Rigidity of Fuchsian hyperbolic polyhedra by variational methods

Roman Prosanov

Technische Universität Wien
Institut für Diskrete Mathematik und Geometrie

Smooth conformality

Definition

Let M be a smooth manifold and g_0, g_1 be two Riemannian metrics on M. They are called ***pointwise conformally equivalent*** *if*

$$g_0 = e^{2u} g_1$$

for a smooth function u on M.

Smooth conformality

Definition

*Let M be a smooth manifold and g_0, g_1 be two Riemannian metrics on M. They are called **pointwise conformally equivalent** if*

$$g_0 = e^{2u} g_1$$

for a smooth function u on M.

Theorem

Two Riemannian metrics on M are pointwise conformally equivalent if and only if the angles between any two smooth curves are the same in both metrics.

Smooth conformality

Definition

Let M be a smooth manifold and g_0, g_1 be two Riemannian metrics on M. They are called ***pointwise conformally equivalent*** *if*

$$g_0 = e^{2u} g_1$$

for a smooth function u on M.

Theorem

Two Riemannian metrics on M are pointwise conformally equivalent if and only if the angles between any two smooth curves are the same in both metrics.

Definition

A **conformal structure** *on M is a pointwise conformal equivalence class of Riemannian metrics on M.*

Cone metrics

Definition

A ***cone metric*** *d on a surface S is locally isometric to a constant curvature model space (up to scaling those are the Euclidean plane $\mathbb{E}^2$, the hyperbolic plane $\mathbb{H}^2$, or the standard sphere $\mathbb{S}^2$) except finitely many points called* ***conical points****. At a conical point v the metric d is locally isometric to a cone with total angle $\lambda_v(d) \neq 2\pi$.*

Cone metrics

Definition

A ***cone metric*** *d on a surface S is locally isometric to a constant curvature model space (up to scaling those are the Euclidean plane* $\mathbb{E}^2$*, the hyperbolic plane* $\mathbb{H}^2$*, or the standard sphere* $\mathbb{S}^2$*) except finitely many points called* ***conical points****. At a conical point v the metric d is locally isometric to a cone with total angle* $\lambda_v(d) \neq 2\pi$*.*

The number $\kappa_v(d) := 2\pi - \lambda_v(d)$ is called the **curvature** of v.
Let $V \subset S$ be a set of marked points. We say that d is a cone-metric on (S, V) if the set of conical points of d is a subset of V.

Cone metrics

Definition

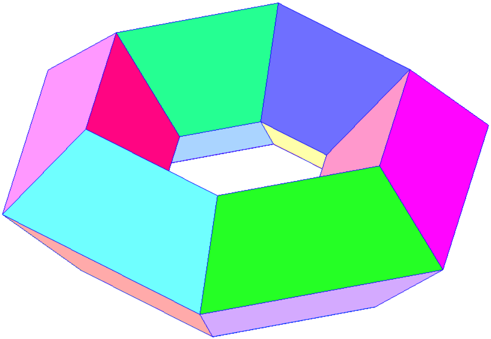
*A **cone metric** d on a surface S is locally isometric to a constant curvature model space (up to scaling those are the Euclidean plane $\mathbb{E}^2$, the hyperbolic plane $\mathbb{H}^2$, or the standard sphere $\mathbb{S}^2$) except finitely many points called **conical points**. At a conical point v the metric d is locally isometric to a cone with total angle $\lambda_v(d) \neq 2\pi$.*

The number $\kappa_v(d) := 2\pi - \lambda_v(d)$ is called the **curvature** of v.
Let $V \subset S$ be a set of marked points. We say that d is a cone-metric on (S, V) if the set of conical points of d is a subset of V.

Example

*The induced metric on the boundary of a convex polytope in $\mathbb{E}^3$ is a Euclidean cone-metric. Moreover, it is **convex**, i.e., for all conical points we have $\kappa_v(d) > 0$.*

Cone metrics



Circle packing metrics

- Let S be a closed surface with a triangulation $\mathcal{T}$ and $l : E(\mathcal{T}) \to \mathbb{R}_{>0}$ be a map assigning lengths to the edges so that the triangle inequalities are satisfied for each face. Replace each triangle by a triangle in a model space with side lengths defined by l. We obtain a cone-metric on $(S, \mathcal{T})$.

