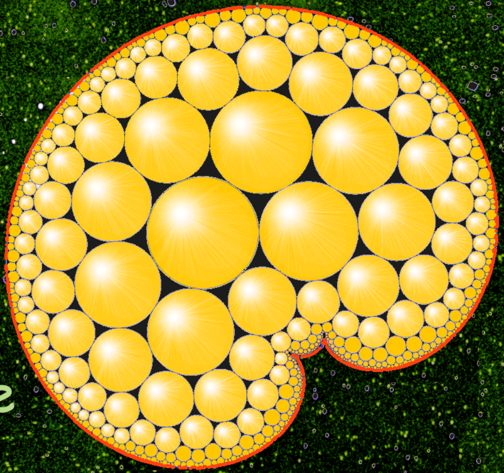
Domain-Filling Circle Agglomerations

Elias Wegert
David Krieg

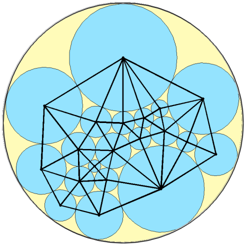


Fields Institute
Toronto 2021

Motivation

Discrete conformal mapping

Koebe-Andreev-Thurston Theorem: For each contact graph which is a topological triangulation of a disk there exists an associated maximal circle packing which "fills" the (complex unit) disk $\mathbb{D}$.



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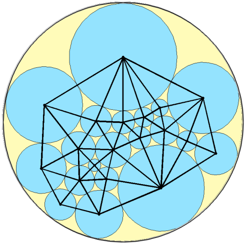
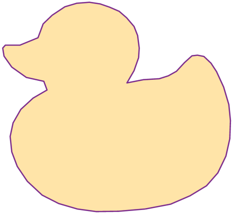
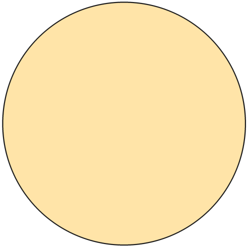


Discrete conformal mapping using the “cookie-cutting” technique





Images created with Ken Stephenson’s software CirclePack

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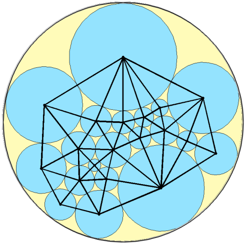

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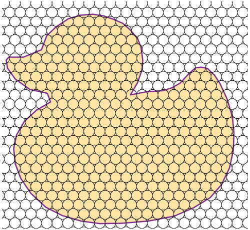

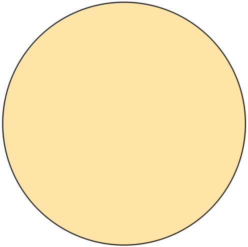

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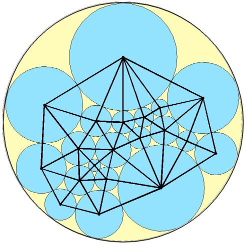

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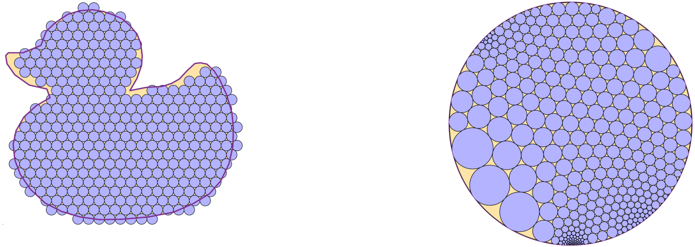

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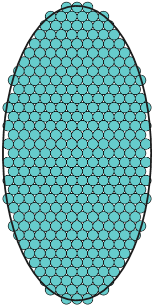
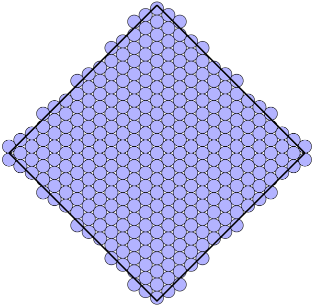
Discrete conformal mapping with given contact graph

disadvantage of cookie-cutting: geometry determines combinatorics



357 disks



323 disks

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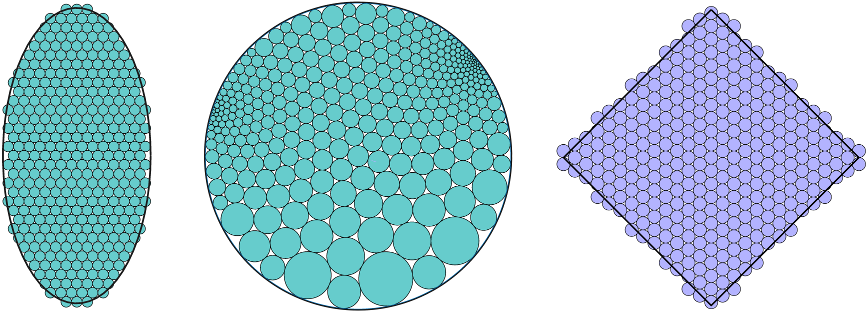

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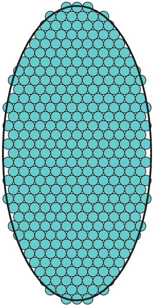

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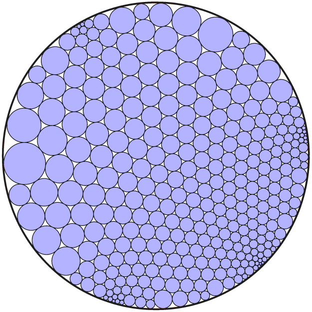

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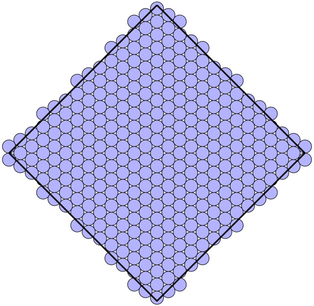

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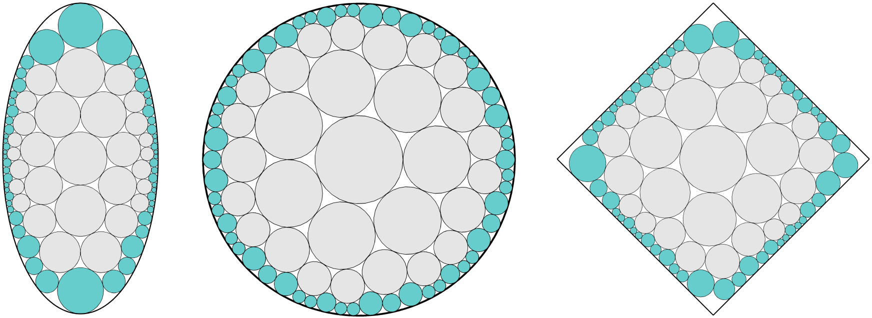

all packings have 55 disks and the same contact graph

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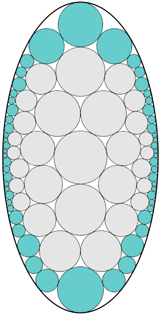
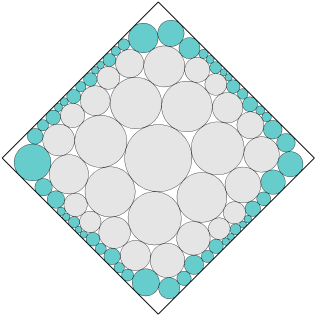


$\longrightarrow$

discrete conformal mapping
between ellipse and square

$\longleftarrow$



study finite circle packings with given contact graph ("combinatorics")
filling arbitrary bounded simply connected domains

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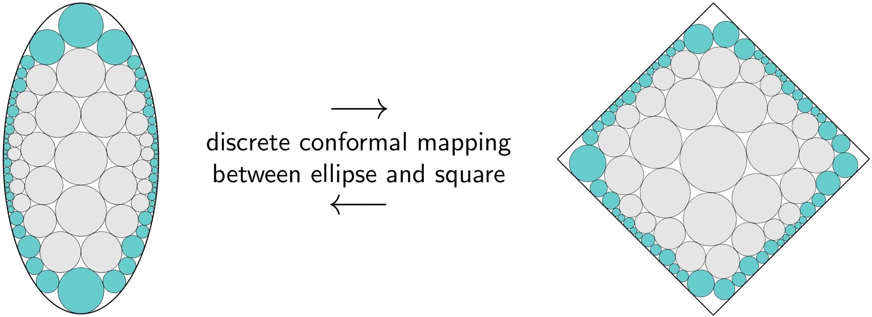

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existence, uniqueness, normalization, ...

Known results, questions, strategy

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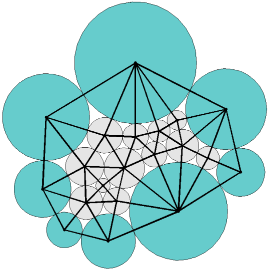
While the approach of He and Schramm relies on Koebe's uniformization theorem and its generalizations, we were interested in an elementary approach based on Sperner's lemma and induction.

The Setting

Circle packings and circle agglomerations

Circle packing: ensemble of circles (disks) with prescribed pattern of tangencies encoded in a simplicial complex K. Those K that are topological disks form the class $\mathscr{K}$ of "admissible complexes".



Circle packing with its complex K

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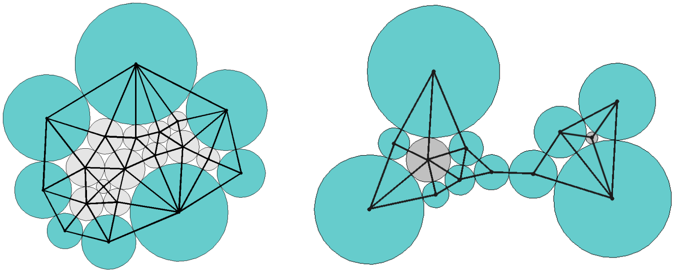
Circle agglomeration: more general than circle packings, underlying complex K in larger class of "acceptable complexes" $\mathscr{K}^*$.



