A sufficient condition for a polyhedron to be rigid and necessary conditions for the extendibility of a first-order flex of a polyhedron to its flex

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In this talk, I will focus on the main thing only, i.e., I will not strive for the most general or rigorous formulations, and the authorship of each result mentioned below.

You can find all omitted technical details, historical information, and further references in the following articles, on which this talk is based:

V. Alexandrov, "A sufficient condition for a polyhedron to be rigid." J. Geom. **110**, Paper No. 38, 11 p. (2019). doi:10.1007/s00022-019-0492-0

V. Alexandrov, "Necessary conditions for the extendibility of a first-order flex of a polyhedron to its flex." Beitr. Algebra Geom. **61**, 355–368 (2020). doi:10.1007/s13366-019-00473-8

Polyhedron = an oriented connected closed polyhedral surface with non-degenerate triangular faces in Euclidean 3-space

Flexible polyhedron = a polyhedron whose spatial shape can be changed continuously by changing its dihedral angles only, i. e., a polyhedron for which there is a continuous family of deformations which does not change the length of its edges, but changes at least one dihedral angle

Flex = a nontrivial flex of a polyhedron = the above-mensioned continuous family of deformations that does not change the length of its edges, but changes at least one dihedral angle

Rigid polyhedron = a polyhedron for which there is no flex

We know quite a lot about flexible polyhedra:

(a) they do exist; moreover, they can be
- self-intersection free (and in this case can have 9 vertices only)
- of arbitrary genius (e.g., can be non-orientable)

(b) during the flex, they necessarily keep unaltered
- the total mean curvature
- the volume of the domain they bound
- the Dehn invariants

(c) the notion of a flexible polyhedron can be introduced in every d-space of constant curvature for $d \geqslant 3$; moreover, in many such spaces flexible polyhedra possess properties similar to (a), (b)

There are many open problems in the theory of flexible polyhedra and around it:

(α) Do there exists a flexible closed smooth surface?

(β) Given a polyhedron, is it flexible?

(γ) Describe all quantities that remain constant during the flex

(δ) Do there exists a self-intersection free flexible polyhedron with 8 verticies?

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Theorem 1 (A.A. Gaifullin – L.S. Ignashchenko, 2018; VA, 2019)

Let P be a closed oriented polyhedron in Euclidean 3-space with nondegenerate triangular faces.
Let the set of the edge lengths of P be $\mathbb{Q}$-linearly independent.
Then P is rigid.

Remarks:

• Theorem 1 is a sufficient condition for a polyhedron to be rigid

• Theorem 1 is not a statement about generic rigidity. It requires the $\mathbb{Q}$-linear independence of edge lengths, not algebraic independence of the coordinates of vertices

• Theorem 1 is an immediate corollary of Theorem 2, which is a tough nut to crack

Theorem 2 (A.A. Gaifullin – L.S. Ignashchenko, 2018)

Let P be a flexible closed oriented polyhedron in Euclidean 3-space with nondegenerate triangular faces.
Then every Dehn invariant of P remains unaltered during any flex.

Dehn invariant of $P = \sum_\sigma \ell_\sigma f(\varphi_\sigma)$,
where summation is taken over all edges σ of P;
$f : \mathbb{R} \to \mathbb{R}$ is any $\mathbb{Q}$-linear function such that $f(\pi) = 0$;
ℓ_σ stands for the length of the edge σ;
φ_σ stands for the internal dihedral angle at the edge σ.

On the next slide: Theorem 3 is yet another corollary of Theorem 2.

Theorem 3 (VA, 2019)

Let P be a flexible closed oriented polyhedron in Euclidean 3-space with nondegenerate triangular faces;
E be the set of the edges σ of P;
L be the set of the lengths ℓ_σ of all edges $\sigma \in E$;
$\{\lambda_1, \dots, \lambda_m\}$ be a $\mathbb{Q}$-basis in the $\mathbb{Q}$-linear span of L;
$\alpha_{\sigma j} \in \mathbb{Q}$ be such that

$$\ell_\sigma = \sum_{j=1}^{m} \alpha_{\sigma j} \lambda_j \qquad \forall \sigma \in E.$$

Then, for every flex of P and every $j = 1, \dots, m$, the expression

$$\sum_{\sigma \in E} \alpha_{\sigma j} \varphi_\sigma$$

is constant during the flex.
Here φ_σ stands for the internal dihedral angle of P at the edge σ.