Circle packing metrics

• Let S be a closed surface with a triangulation $\mathcal{T}$ and $l : E(\mathcal{T}) \to \mathbb{R}_{>0}$ be a map assigning lengths to the edges so that the triangle inequalities are satisfied for each face. Replace each triangle by a triangle in a model space with side lengths defined by l. We obtain a cone-metric on $(S, \mathcal{T})$.

Lemma

For each cone metric d on (S, V) there exists a geodesic triangulation of (S, V, d) with vertices at V.

Circle packing metrics

- Let S be a closed surface with a triangulation $\mathcal{T}$ and $l : E(\mathcal{T}) \to \mathbb{R}_{>0}$ be a map assigning lengths to the edges so that the triangle inequalities are satisfied for each face. Replace each triangle by a triangle in a model space with side lengths defined by l. We obtain a cone-metric on $(S, \mathcal{T})$.

Lemma

For each cone metric d on (S, V) there exists a geodesic triangulation of (S, V, d) with vertices at V.

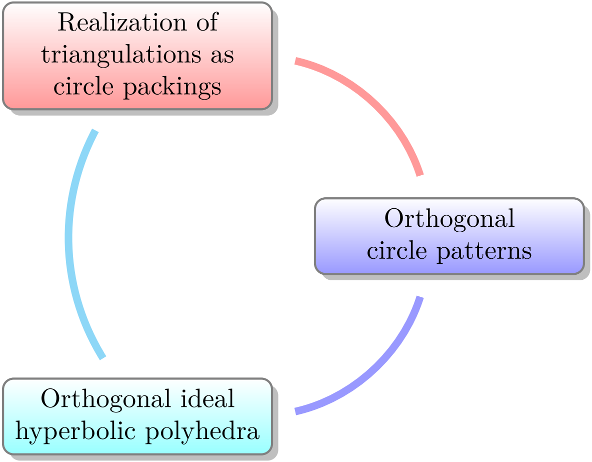
Definition

*A cone metric on $(S, \mathcal{T})$ is a **circle packing metric** if*

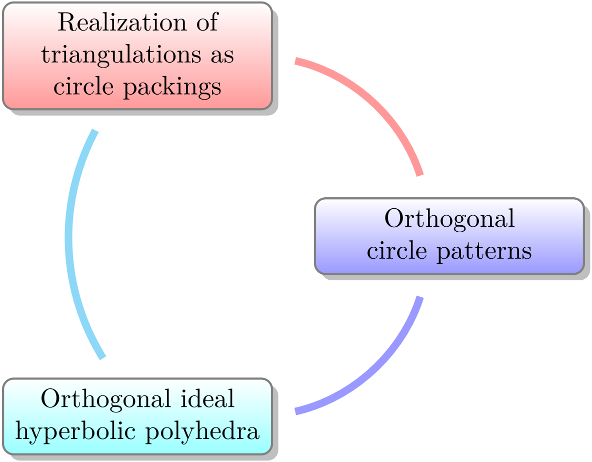
$$l_e = r_v + r_w$$

for each edge e with endpoints v and w, and for a function $r : V(\mathcal{T}) \to \mathbb{R}_{>0}$.

Circle packings after Thurston



Circle packings after Thurston



Based on the fact that conformal maps are characterized by sending infinitesimal circles to infinitesimal circles, it was proposed to use circle packings for discrete conformal maps. Rodin–Sullivan showed the convergence for disk-shaped domains.

Uniformization

Theorem (Smooth Uniformization Theorem)

Let S be a closed orientable surface with a Riemannian metric g. Then (S, g) is pointwise conformally equivalent to a unique up to scaling metric of constant curvature.

Uniformization

Theorem (Smooth Uniformization Theorem)

Let S be a closed orientable surface with a Riemannian metric g. Then (S, g) is pointwise conformally equivalent to a unique up to scaling metric of constant curvature.

Definition

We say that two circle packing metrics on $(S, \mathcal{T})$ are cp-conformally equivalent with respect to $\mathcal{T}$.