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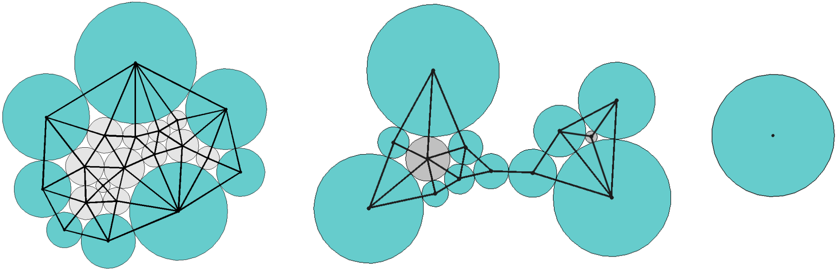

Circle packing (left) and circle agglomerations (middle, right).

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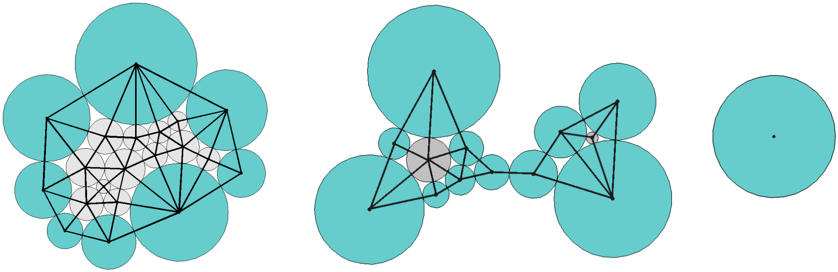
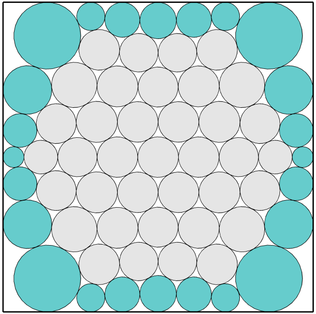


Circle packing (left) and circle agglomerations (middle, right).
All agglomerations consist of finitely many non-overlapping disks.

Circle packings filling Jordan domains

A circle packing $\mathscr{P}$ fills a Jordan domain G if

- all (open) disks D of $\mathscr{P}$ are contained in G, $D \subset G$,
- all boundary disks D of $\mathscr{P}$ touch ∂G, $\overline{D} \cap \partial G \neq \emptyset$.



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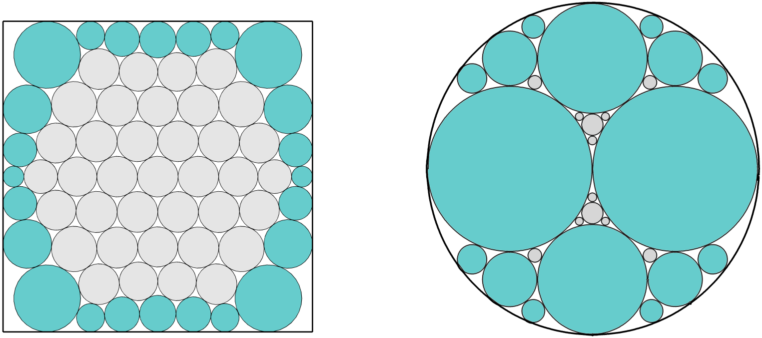

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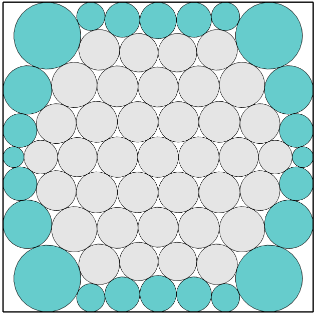

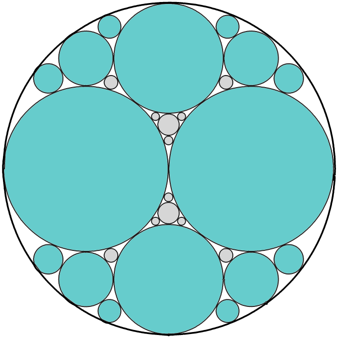

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For non-Jordan domains the definition is inappropriate.

Uniqueness and normalization

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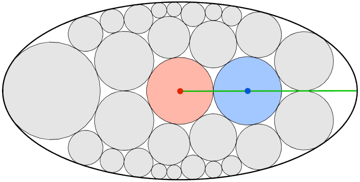

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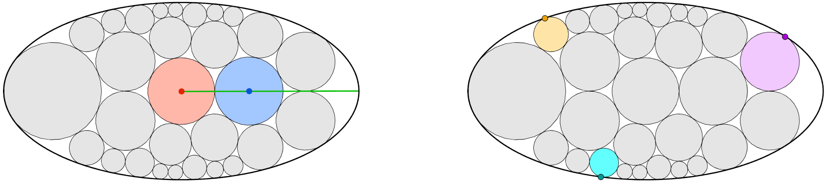

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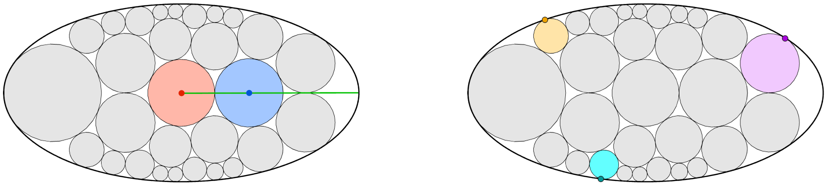

If G is not Jordan, the *points* z_j must be replaced by prime ends.

Trilaterals

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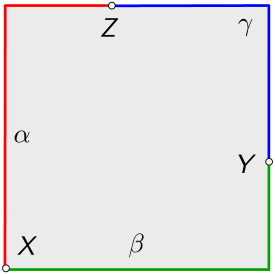
A trilateral $G(\alpha,\beta,\gamma)$ is a domain G whose boundary ∂G
is decomposed into three *closed* arcs α, β and γ .
The vertices of $G(\alpha,\beta,\gamma)$ are $X := \alpha \cap \beta$, $Y := \beta \cap \gamma$ and $Z := \gamma \cap \alpha$.



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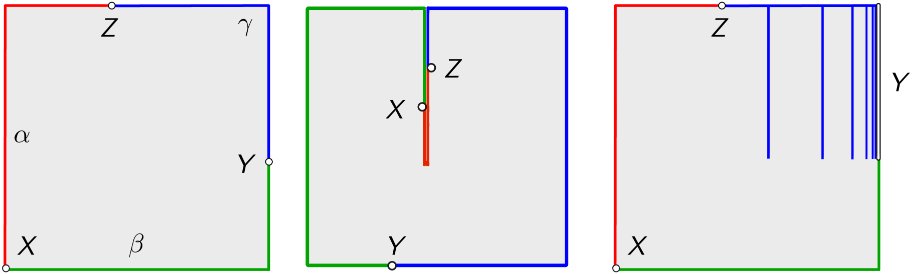

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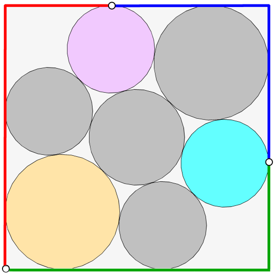
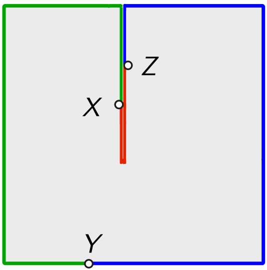
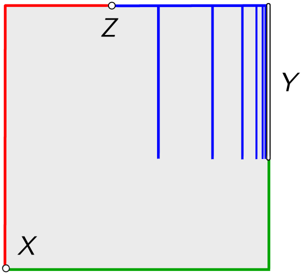


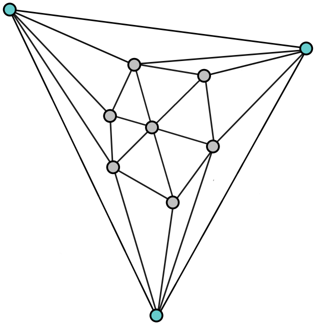




A disk D meets a prime end X if it touches two arcs of prime ends with common endpoint X.

Tri-complexes and framing

A tri-complex $T = T(a, b, c)$ is an admissible complex with exactly three boundary vertices a, b, c and at least one interior vertex. The class of all tri-complexes is denoted by $\mathscr{T}$.



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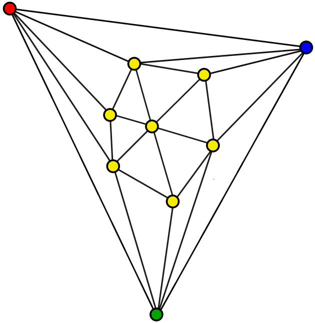

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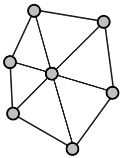

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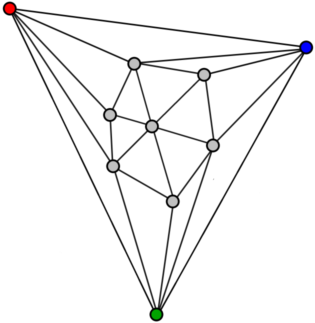
Framing is the conversion of a complex K to a tri-complex T by associating three boundary vertices.