Remarks:

• Each linear combination

$$\sum_{\sigma \in E} \alpha_{\sigma j} \varphi_\sigma,$$

that is constant during the flex according to Theorem 3, is equivalent to a similar linear combination with integer coefficients, not all of which are zero

• For Bricard octahedra of all types, such linear combinations with integer coefficients are known explicitly for a long time

• Theorem 3 makes it possible to take a fresh look at the question of whether it is possible to extend a given first-order flex of a polyhedron into its flex?

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A first-order flex of a polyhedron P with triangular faces and with vertices $\boldsymbol{x}_1, \dots, \boldsymbol{x}_V$ in Euclidean 3-space is a finite set of labeled vectors $\boldsymbol{x}'_1, \dots, \boldsymbol{x}'_V$, where each $\boldsymbol{x}_i$ is in $\mathbb{R}^3$, such that the equations

$$(\boldsymbol{x}_i - \boldsymbol{x}_j) \cdot (\boldsymbol{x}'_i - \boldsymbol{x}'_j) = 0$$

are satisfied for all i, j such that the vertices $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ are incident to an edge.

These equations come from the formal derivative of the expression $(\boldsymbol{x}_i - \boldsymbol{x}_j) \cdot (\boldsymbol{x}_i - \boldsymbol{x}_j)$ for the squares of the lengths of the edge incident to $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$, which are supposed to be constant.

Intuitively, $\boldsymbol{x}'_1, \dots, \boldsymbol{x}'_V$ can be regarded as formal velocities permitted by the edge length constraints of the polyhedron P.

Similarly, the notion of an Nth-order flex can be introduced for all $N \geqslant 2$.

The standard idea to answer the question of whether it is possible to extend a given first-order flex of a polyhedron into its flex? is to split it into the following sequence of sub-problems:

(A) Given $N \geqslant 1$ and an Nth-order flex of a polyhedron P, can we extend it to an $(N+1)$th-order flex?

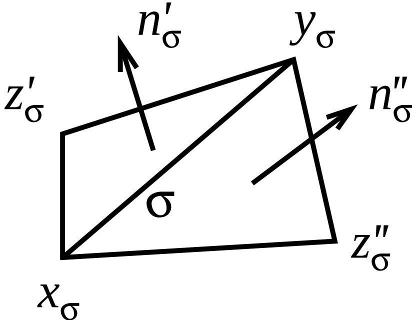
(B) Suppose, the answer to sub-problem (A) is positive for all N, $1 \leqslant N \leqslant N^*$, for some large enough N^*. Is it true that there is a continuous family of deformations of P which preserves its edge lengths and which generates all Nth-order flexes, $1 \leqslant N \leqslant N^*$?

(C) Is it true that the continuous family of deformations of P obtained in sub-problem (B) is its flex, i.e., that it is a nontrivial flex of P?

Each of sub-problems (A)–(C) is non-trivial and has not been fully resolved yet.

Our approach, based on Theorem 3, allows us to exclude all high-order flexes from consideration.

We need the following specific notation:



This figure shows:

- an edge σ
- the faces adjacent to σ
- the vertices $\boldsymbol{x}_\sigma$, $\boldsymbol{y}_\sigma$, $\boldsymbol{z}'_\sigma$, $\boldsymbol{z}''_\sigma$ of these faces
- the outward normals $\boldsymbol{n}'_\sigma$ and $\boldsymbol{n}''_\sigma$ to the faces

Theorem 4 (VA, 2020)

Suppose the conditions and notation of Theorem 3 are fulfilled. Suppose also that $\cup_{\sigma\in E}\{\boldsymbol{a}_\sigma, \boldsymbol{b}_\sigma, \boldsymbol{c}'_\sigma, \boldsymbol{c}''_\sigma\}$ *is a first-order flex of* P *which can be extended to a flex of* P.
Then, for every $j = 1, \dots, m$, *the equality*