Uniformization

Theorem (Smooth Uniformization Theorem)

Let S be a closed orientable surface with a Riemannian metric g. Then (S, g) is pointwise conformally equivalent to a unique up to scaling metric of constant curvature.

Definition

We say that two circle packing metrics on $(S, \mathcal{T})$ are cp-conformally equivalent with respect to $\mathcal{T}$.

Theorem (KAT Theorem $\sim$ Circle packing uniformization)

Let S be a closed orientable surface with a simplicial triangulation $\mathcal{T}$ and a circle packing metric d on $(S, \mathcal{T})$. Then (S, d) is cp-conformally equivalent w.r.t. $\mathcal{T}$ to a unique up to scaling metric of constant curvature.

Discrete conformality

Definition

Two Euclidean cone-metrics d, d' *on* $(S, \mathcal{T})$ *are discretely conformally equivalent with respect to* $\mathcal{T}$ *if there exists a function* $u : V \to \mathbb{R}$ *such that for every edge* e *with endpoints* v *and* w *we have*

$$l_e(d) = e^{u_v + u_w} l_e(d').$$

Delaunay triangulations

Definition

Let d be a cone-metric on (S, V). A decomposition of (S, d, V) into geodesic polygons with vertices at V is called ***Delaunay*** *if each polygon can be inscribed in a circle and all vertices of V except the vertices of the polygon lie outside the circle.*

Delaunay triangulations

Definition

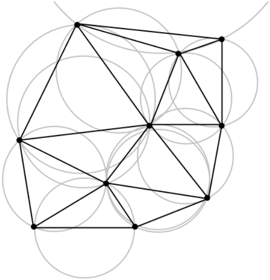
*Let d be a cone-metric on (S, V). A decomposition of (S, d, V) into geodesic polygons with vertices at V is called **Delaunay** if each polygon can be inscribed in a circle and all vertices of V except the vertices of the polygon lie outside the circle.*
*A **Delaunay triangulation** of (S, d, V) is any triangulation refining the Delaunay decomposition.*

Delaunay triangulations

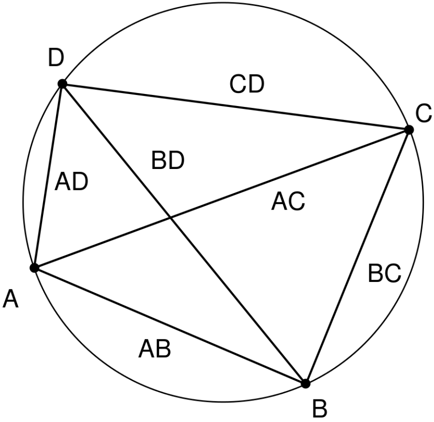
Definition

*Let d be a cone-metric on (S, V). A decomposition of (S, d, V) into geodesic polygons with vertices at V is called **Delaunay** if each polygon can be inscribed in a circle and all vertices of V except the vertices of the polygon lie outside the circle.*
*A **Delaunay triangulation** of (S, d, V) is any triangulation refining the Delaunay decomposition.*

A Delaunay decomposition always exists and is unique.

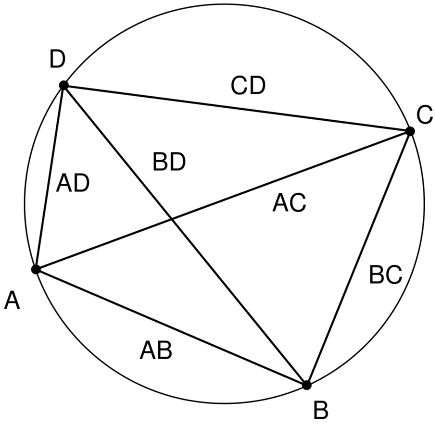


Ptolemy relation



Ptolemy's relation
$ABCD$ is cyclic iff
$AC \cdot BD = AB \cdot CD + BC \cdot AD.$

Ptolemy relation



Ptolemy's relation
$ABCD$ is cyclic iff
$AC \cdot BD = AB \cdot CD + BC \cdot AD.$

If $\mathcal{T}_1$, $\mathcal{T}_2$ be two triangulations of (S, V, d) distinct by a diagonal flip in a quadrilateral Q and d' is d.c.e. to d with respect to $\mathcal{T}_1$, then d' is d.c.e. to d with respect to $\mathcal{T}_2$ iff Q is cyclic.