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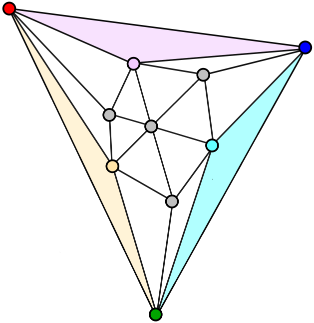
Framing is the conversion of a complex K to a tri-complex T by associating three boundary vertices. Leading vertices are those vertices of K which together with *two* boundary vertices of T form a face of T.



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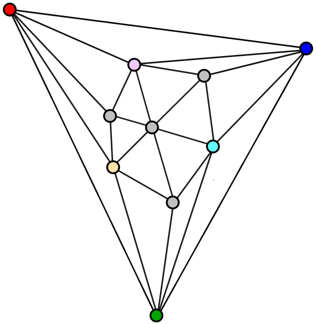

The class $\mathscr{K}^*$ of acceptable complexes consists of all complexes K which are convertible to a tri-complex by framing.

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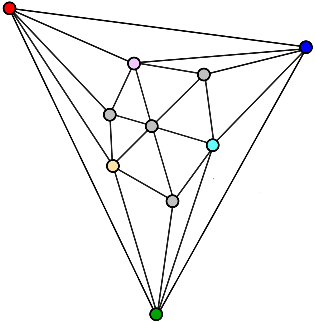
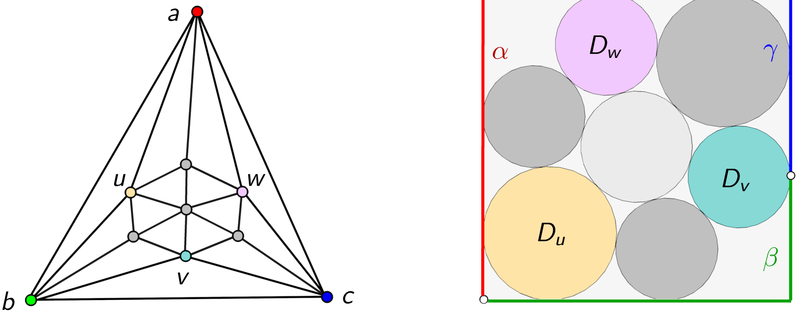


The class $\mathscr{K}^*$ of acceptable complexes consists of all complexes K which are convertible to a tri-complex by framing.
Admissible complexes are acceptable, $\mathscr{K} \subset \mathscr{K}^*$.

Circle agglomerations filling Jordan trilaterals

A circle agglomeration $\mathscr{P}$ associated with a tri-complex $T(a, b, c)$ fills a Jordan trilateral $G(\alpha, \beta, \gamma)$ if its disks lie in G and each boundary disk D_v touches the corresponding arcs α, β, γ associated with those vertices a, b or c which are neighbors of v in T.



Circle packing associated with tri-complex filling a trilateral

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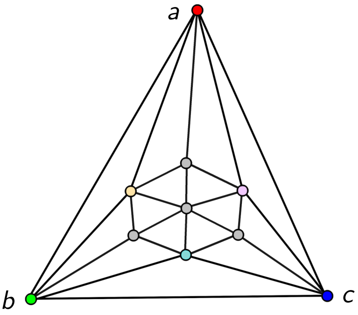

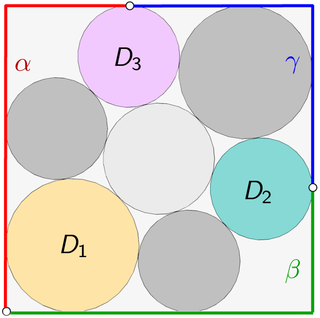

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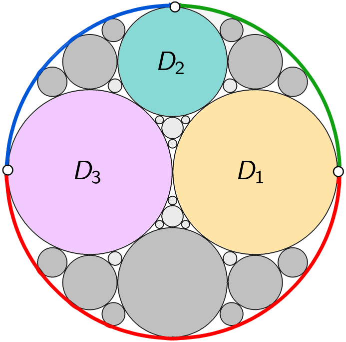


Circle agglomeration associated with tri-complex filling a trilateral

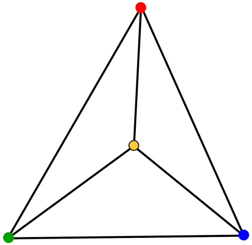
Incircles

Incircle of a trilateral

The simplest acceptable complex has a single vertex

Incircle of a trilateral

The simplest acceptable complex has a single vertex and can be framed in just one way.



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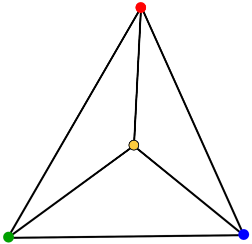
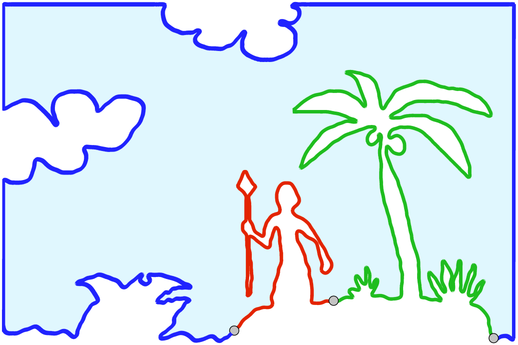


An incircle touches all three arcs of a trilateral.



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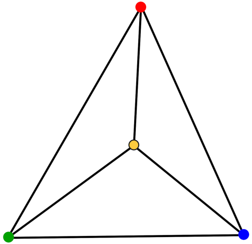

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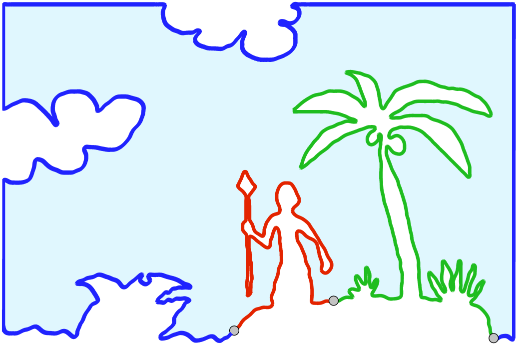

Theorem (David Krieg, EW)

Every trilateral $G(\alpha, \beta, \gamma)$ has an incircle.
If the trilateral is tame, its incircle is uniquely determined.

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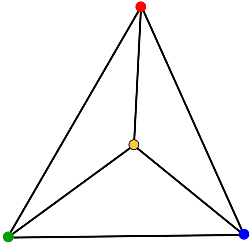

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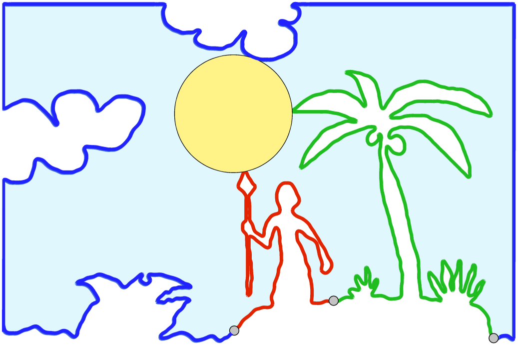
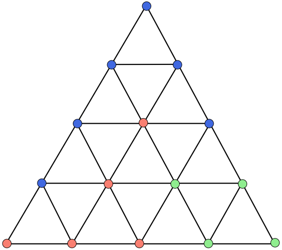


Theorem (David Krieg, EW)

Every trilateral $G(\alpha, \beta, \gamma)$ has an incircle.
If the trilateral is tame, its incircle is uniquely determined.

Idea of proof: Sperner's lemma

Sperner's Lemma. Let Δ be a triangle with vertices r, g, b, and let T be a triangulation of Δ. Assume that every vertex of T is colored with one of three colors, such that r, g and b are colored red, green and blue, respectively, and each vertex on an edge of Δ is colored with one of the two colors at the ends of that edge.



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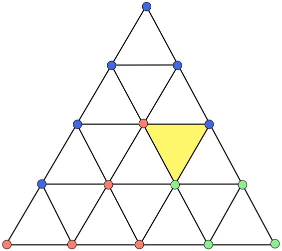

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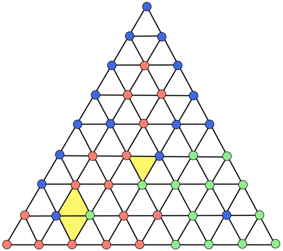

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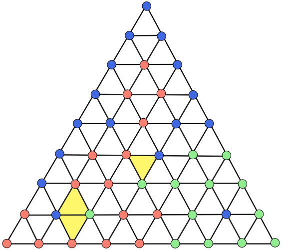

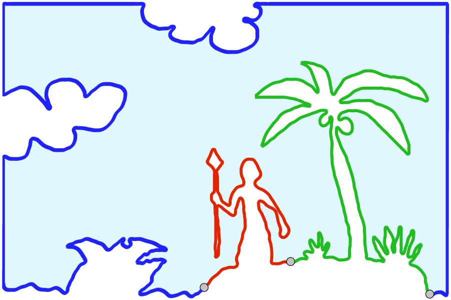

Color the points z of G like the closest of the arcs α, β, γ (with preferences $R \succ G \succ B$).

Idea of proof: Sperner's lemma

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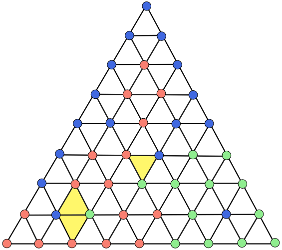

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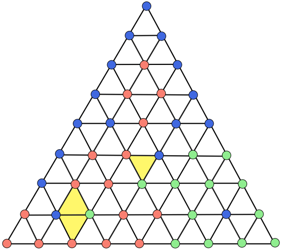
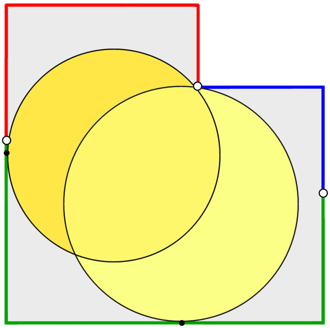


Color the points z of G like the closest of the arcs α, β, γ (with preferences $R \succ G \succ B$). Use Sperner's lemma and a compactness argument.