$$\sum_{\sigma\in E} \alpha_{\sigma j}\big[\boldsymbol{g}_\sigma \cdot (\boldsymbol{b}_\sigma - \boldsymbol{a}_\sigma) + \boldsymbol{g}'_\sigma \cdot (\boldsymbol{c}'_\sigma - \boldsymbol{a}_\sigma) + \boldsymbol{g}''_\sigma \cdot (\boldsymbol{c}''_\sigma - \boldsymbol{a}_\sigma)\big] = 0$$

holds true, where the vectors $\boldsymbol{g}_\sigma$, $\boldsymbol{g}'_\sigma$, $\boldsymbol{g}''_\sigma$ *are defined by the following long but explicit formulas via position vectors* $\boldsymbol{x}_\sigma$, $\boldsymbol{y}_\sigma$, $\boldsymbol{z}'_\sigma$, $\boldsymbol{z}''_\sigma$ *of the vertices of the polyhedron:*

$$
\boldsymbol{g}_\sigma = \begin{cases} \boldsymbol{r}_\sigma, & \text{if } \cos\varphi_\sigma = 0; \\ \boldsymbol{s}_\sigma, & \text{if } \sin\varphi_\sigma = 0; \\ \text{either } \boldsymbol{r}_\sigma \text{ or } \boldsymbol{s}_\sigma, & \text{if } (\cos\varphi_\sigma)(\sin\varphi_\sigma) \neq 0, \end{cases}
$$

$$
\boldsymbol{g}'_\sigma = \begin{cases} \boldsymbol{r}'_\sigma, & \text{if } \cos\varphi_\sigma = 0; \\ \boldsymbol{s}'_\sigma, & \text{if } \sin\varphi_\sigma = 0; \\ \text{either } \boldsymbol{r}'_\sigma \text{ or } \boldsymbol{s}'_\sigma, & \text{if } (\cos\varphi_\sigma)(\sin\varphi_\sigma) \neq 0, \end{cases}
$$

$$
\boldsymbol{g}''_\sigma = \begin{cases} \boldsymbol{r}''_\sigma, & \text{if } \cos\varphi_\sigma = 0; \\ \boldsymbol{s}''_\sigma, & \text{if } \sin\varphi_\sigma = 0; \\ \text{either } \boldsymbol{r}''_\sigma \text{ or } \boldsymbol{s}''_\sigma, & \text{if } (\cos\varphi_\sigma)(\sin\varphi_\sigma) \neq 0, \end{cases}
$$

where, in turn,

$$
\begin{aligned}
\boldsymbol{r}_\sigma &= -\frac{1}{\eta_\sigma}\big[(\boldsymbol{y}_\sigma - \boldsymbol{x}_\sigma)\cdot(\boldsymbol{z}''_\sigma - \boldsymbol{x}_\sigma)\big](\boldsymbol{z}'_\sigma - \boldsymbol{x}_\sigma) \\
&\qquad - \frac{1}{\eta_\sigma}\big[(\boldsymbol{y}_\sigma - \boldsymbol{x}_\sigma)\cdot(\boldsymbol{z}'_\sigma - \boldsymbol{x}_\sigma)\big](\boldsymbol{z}''_\sigma - \boldsymbol{x}_\sigma), \\
\boldsymbol{r}'_\sigma &= -\frac{1}{\eta_\sigma}\big[(\boldsymbol{y}_\sigma - \boldsymbol{x}_\sigma)\cdot(\boldsymbol{z}''_\sigma - \boldsymbol{x}_\sigma)\big](\boldsymbol{y}_\sigma - \boldsymbol{x}_\sigma) \\
&\qquad + \frac{\ell_\sigma^2}{\eta_\sigma}(\boldsymbol{z}''_\sigma - \boldsymbol{x}_\sigma)(\boldsymbol{y}_\sigma - \boldsymbol{x}_\sigma)\cdot\big[(\boldsymbol{z}'_\sigma - \boldsymbol{x}_\sigma)\times(\boldsymbol{z}''_\sigma - \boldsymbol{x}_\sigma)\big], \\
\boldsymbol{r}''_\sigma &= -\frac{1}{\eta_\sigma}\big[(\boldsymbol{y}_\sigma - \boldsymbol{x}_\sigma)\cdot(\boldsymbol{z}'_\sigma - \boldsymbol{x}_\sigma)\big](\boldsymbol{y}_\sigma - \boldsymbol{x}_\sigma) + \frac{\ell_\sigma^2}{\eta_\sigma}(\boldsymbol{z}'_\sigma - \boldsymbol{x}_\sigma), \\
\eta_\sigma &= \ell_\sigma(\boldsymbol{y}_\sigma - \boldsymbol{x}_\sigma)\cdot\big[(\boldsymbol{z}'_\sigma - \boldsymbol{x}_\sigma)\times(\boldsymbol{z}''_\sigma - \boldsymbol{x}_\sigma)\big],
\end{aligned}
$$