Discrete conformality

Definition

Two Euclidean cone-metrics d *and* d' *on* (S, V) *are discretely conformally equivalent if there is a sequence of metrics with triangulations* $(d_1, \mathcal{T}_1), \ldots, (d_m, \mathcal{T}_m)$ *on* (S, V) *such that*

(i) $d_1 = d$, $d_m = d'$;

(ii) $\mathcal{T}_i$ *is Delaunay for* d_i;

(iii) either $\mathcal{T}_i = \mathcal{T}_{i+1}$ *and* d_i, d_{i+1} *are discretely conformally equivalent with respect to* $\mathcal{T}_i$;

(iv) or $\mathcal{T}_i \neq \mathcal{T}_{i+1}$, *but* $d_i = d_{i+1}$ *and* $\mathcal{T}_i$, $\mathcal{T}_{i+1}$ *are two different Delaunay triangulations of the same metric.*

Discrete conformality: curvature

Theorem (The Gauss–Bonnet theorem for Euclidean cone-metrics)

$$\sum_{v \in V} \kappa_v(d) = 2\pi\chi(S).$$

Discrete conformality: curvature

Theorem (The Gauss–Bonnet theorem for Euclidean cone-metrics)

$$\sum_{v \in V} \kappa_v(d) = 2\pi\chi(S).$$

Theorem (Gu–Luo–Sun–Wu, 2018)

Let $\widetilde{\kappa} : V \to (-\infty; 2\pi)$ be a function such that $\sum_{v \in V} \widetilde{\kappa}_v(d) = 2\pi\chi(S)$. Then in every discrete conformal class of Euclidean cone-metrics on (S, V) there exists a unique up to scaling metric d with $\kappa_v(d) = \widetilde{\kappa}(v)$ for each $v \in V$. Moreover, there exists an algorithm to construct d.

Discrete conformality: curvature

Theorem (The Gauss–Bonnet theorem for Euclidean cone-metrics)

$$\sum_{v \in V} \kappa_v(d) = 2\pi\chi(S).$$

Theorem (Gu–Luo–Sun–Wu, 2018)

Let $\widetilde{\kappa} : V \to (-\infty; 2\pi)$ *be a function such that* $\sum_{v \in V} \widetilde{\kappa}_v(d) = 2\pi\chi(S)$. *Then in every discrete conformal class of Euclidean cone-metrics on* (S, V) *there exists a unique up to scaling metric* d *with* $\kappa_v(d) = \widetilde{\kappa}(v)$ *for each* $v \in V$. *Moreover, there exists an algorithm to construct* d.

Corollary (Discrete uniformization on a torus)

Each Euclidean cone-metric on a torus is discretely conformally equivalent to a unique up to scaling Euclidean metric.

Discrete conformality: convergence

Theorem (Smooth uniformization)

Let S be the (open) disk and g be a Riemannian metric on the closure of S. Then (S, g) is conformally diffeomorphic to the (open) unit disk in $\mathbb{E}^2$.

Discrete conformality: convergence

Theorem (Smooth uniformization)

Let S be the (open) disk and g be a Riemannian metric on the closure of S. Then (S, g) is conformally diffeomorphic to the (open) unit disk in $\mathbb{E}^2$.

Theorem (Gu–Luo–Wu, 2019)

There exists an algorithm, which computes the uniformization map from Theorem above.