Uniqueness of incircle and tame trilaterals

The incircle of a trilateral need not be unique.



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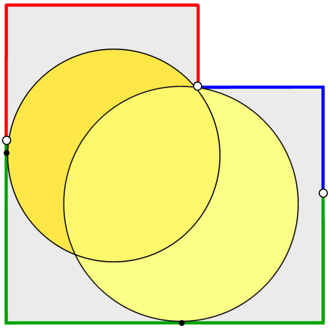

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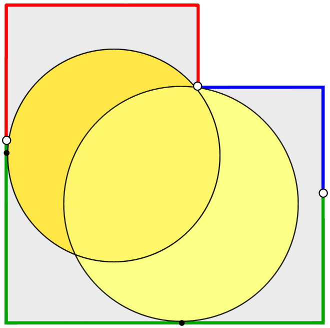

A prime end X of G is said to be regular if for any two disks $D_1, D_2 \subset G$ which touch X necessarily $D_1 \subset D_2$ or $D_2 \subset D_1$. A trilateral is tame if all its vertices are regular.

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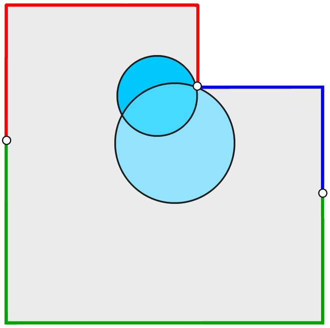

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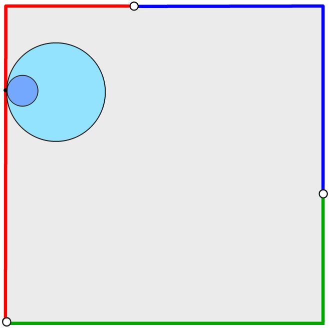
A tame trilateral
all boundary points are regular



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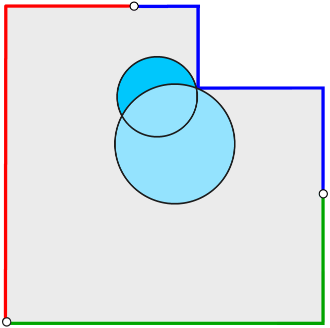

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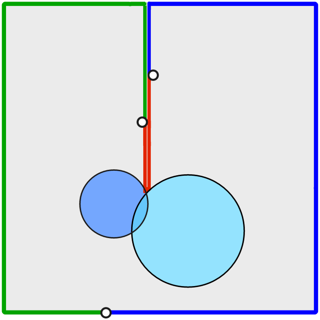

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Uniqueness of incircle and tame trilaterals

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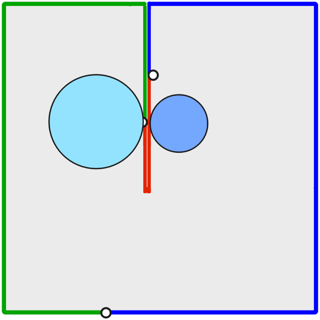
A tame trilateral
disks touch the same *point*,
but different *prime ends*



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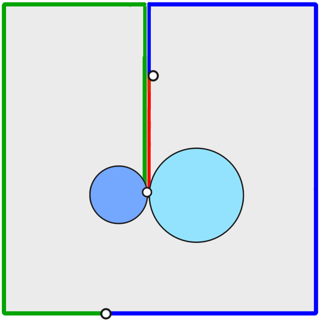
This trilateral is not tame
disks touch the same prime end



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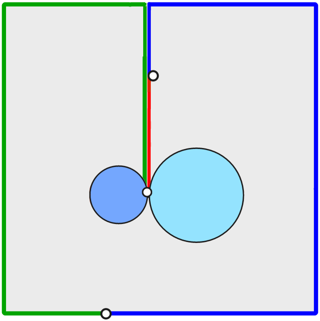

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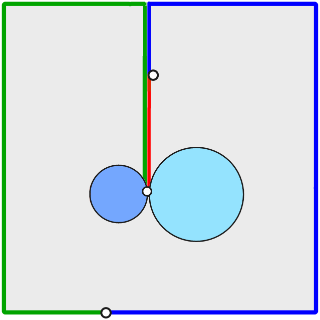
Uniqueness of incircle and tame trilaterals

The incircle of a trilateral need not be unique.

This trilateral is spiky
disks touch the same vertex



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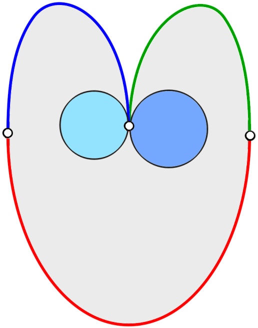
A spiky trilateral has a vertex that can be touched by *two disjoint disks*.

Uniqueness of incircle and tame trilaterals

The incircle of a trilateral need not be unique.

A spiky Jordan trilateral



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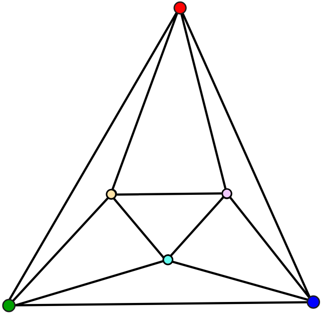
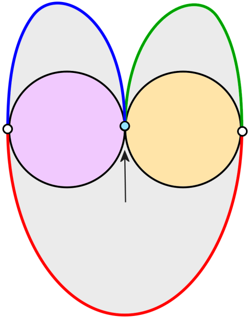
Degeneration

Degenerate circle agglomerations: spiky trilaterals

Some spiky trilaterals cannot be filled by circle packings with given admissible complex.





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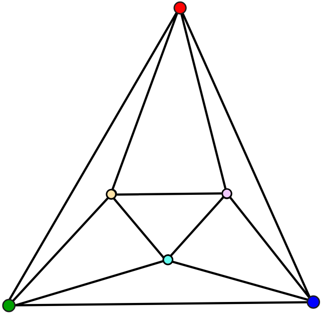

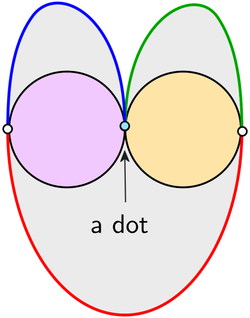

Two leading disks must be positioned as shown, the third degenerates.

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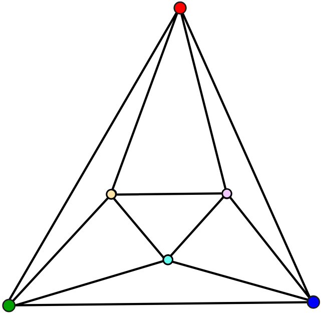
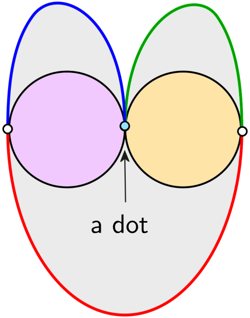
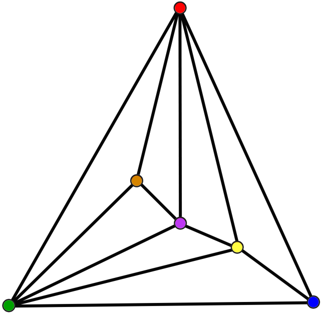


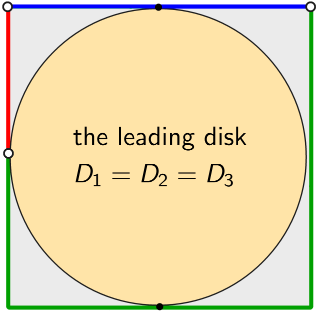


Two leading disks must be positioned as shown, the third degenerates.
To fill general trilaterals, we must admit degenerate packings.

Degenerate circle agglomerations: reducible complexes

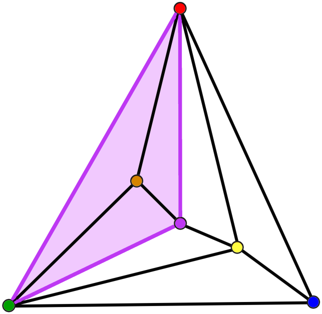
There are even tame trilaterals that cannot be filled by (proper) circle agglomerations associated with some tri-complex

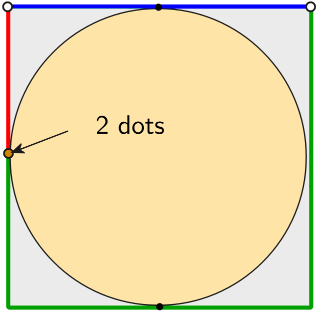




Degenerate circle agglomerations: reducible complexes

There are even tame trilaterals that cannot be filled by (proper) circle agglomerations associated with some tri-complex (boundary reducible)





A tri-complex is boundary reducible if it has three vertices that are not all interior and form a triangle but not a face.

