot{c})$$
$$(u \circ \gamma_2) = u + \mathrm{Im}(\dot{c}\tau).$$

while the conguate $u^* : F \to \mathbb{R}$ satisfies

$$(u^* \circ \gamma_1) = u^* - \mathrm{Re}(\dot{c})$$
$$(u^* \circ \gamma_2) = u^* - \mathrm{Re}(\dot{c}\tau).$$

Moreover

$$\mathcal{E}(u) = |\dot{c}|^2 \,\mathrm{Im}(\tau)$$

Q: Is this energy really achievable by a discrete harmonic function?

If $\dot{c} \neq 0$, consider smooth harmonic function on the universal cover of $\mathbb{C} - \{0\}$

$$u^{\dagger} := \mathrm{Re}(-\mathbf{i}\frac{\dot{c}}{c}\log z)$$

Pulled back by the affine developing map, we have

$$(u^{\dagger} \circ \gamma_1) = u^{\dagger} + \mathrm{Im}(\dot{c})$$
$$(u^{\dagger} \circ \gamma_2) = u^{\dagger} + \mathrm{Im}(\dot{c}\tau).$$

and Dirichlet energy

$$\mathcal{E}(u^{\dagger}) = |\dot{c}|^2 \,\mathrm{Im}(\tau) = \mathcal{E}(u)$$

which is impossible since $u^{\dagger}$ is the unique minimizer and smooth.

If $\dot{c} = 0$, then u, u^* are constant by maximum principle and thus the deformation is trivial.

We have $\pi \circ f : P(\Theta) \to \mathcal{T}(M)$ local homeomorphism for non-Euclidean affine torus.

$\implies$ Altogether, Covering map with at most 1 branch point at Euclidean torus.

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No branching at Euclidean torus

Circle patterns on affine tori parametrized by scaling part of holonomy $A_1, A_2 \in \mathbb{R}$.

$$\pi \circ f(A_1, A_2) = \frac{c\tau}{c} = \frac{A_1 + \mathbf{i}\,\mathrm{Im}(c\tau)}{A_2 + \mathbf{i}\,\mathrm{Im}(c)} \in \mathbb{H}$$

Given $R > 0$, consider the loop $(A_1(t), A_2(t)) = (R\cos t, R\sin t)$ where $t \in [0, \pi]$.
It is a generator of $\pi_1(P(\Theta) - \{(0,0)\})$.

Map the upper half-plane to the unit disk $g(z) = \frac{z-\mathbf{i}}{z+\mathbf{i}}$.

Note: $|\mathrm{Im}(c)|, |\mathrm{Im}(c\tau)| < \infty$. For $R >> 1$,

$$g \circ \pi \circ f(A_1(t), A_2(t)) = \frac{R\cos t - \mathbf{i}R\sin t + O(1)}{R\cos t + \mathbf{i}R\sin t + O(1)} \sim e^{-2t\mathbf{i}}.$$

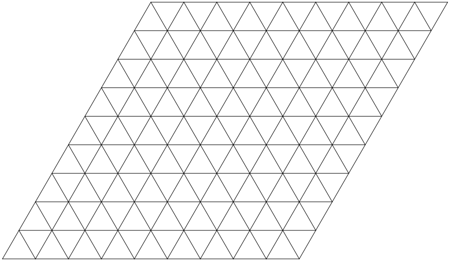
Thus $\pi \circ f : P(\Theta) - \{(0,0)\} \to \mathcal{T}(M) - \{\tau_0\}$ is a degree-1 map.

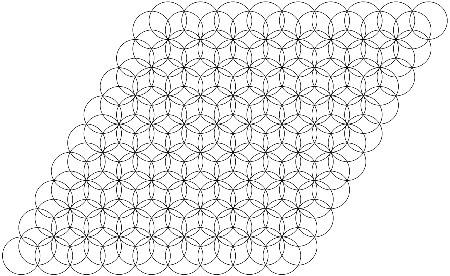
Hence $\pi \circ f : P(\Theta) \to \mathcal{T}(M)$ is a homeomorphism.

(Credit: Tianqi Wu)

ELEMENTS OF $P(\Theta)$

$\Theta \equiv \pi/3$ on a triangulated torus.





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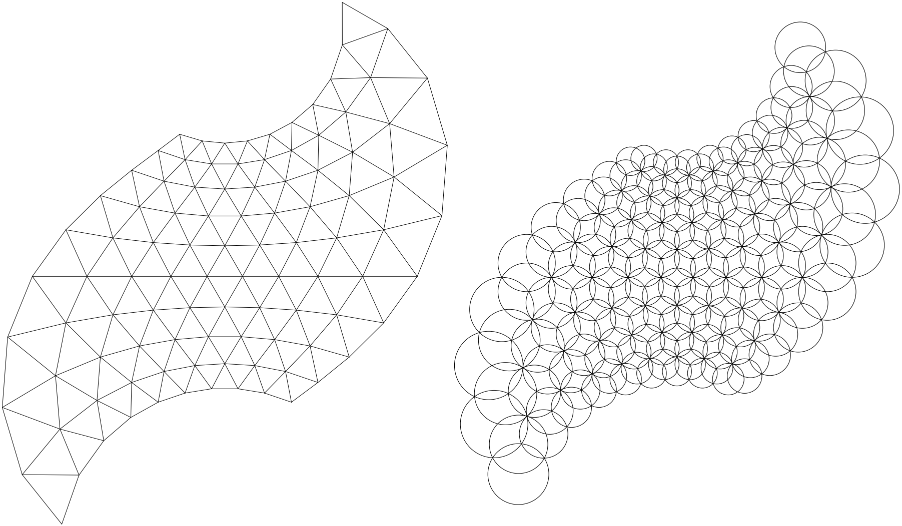

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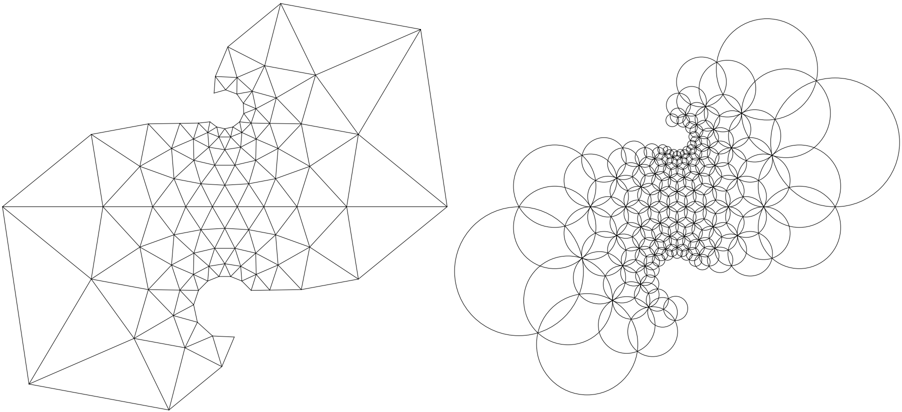
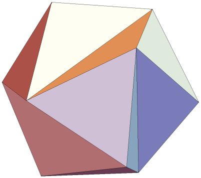
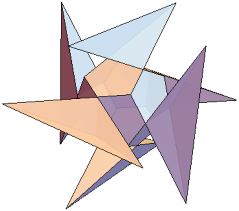


Good discretization = Rich in mathematical structures

Thank you!





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