Discrete conformality: the hyperbolic case

Definition

Two hyperbolic cone-metrics d and d' on (S, V) are discretely conformally equivalent if there is a sequence of metrics with triangulations $(d_1, \mathcal{T}_1), \ldots, (d_m, \mathcal{T}_m)$ on (S, V) such that

(i) $d_1 = d$, $d_m = d'$;

(ii) $\mathcal{T}_i$ *is Delaunay for* d_i;

(iii) either $\mathcal{T}_i = \mathcal{T}_{i+1}$ *and there exists a function* $u : V \to \mathbb{R}$ *such that for every edge* e *with the endpoints* v *and* w *we have*

$$\sinh \frac{l_e(d_i)}{2} = e^{u_v + u_w} \sinh \frac{l_e(d_{i+1})}{2};$$

(iv) or $\mathcal{T}_i \neq \mathcal{T}_{i+1}$, *but* $d_i = d_{i+1}$ *and* $\mathcal{T}_i$, $\mathcal{T}_{i+1}$ *are two different Delaunay triangulations of the same metric.*

Discrete conformality: curvature

Theorem (The Gauss–Bonnet theorem for hyperbolic cone-metrics)

$$\sum_{v\in V} \kappa_v(d) = 2\pi\chi(S) + \mathrm{area}(S, d).$$

Discrete conformality: curvature

Theorem (The Gauss–Bonnet theorem for hyperbolic cone-metrics)

$$\sum_{v \in V} \kappa_v(d) = 2\pi\chi(S) + \mathrm{area}(S, d).$$

Theorem (Gu–Guo–Luo–Sun–Wu, 2018)

Let $\widetilde{\kappa} : V \to (-\infty; 2\pi)$ *be a function such that* $\sum_{v \in V} \widetilde{\kappa}_v(d) > 2\pi\chi(S)$. *Then in every discrete conformal class of hyperbolic cone-metrics on* (S, V) *there exists a unique metric* d *with* $\kappa_v(d) = \widetilde{\kappa}(v)$ *for each* $v \in V$. *Moreover, there exists an algorithm to construct* d.

Discrete conformality: curvature

Theorem (The Gauss–Bonnet theorem for hyperbolic cone-metrics)

$$\sum_{v \in V} \kappa_v(d) = 2\pi\chi(S) + \mathrm{area}(S, d).$$

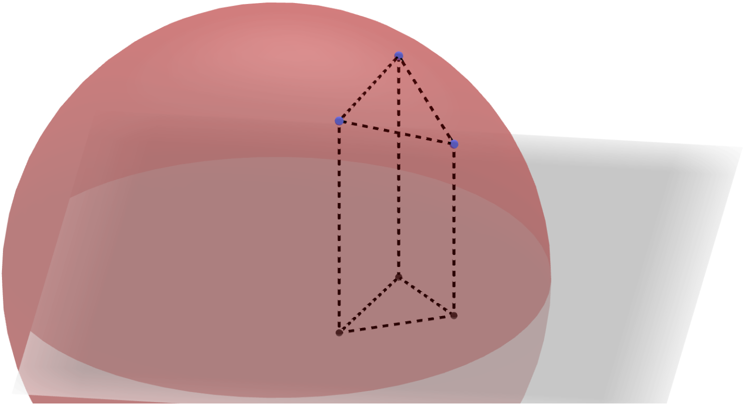
Theorem (Gu–Guo–Luo–Sun–Wu, 2018)

Let $\widetilde{\kappa} : V \to (-\infty; 2\pi)$ be a function such that $\sum_{v \in V} \widetilde{\kappa}_v(d) > 2\pi\chi(S)$. Then in every discrete conformal class of hyperbolic cone-metrics on (S, V) there exists a unique metric d with $\kappa_v(d) = \widetilde{\kappa}(v)$ for each $v \in V$. Moreover, there exists an algorithm to construct d.

Corollary (Discrete uniformization with genus ≥ 2)

Each hyperbolic cone-metric on S_g, $g \geq 2$, is discretely conformally equivalent to a unique hyperbolic metric.

Approach: Ideal prisms



- An **ideal prism** is a prism in $\mathbb{H}^3$ with two triangular faces, all vertices of the upper face are ideal, all lateral edges are orthogonal to the lower face. It is uniquely determined by the lower face.

Approach: Ideal Fuchsian cone-manifolds

-
  - S_g is a closed surface of genus $g \geq 2$;
  - V is a set of marked points;
  - d is a hyperbolic cone-metric on (S_g, V);
  - $\mathcal{T}$ be a geodesic triangulation of (S_g, V, d).