Degenerate circle agglomerations: reducible complexes

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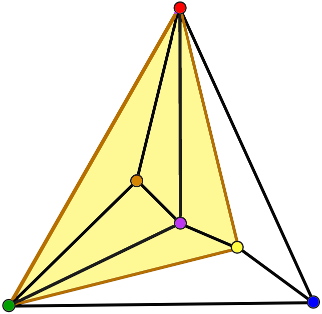

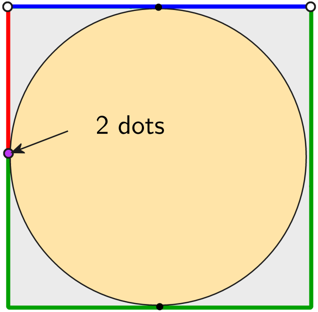

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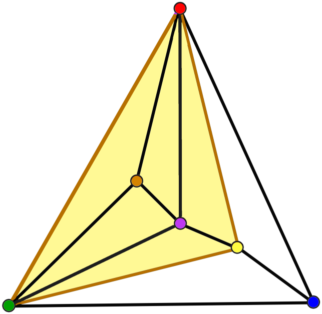
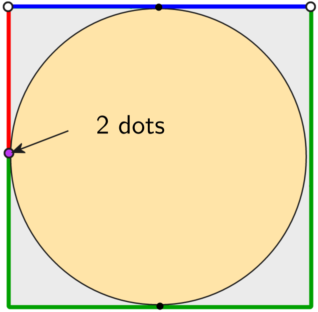
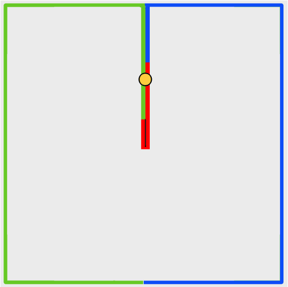




Since boundary reducible complexes show up in the proof, we must admit degenerate agglomerations containing dots.

Generalized circle agglomerations: disks and dots

Dots result as *limits of disks* with radius converging to zero.
A dot $\{s\}$ "sitting at s" is to be distinguished from the point s.



Generalized circle agglomerations: disks and dots

Dots result as *limits of disks* with radius converging to zero.
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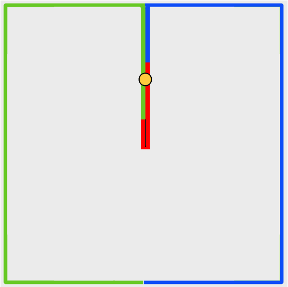
This distinction is essential if G is not a Jordan domain:
we work with the intrinsic boundary ∂G^* of G, formed by prime ends.



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While it is clear which prime ends a *disk* $D \subset G$ touches (if any),
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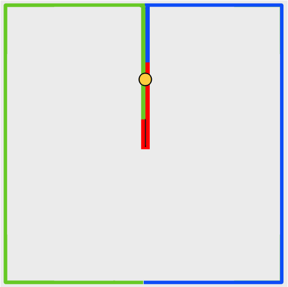

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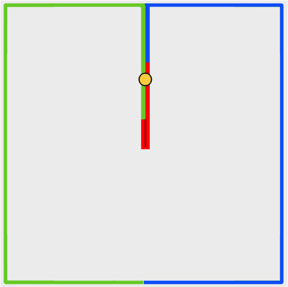

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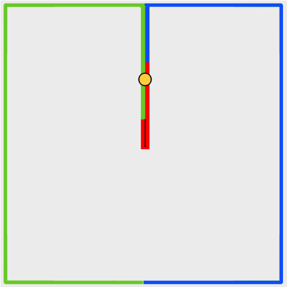

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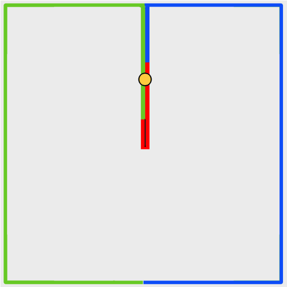

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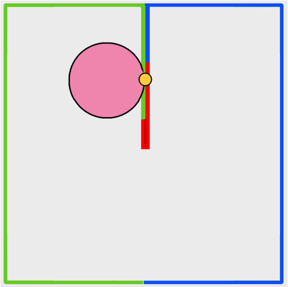

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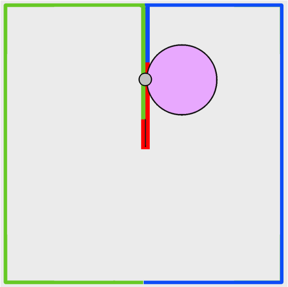

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When attached to this disk, $\{s\}$ touches the "red prime end".

Pseudo contact points

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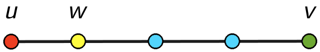

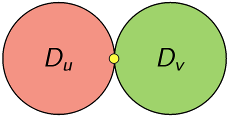

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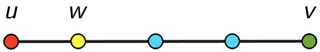

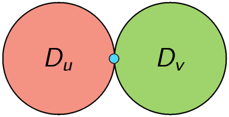

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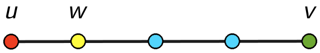

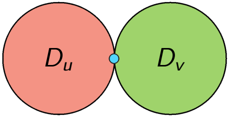

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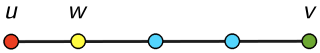

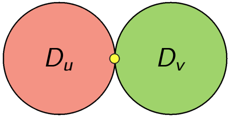

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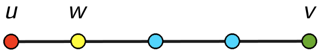

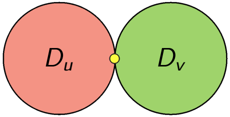

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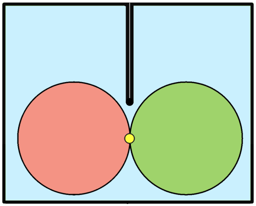
Pseudo-contact points are important to decide whether a generalized circle agglomeration lies in a domain which is not Jordan.



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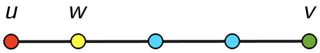

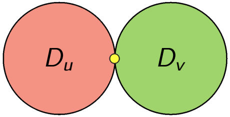

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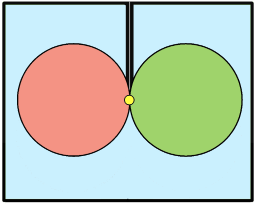

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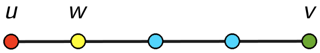

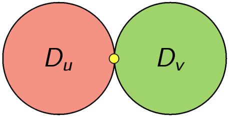

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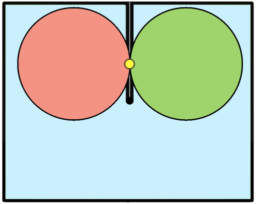

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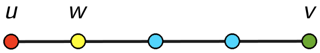

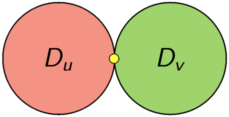

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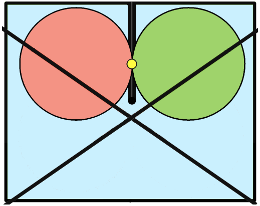

Main Result

Generalized circle agglomerations filling trilaterals

Let $\mathscr{P}$ be a non-collapsed generalized circle agglomeration with acceptable complex $K(V, E, F)$ framed by a tri-complex $T(a, b, c)$.
We say that $\mathscr{P}$ is associated with T and fills the trilateral $G(\alpha, \beta, \gamma)$ if the following conditions are satisfied:

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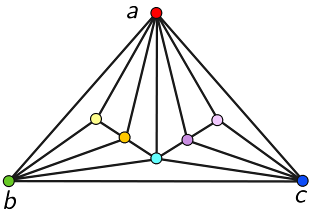

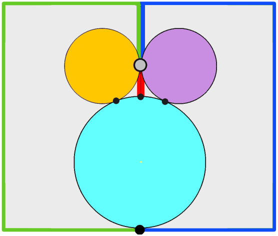

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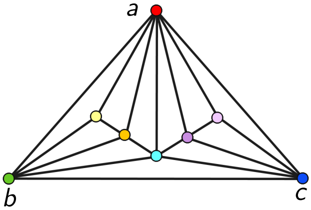

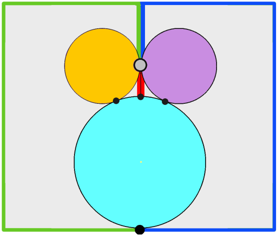

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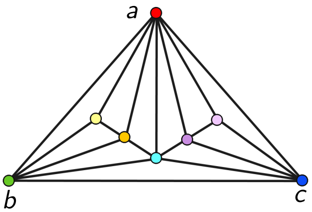

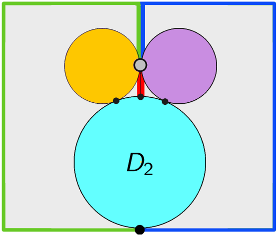

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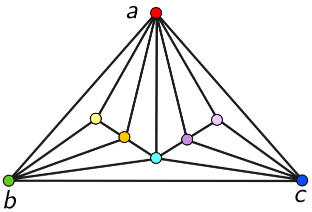

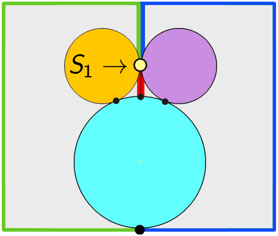

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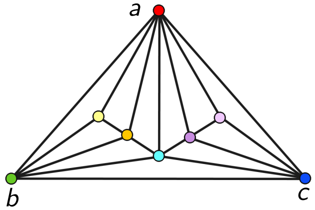

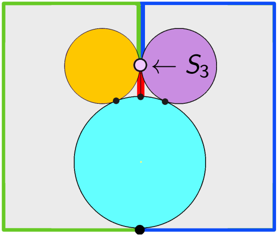

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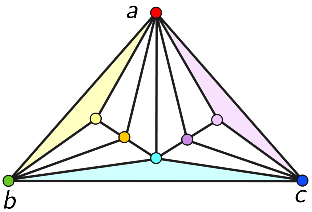