and

$$
\begin{aligned}
\boldsymbol{s}_\sigma &= -\frac{1}{\vartheta_\sigma}(\boldsymbol{z}'_\sigma - \boldsymbol{x}_\sigma) \times (\boldsymbol{z}''_\sigma - \boldsymbol{x}_\sigma), \\
\boldsymbol{s}'_\sigma &= -\frac{1}{\vartheta_\sigma}(\boldsymbol{y}_\sigma - \boldsymbol{x}_\sigma) \times (\boldsymbol{z}''_\sigma - \boldsymbol{x}_\sigma), \\
\boldsymbol{s}''_\sigma &= -\frac{1}{\vartheta_\sigma}(\boldsymbol{y}_\sigma - \boldsymbol{x}_\sigma) \times (\boldsymbol{z}'_\sigma - \boldsymbol{x}_\sigma), \\
\vartheta_\sigma &= \ell_\sigma^2(\boldsymbol{z}'_\sigma - \boldsymbol{x}_\sigma) \cdot (\boldsymbol{z}''_\sigma - \boldsymbol{x}_\sigma) \\
&\quad -\big[(\boldsymbol{z}'_\sigma - \boldsymbol{x}_\sigma) \cdot (\boldsymbol{y}_\sigma - \boldsymbol{x}_\sigma)\big]\big[(\boldsymbol{z}''_\sigma - \boldsymbol{x}_\sigma) \cdot (\boldsymbol{y}_\sigma - \boldsymbol{x}_\sigma)\big].
\end{aligned}
$$

Theorem 4 (VA, 2020)

Suppose the conditions and notation of Theorem 3 are fulfilled. Suppose also that $\cup_{\sigma\subset P}\{\boldsymbol{a}_\sigma, \boldsymbol{b}_\sigma, \boldsymbol{c}'_\sigma, \boldsymbol{c}''_\sigma\}$ *is a first-order flex of* P *which can be extended to a flex of* P.
Then, for every $j = 1, \dots, m$, *the equality*

$$\sum_{\sigma\in E} \alpha_{\sigma j}\big[\boldsymbol{g}_\sigma\cdot(\boldsymbol{b}_\sigma-\boldsymbol{a}_\sigma)+\boldsymbol{g}'_\sigma\cdot(\boldsymbol{c}'_\sigma-\boldsymbol{a}_\sigma)+\boldsymbol{g}''_\sigma\cdot(\boldsymbol{c}''_\sigma-\boldsymbol{a}_\sigma)\big]=0$$

holds true, where the vectors $\boldsymbol{g}_\sigma$, $\boldsymbol{g}'_\sigma$, $\boldsymbol{g}''_\sigma$ *are defined by the following long but explicit formulas via position vectors* $\boldsymbol{x}_\sigma$, $\boldsymbol{y}_\sigma$, $\boldsymbol{z}'_\sigma$, $\boldsymbol{z}''_\sigma$ *of the vertices of the polyhedron.*

Remarks:

In (VA, 2020), an example of a polyhedron is given for which the system of red equations is not a consequence of the system of equations in the definition of the first-order flex.

Hence, for some polyhedra, it can occur that some of the red equations are violated for some of their first-order flexes.

In this case, we can answer in negative to our question of whether it is possible to extend the given first-order flex of a polyhedron into its flex?

I am not aware about any previously published results similar to Theorem 4 neither for polyhedra, nor for bar-and-joint frameworks.