Approach: Ideal Fuchsian cone-manifolds

- - S_g is a closed surface of genus $g \geq 2$;
  - V is a set of marked points;
  - d is a hyperbolic cone-metric on (S_g, V);
  - $\mathcal{T}$ be a geodesic triangulation of (S_g, V, d).
- Take each triangle T of $\mathcal{T}$ and construct the ideal prism with T as the lower boundary. Glue all these prisms according to $\mathcal{T}$. We obtain a hyperbolic cone-3-manifold. It is called an *ideal Fuchsian cone-manifold* $P_{\downarrow}(d, \mathcal{T})$. It is homeomorphic to $S_g \times [0; 1]$ minus points at the upper boundary.

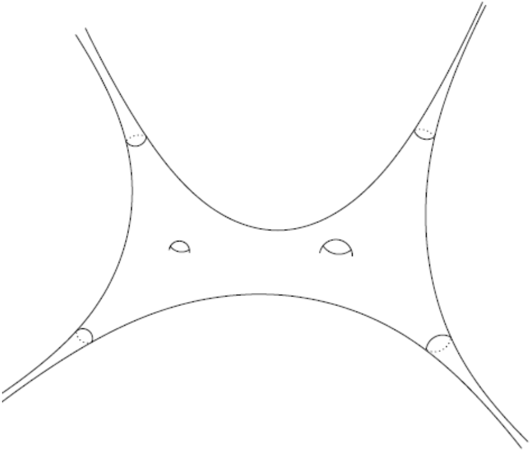
Approach: Ideal Fuchsian cone-manifolds

-
  - S_g is a closed surface of genus $g \geq 2$;
  - V is a set of marked points;
  - d is a hyperbolic cone-metric on (S_g, V);
  - $\mathcal{T}$ be a geodesic triangulation of (S_g, V, d).
- Take each triangle T of $\mathcal{T}$ and construct the ideal prism with T as the lower boundary. Glue all these prisms according to $\mathcal{T}$. We obtain a hyperbolic cone-3-manifold. It is called an *ideal Fuchsian cone-manifold* $P_{\downarrow}(d, \mathcal{T})$. It is homeomorphic to $S_g \times [0; 1]$ minus points at the upper boundary.
- If $\mathcal{T}$ is Delaunay, then it is denoted just as $P_{\downarrow}(d)$ and it is convex.

Approach: Ideal Fuchsian cone-manifolds

- - S_g is a closed surface of genus $g \geq 2$;
  - V is a set of marked points;
  - d is a hyperbolic cone-metric on (S_g, V);
  - $\mathcal{T}$ be a geodesic triangulation of (S_g, V, d).
- Take each triangle T of $\mathcal{T}$ and construct the ideal prism with T as the lower boundary. Glue all these prisms according to $\mathcal{T}$. We obtain a hyperbolic cone-3-manifold. It is called an *ideal Fuchsian cone-manifold* $P_\downarrow(d, \mathcal{T})$. It is homeomorphic to $S_g \times [0; 1]$ minus points at the upper boundary.
- If $\mathcal{T}$ is Delaunay, then it is denoted just as $P_\downarrow(d)$ and it is convex.
- The upper boundary of $P_\downarrow(d, \mathcal{T})$ is glued from ideal hyperbolic triangles. The obtained metric on the upper boundary is called a *hyperbolic cusp-metric* on S_g.
  In the case of $P_\downarrow(d)$ we denote it by $\tilde{d}$.

Approach: Hyperbolic cusp-metric



On a hyperbolic cusp-metric $\tilde{d}$ marked points are at infinite distance from each other.

Approach: Ideal Fuchsian cone-manifolds

Lemma

Let d, d' be two hyperbolic cone-metrics on (S_g, V). Then they are discretely conformally equivalent iff $\tilde{d}$ is isometric to $\tilde{d}'$.

Approach: Ideal Fuchsian cone-manifolds

Lemma

Let d, d' be two hyperbolic cone-metrics on (S_g, V). Then they are discretely conformally equivalent iff $\tilde{d}$ is isometric to $\tilde{d}'$.

- Thus, to study a discrete conformal class one can study convex ideal Fuchsian cone-manifolds with a fixed upper boundary metric.