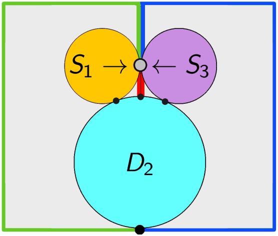

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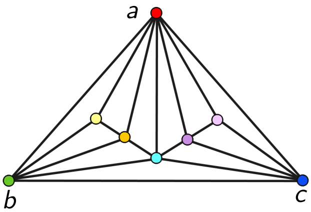

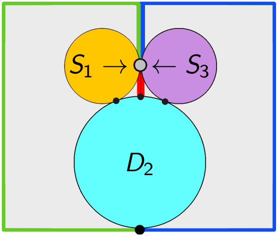

Main result

Theorem (Domain-Filling Circle Agglomerations, David Krieg & EW)

Let T be a tri-complex and let $G(\alpha, \beta, \gamma)$ be a trilateral for a bounded, simply connected domain G. Then:

(i) *There exists a generalized circle agglomeration P which is associated with T and fills $G(\alpha, \beta, \gamma)$.*

(ii) *If the trilateral $G(\alpha, \beta, \gamma)$ is tame, P is unique.*

(iii) *If T is boundary irreducible and the trilateral $G(\alpha, \beta, \gamma)$ is not spiky, then (any such) P is non-degenerate.*

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If K is admissible and $K = \mathrm{int}\, T$, then T is boundary irreducible. Hence any non-spiky trilateral can be filled by a proper *circle packing* with complex K associated with T.

Main result

Theorem (Domain-Filling Circle Agglomerations, David Krieg & EW)

Let T be a tri-complex and let $G(\alpha, \beta, \gamma)$ be a trilateral for a bounded, simply connected domain G. Then:

(i) *There exists a generalized circle agglomeration P which is associated with T and fills $G(\alpha, \beta, \gamma)$.*

(ii) *If the trilateral $G(\alpha, \beta, \gamma)$ is tame, P is unique.*

(iii) *If T is boundary irreducible and the trilateral $G(\alpha, \beta, \gamma)$ is not spiky, then (any such) P is non-degenerate.*

If K is admissible and $K = \operatorname{int} T$, then T is boundary irreducible.
Hence any non-spiky trilateral can be filled by a proper *circle packing* with complex K associated with T.
Recall that we did not require irreducibility of admissible complexes, so our concept of circle packings is slightly more general than usual.

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(1) Induction with respect to the number n of vertices in K

(2) Existence proof using Sperner's Lemma

(3) Exhaustion of arbitrary domains by smooth domains

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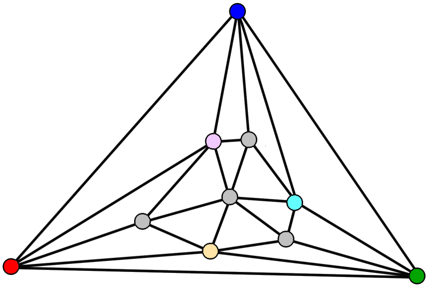
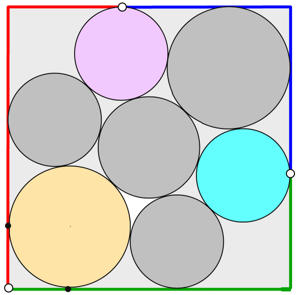
present only ideas of (1) and (2)

Induction: combinatoric and geometric surgery

Induction requires simultaneously to modify the *complex* and the *trilateral*. We demonstrate this with a simple example.





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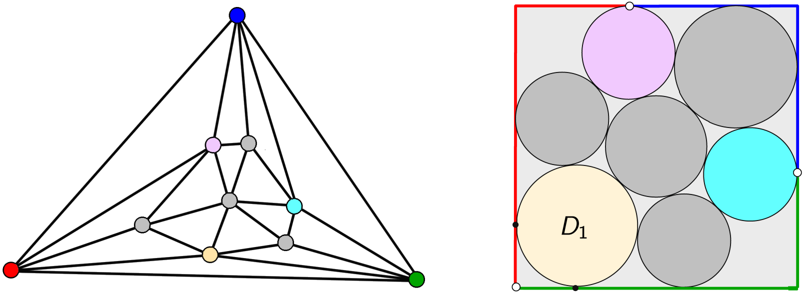

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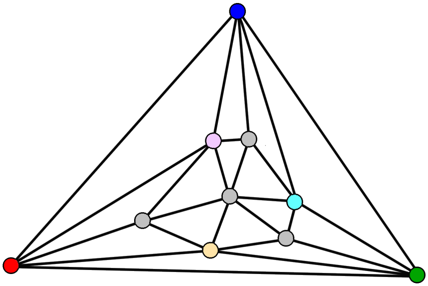

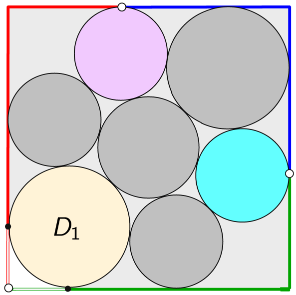

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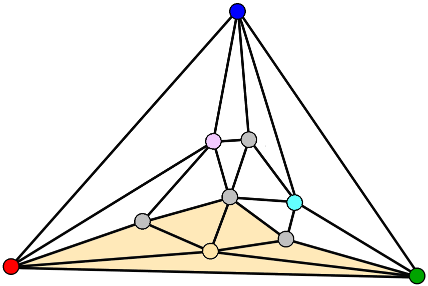

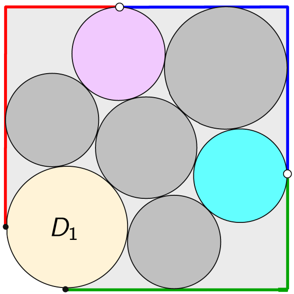

We want to eliminate the leading disk D_1.
The part of G "behind" D_1 will be removed.
Removing the disk D_1 will require modifications of K.

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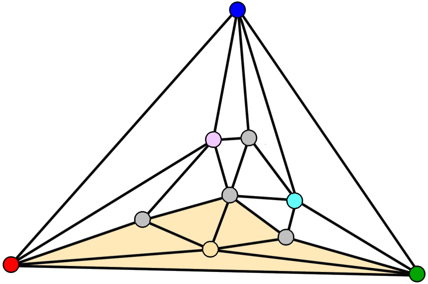

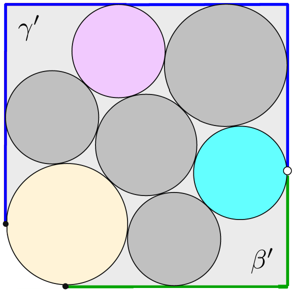

The arc β is reduced to β' and the arc γ is extended to γ'.

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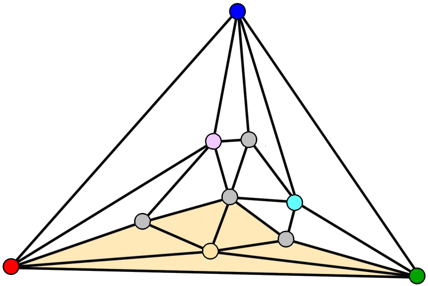

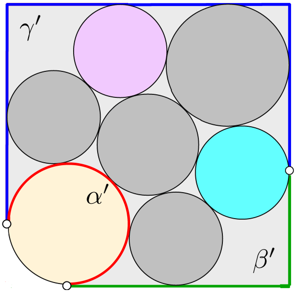

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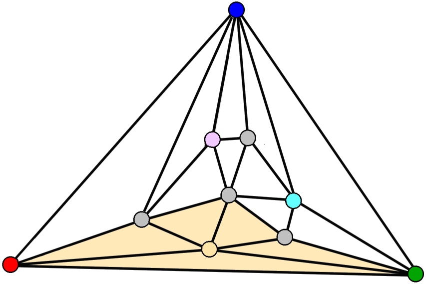

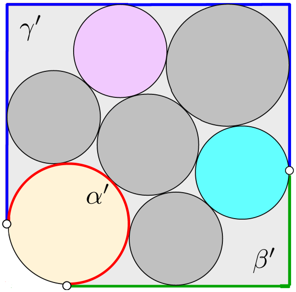

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The complex K must be modified accordingly.