Let us recall the logical relationship of the above presented results:

Theorem 2 (A.A. Gaifullin – L.S. Ignashchenko, 2018)

. . . the Dehn invariants remain unaltered during the flex . . .

$$\Downarrow$$

Theorem 3 (VA, 2019)

. . . the expression $\displaystyle\sum_{\sigma\in E}\alpha_{\sigma j}\varphi_\sigma$ *is constant during the flex . . .*

$$\Downarrow$$

Theorem 4 (VA, 2020)

. . . for every $j = 1, \dots, m$,

$$\sum_{\sigma\in E}\alpha_{\sigma j}\big[\boldsymbol{g}_\sigma\cdot(\boldsymbol{b}_\sigma-\boldsymbol{a}_\sigma)+\boldsymbol{g}'_\sigma\cdot(\boldsymbol{c}'_\sigma-\boldsymbol{a}_\sigma)+\boldsymbol{g}''_\sigma\cdot(\boldsymbol{c}''_\sigma-\boldsymbol{a}_\sigma)\big]=0$$

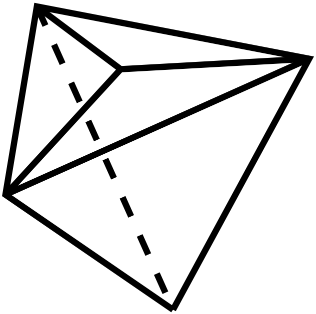
Note that the Dehn invariants are not the only source of equations similar to the red ones, that is of necessary conditions for a given first-order flex of a polyhedron P to be extendable to a flex.

Another source for such conditions is the rigidity matrix. More precisely, every $(3V-6)\times(3V-6)$ minor of the rigidity matrix, where V is the number of vertices of P, can be considered as a source for such conditions.

Such a minor is a homogeneous polynomial in the coordinates of the vertices of P that is identically equal to zero during the flex. Differentiating the minor with respect to the parameter of the flex at the starting point of the flex and replacing the derivatives of the variables with the correstonding vectors of the first-order flex of P, we obtain some new equations similar to the red ones.

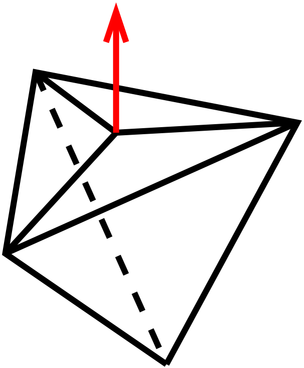
Necessary conditions for a given first-order flex of a polyhedron to be extendable to a flex that arise from the rigidity matrix are too complicated to be written out here explicitly.

Note that, for some polyhedra, some of the equations arising from the rigidity matrix are violated for some of their first-order flexes.

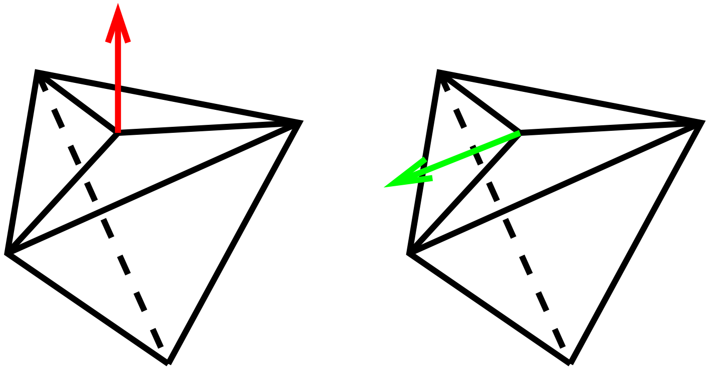


This figure shows a tetrahedron T, one of whose faces is additionally triangulated.

For such a polyhedron at least one of the equations described on the previous slide is not a consequence of the "standard" system of equations for the first-order flex.



Red arrow is perpendicular to the additionally triangulated face of T. It represents a first-order flex (the velocities of all other vertices are equal to zero).



Green arrow lies in the additionally triangulated face of T.
It represents an infinitesimal deformation along the set of all first-order non-rigid polyhedra (the velocities of all other vertices are equal to zero).

Hence, for some polyhedra, some of the equations arising from the rigidity matrix are independent of the equations from the definition of the first-order flex.

In conclusion, I would like to note once again that many problems related to the topic of this talk are still awaiting their solution ...

For example: Do these new equations imply that the volume bounded by a flexible polyhedron is stationary?

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In conclusion, I would like to note once again that many problems related to the topic of this talk are still awaiting their solution ...

For example: Do these new equations imply that the volume bounded by a flexible polyhedron is stationary?

`Thank you for attention!`