Approach: Ideal Fuchsian cone-manifolds

Lemma

Let d, d' be two hyperbolic cone-metrics on (S_g, V). Then they are discretely conformally equivalent iff $\tilde{d}$ is isometric to $\tilde{d}'$.

- Thus, to study a discrete conformal class one can study convex ideal Fuchsian cone-manifolds with a fixed upper boundary metric.
- *Discrete uniformization.* We have a hyperbolic cone-metric d and we try to uniformize it. Construct $P_\downarrow(d)$, obtain $\tilde{d}$. To uniformize d means to deform $P_\downarrow(d)$, while preserving the upper boundary, so that cone-singularities dissolve.

Approach: Ideal Fuchsian cone-manifolds

Lemma

Let d, d' be two hyperbolic cone-metrics on (S_g, V). Then they are discretely conformally equivalent iff $\tilde{d}$ is isometric to $\tilde{d}'$.

- Thus, to study a discrete conformal class one can study convex ideal Fuchsian cone-manifolds with a fixed upper boundary metric.
- *Discrete uniformization.* We have a hyperbolic cone-metric d and we try to uniformize it. Construct $P_{\downarrow}(d)$, obtain $\tilde{d}$. To uniformize d means to deform $P_{\downarrow}(d)$, while preserving the upper boundary, so that cone-singularities dissolve.
- The discrete uniformization of d is equivalent to an isometric embedding of $\tilde{d}$ to a Fuchsian manifold.

Approach: Ideal Fuchsian cone-manifolds

Lemma

Let d, d' be two hyperbolic cone-metrics on (S_g, V). Then they are discretely conformally equivalent iff $\tilde{d}$ is isometric to $\tilde{d}'$.

- Thus, to study a discrete conformal class one can study convex ideal Fuchsian cone-manifolds with a fixed upper boundary metric.
- *Discrete uniformization.* We have a hyperbolic cone-metric d and we try to uniformize it. Construct $P_\downarrow(d)$, obtain $\tilde{d}$. To uniformize d means to deform $P_\downarrow(d)$, while preserving the upper boundary, so that cone-singularities dissolve.
- The discrete uniformization of d is equivalent to an isometric embedding of $\tilde{d}$ to a Fuchsian manifold.
- A *Fuchsian manifold* is a complete hyperbolic manifold homeomorphic to $S_g \times [0;1)$ with geodesic boundary $S_g \times \{0\}$.

Approach: the discrete curvature

- Let $H(\tilde{d})$ be the set of convex ideal Fuchsian cone-manifolds having $\tilde{d}$ at its upper boundary.

Approach: the discrete curvature

- Let $H(\tilde{d})$ be the set of convex ideal Fuchsian cone-manifolds having $\tilde{d}$ at its upper boundary.
- $H(\tilde{d})$ can be identified with $\mathbb{R}^V$ with the help of hyperbolic decorations.

Approach: the discrete curvature

- Let $H(\tilde{d})$ be the set of convex ideal Fuchsian cone-manifolds having $\tilde{d}$ at its upper boundary.
- $H(\tilde{d})$ can be identified with $\mathbb{R}^V$ with the help of hyperbolic decorations.
- Let P be an ideal Fuchsian cone-manifold. Define the *discrete curvature* $S(P)$ as the sum of its total discrete scalar curvature and the total discrete mean curvature of its boundary.

Approach: the discrete curvature

- Let $H(\tilde{d})$ be the set of convex ideal Fuchsian cone-manifolds having $\tilde{d}$ at its upper boundary.
- $H(\tilde{d})$ can be identified with $\mathbb{R}^V$ with the help of hyperbolic decorations.
- Let P be an ideal Fuchsian cone-manifold. Define the *discrete curvature* $S(P)$ as the sum of its total discrete scalar curvature and the total discrete mean curvature of its boundary.
- Consider S as a function over $H(\tilde{d}) = \mathbb{R}^V$.

Approach: the discrete curvature

- *Fact*: $\frac{\partial S}{\partial h_v} = \kappa_v$. *Proof*: The Schläfli formula.