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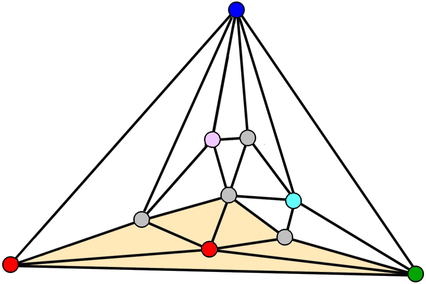

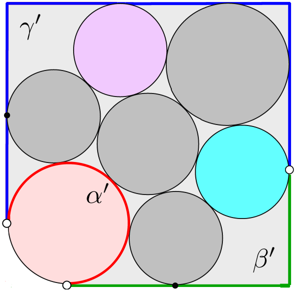

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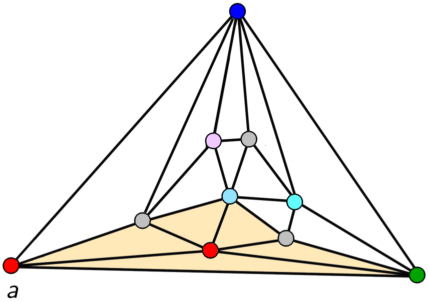

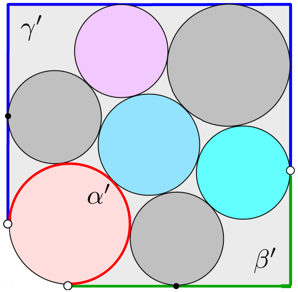

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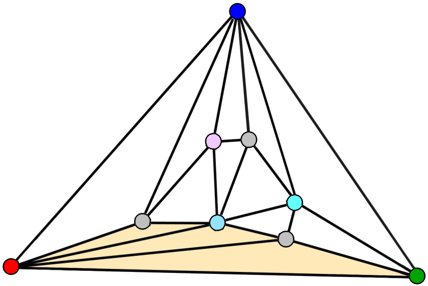

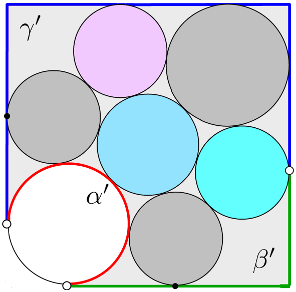

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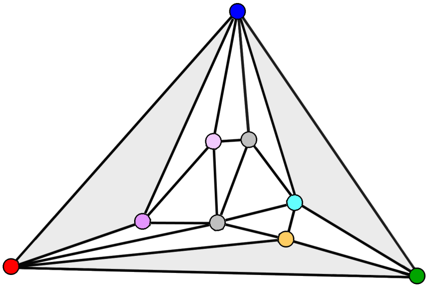

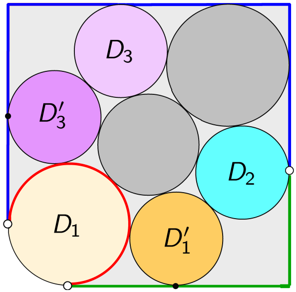

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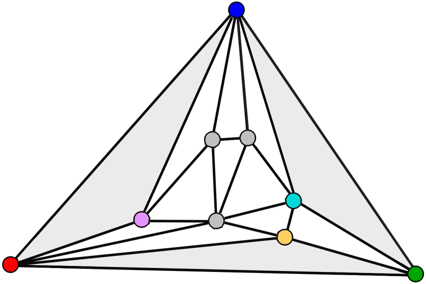

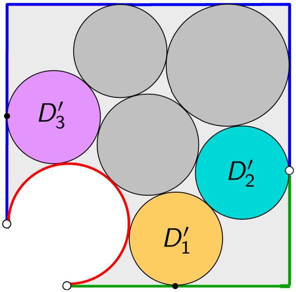

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The leading disks D_1 and D_3 of the old packing are replaced, while D_2 remains.

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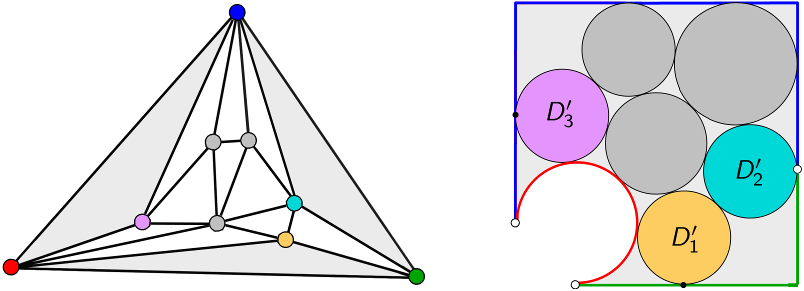

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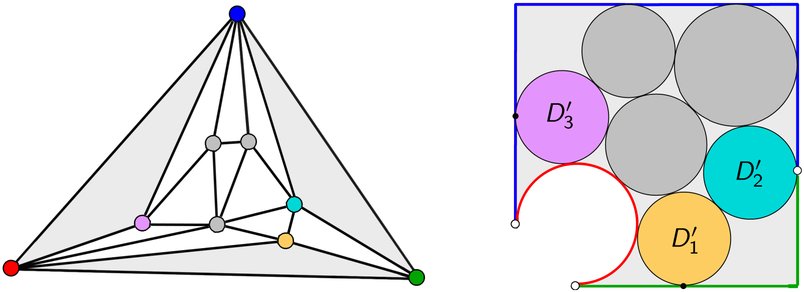

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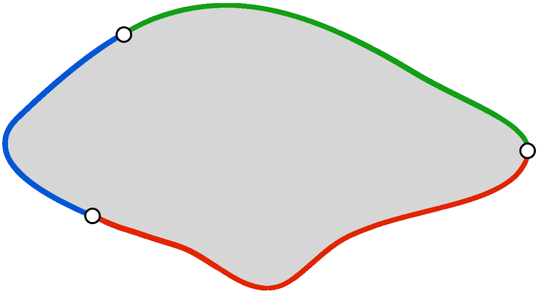
Reducing the number of disks was the easy part –
let's try to do it the other way around.

Geometric surgery: construction of new trilateral

We pick a point $z \in G$ and build a new trilateral $G'_z = G'_z(\alpha', \beta', \gamma')$ removing from G the maximal disk $D_z \subset G$ centered at z.



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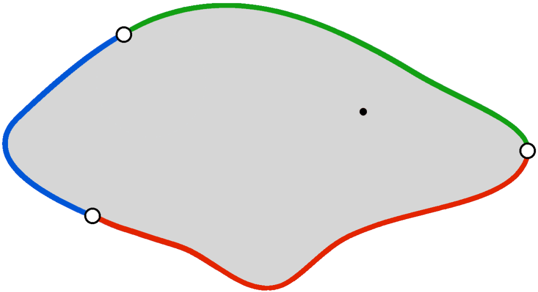

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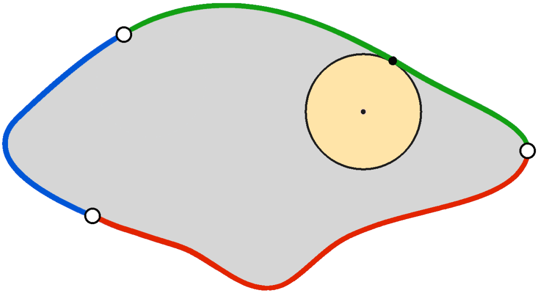

Construction of $G'_z(\alpha', \beta', \gamma')$ if D_z touches one prime end of G.

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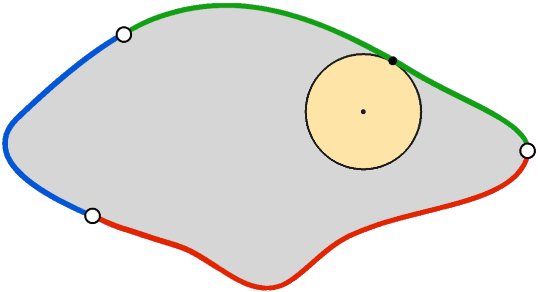

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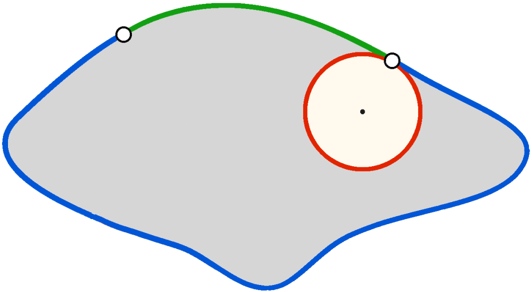

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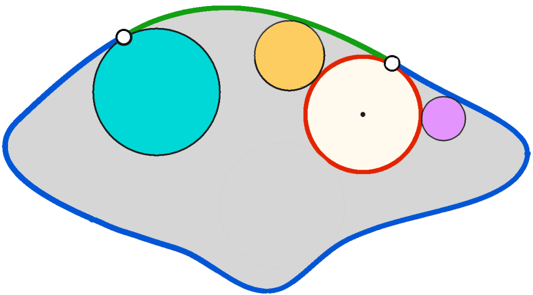

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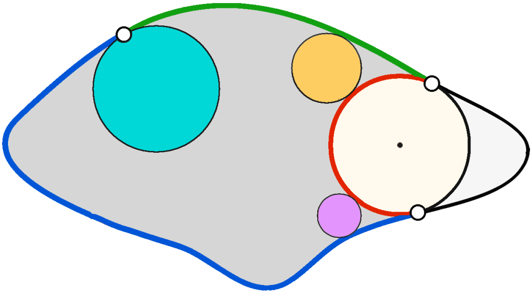

Construction of $G'_z(\alpha', \beta', \gamma')$ if D_z touches several prime ends of G.

Geometric surgery: construction of new trilateral

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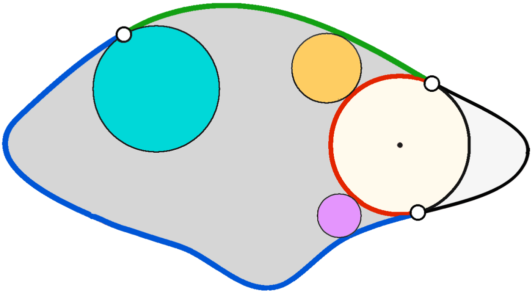

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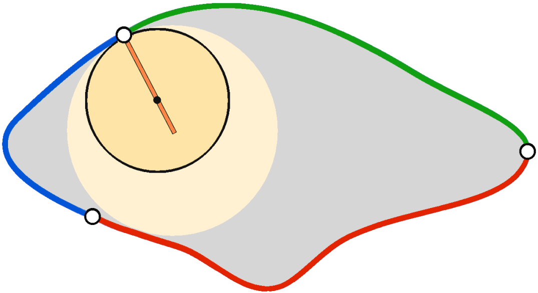

There is an exceptional set E for which the construction does not work.