Approach: the discrete curvature

- *Fact*: $\frac{\partial S}{\partial h_v} = \kappa_v$. *Proof*: The Schläfli formula.
- *Corollary*: critical points of S in $\mathbb{R}^V$ correspond to metrics in the given discrete conformal class without cone-singularities.

Approach: the discrete curvature

- *Fact*: $\frac{\partial S}{\partial h_v} = \kappa_v$. *Proof*: The Schläfli formula.
- *Corollary*: critical points of S in $\mathbb{R}^V$ correspond to metrics in the given discrete conformal class without cone-singularities.
- *Fact*: S is strictly concave over $\mathbb{R}^V$.

Approach: the discrete curvature

- *Fact*: $\frac{\partial S}{\partial h_v} = \kappa_v$. *Proof*: The Schläfli formula.
- *Corollary*: critical points of S in $\mathbb{R}^V$ correspond to metrics in the given discrete conformal class without cone-singularities.
- *Fact*: S is strictly concave over $\mathbb{R}^V$.
- *Corollary*: uniqueness in the discrete uniformization.

Approach: the discrete curvature

- *Fact*: $\frac{\partial S}{\partial h_v} = \kappa_v$. *Proof*: The Schläfli formula.
- *Corollary*: critical points of S in $\mathbb{R}^V$ correspond to metrics in the given discrete conformal class without cone-singularities.
- *Fact*: S is strictly concave over $\mathbb{R}^V$.
- *Corollary*: uniqueness in the discrete uniformization.
- *Task*: Prove existence.

Approach: the discrete curvature

- *Fact*: $\frac{\partial S}{\partial h_v} = \kappa_v$. *Proof*: The Schläfli formula.
- *Corollary*: critical points of S in $\mathbb{R}^V$ correspond to metrics in the given discrete conformal class without cone-singularities.
- *Fact*: S is strictly concave over $\mathbb{R}^V$.
- *Corollary*: uniqueness in the discrete uniformization.
- *Task*: Prove existence.
- Follow the gradient flow of S!
  Prove that it stays in a compact convex subset of $\mathbb{R}^V$.

Step aside: isometric realizations

Problem

Let M be a compact 3-manifold with boundary and d be a metric on ∂M. Does there exist a constant curvature metric on M such that the induced metric on ∂M is d? Is it unique?

Step aside: isometric realizations

Problem

Let M be a compact 3-manifold with boundary and d be a metric on ∂M. Does there exist a constant curvature metric on M such that the induced metric on ∂M is d? Is it unique?

Theorem (Alexandrov, 1942; Volkov, $\sim$ 50s; Bobenko–Izmestiev, 2008)

For every convex Euclidean (resp. hyperbolic) cone-metric d on S^2 there exists a unique convex polyhedron $P \subset \mathbb{E}^3$ (resp. $P \subset \mathbb{H}^3$) such that (S^2, d) is isometric to the boundary of P.

Step aside: isometric realizations

Theorem (Fillastre, 2007)

For every convex hyperbolic cone-metric d on S_g there exists a unique Fuchsian manifold with convex polyhedral boundary such that its upper boundary is isometric to (S_g, d).

Step aside: isometric realizations

Theorem (Fillastre, 2007)

For every convex hyperbolic cone-metric d on S_g there exists a unique Fuchsian manifold with convex polyhedral boundary such that its upper boundary is isometric to (S_g, d).

Theorem (Slutskiy, 2018 + P. 2020)

For every CBB(-1) metric d on S_g there exists a unique Fuchsian manifold with convex boundary such that its upper boundary is isometric to (S_g, d).

Step aside: isometric realizations

Theorem (Fillastre, 2007)

For every convex hyperbolic cone-metric d on S_g there exists a unique Fuchsian manifold with convex polyhedral boundary such that its upper boundary is isometric to (S_g, d).

Theorem (Slutskiy, 2018 + P. 2020)

For every CBB(-1) metric d on S_g there exists a unique Fuchsian manifold with convex boundary such that its upper boundary is isometric to (S_g, d).

Problem

Let M be a compact hyperbolic manifold with convex boundary. Is it true that for every convex hyperbolic cone-metric d on ∂M there is a unique hyperbolic metric on M with convex boundary inducing d on ∂M ?

The end

Thank you!

Roman Prosanov