Sperner coloring

Fill $G'_z(\alpha', \beta', \gamma')$ by packing $\mathscr{P}'_z$, let $\mathscr{P}_z = \mathscr{P}'_z \cup \{D_z\}$. Depending on the arcs touched by leading disks $D_1(z) = D_z, D_2(z), D_3(z)$ of $\mathscr{P}_z$ we build

$$\mathscr{R} := \{z \in G \setminus E : D_3(z) \frown (\alpha \cup \beta),\ D_1(z) \frown \alpha\}$$

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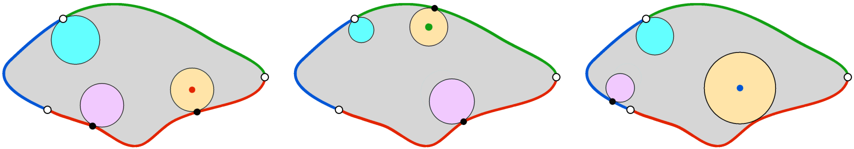

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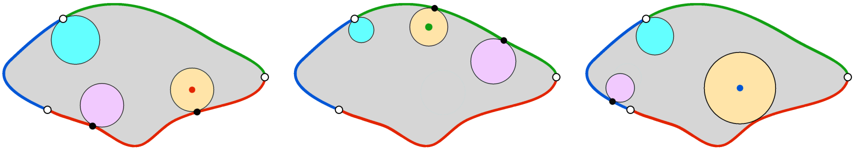

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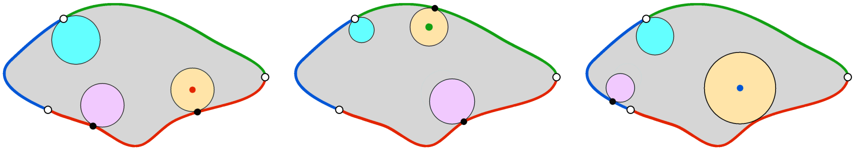

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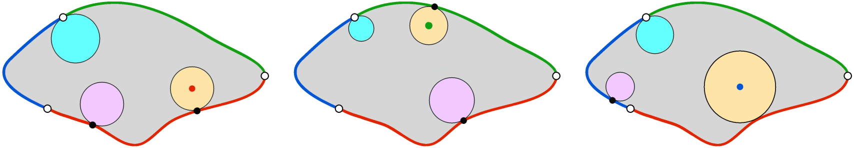
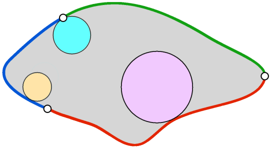


Missing case $D_3(z) \frown (\alpha \cup \beta),\ D_1(z) \frown \gamma$ is impossible.



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Fill $G'_z(\alpha', \beta', \gamma')$ by packing $\mathscr{P}'_z$, let $\mathscr{P}_z = \mathscr{P}'_z \cup \{D_z\}$. Depending on the arcs touched by leading disks $D_1(z) = D_z, D_2(z), D_3(z)$ of $\mathscr{P}_z$ we build

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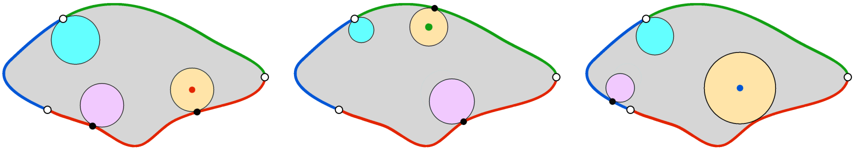

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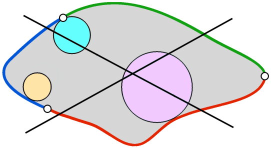

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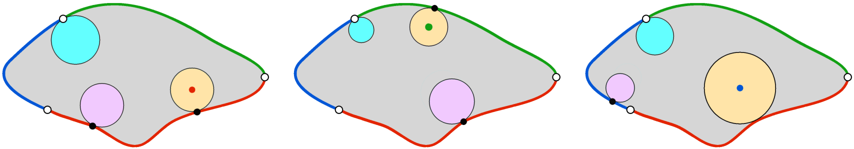

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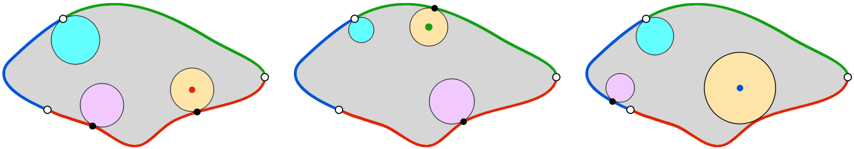

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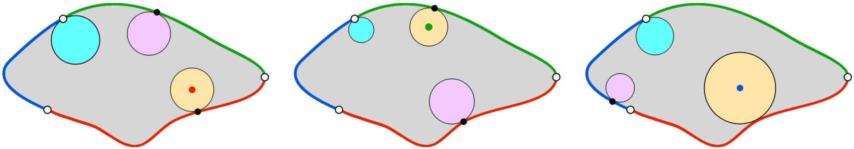

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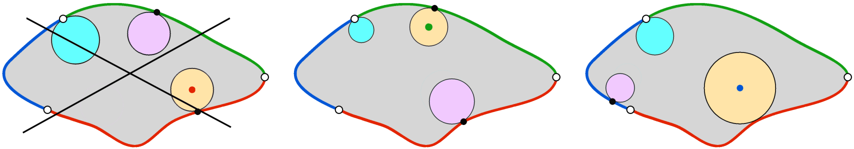

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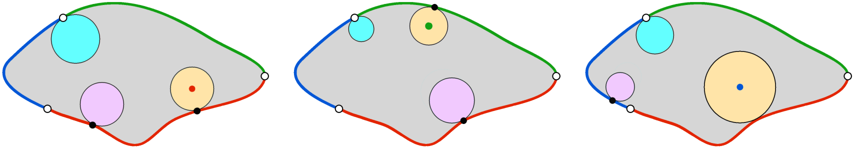

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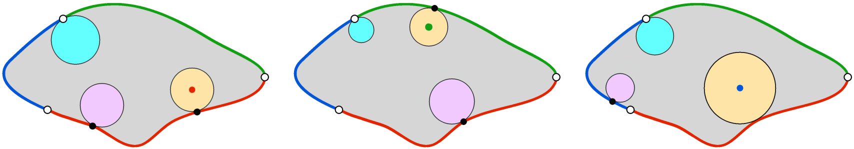

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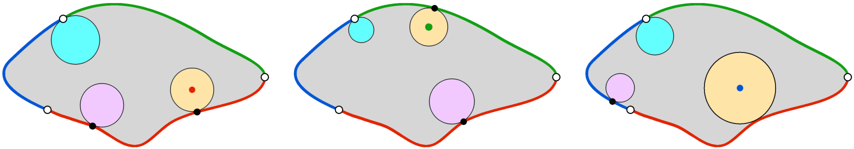

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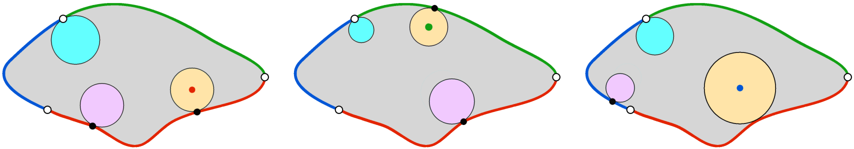

Let $E \subset \mathscr{B}$ and $\mathscr{R} \succ \mathscr{G} \succ \mathscr{B} \Longrightarrow$ Sperner coloring, $\mathscr{R} \cap \mathscr{G} \cap \mathscr{B} \neq \emptyset$

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Regularity, uniqueness and continuity

Regularity ($G(\alpha, \beta, \gamma)$ not spiky, T boundary irreducible):
Assume $\mathscr{P}$ contains dot
$\Rightarrow$ $\mathscr{P}$ contains dot associated with boundary vertex
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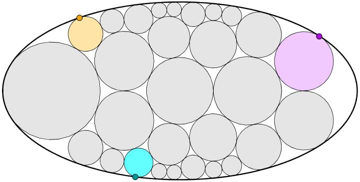
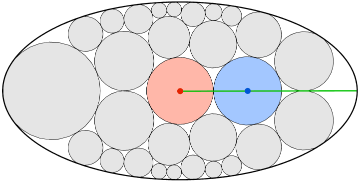
Continuity (of $\mathscr{P}_s$ filling tame trilaterals depending on parameter s):
tedious geometric and combinatorial considerations (many lemmas)
in particular $z \mapsto \mathscr{P}_z$ is continuous (not obvious, even surprising)

Modification of the Setting

Uniqueness and normalization

Why have we chosen Carathéodory's three point normalization at the boundary, and not the standard normalization on the right-hand side?





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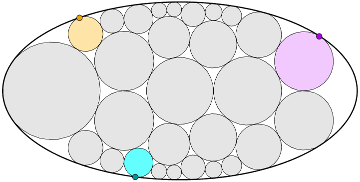

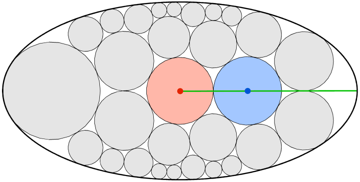

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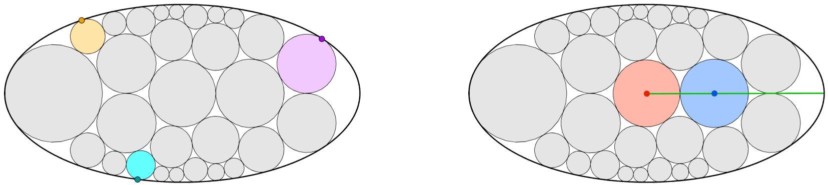

Technical reason: in the inductive step, domain and packing can be more easily modified at their boundaries.

Textual reason: for the standard normalization, uniqueness cannot be guaranteed even for smooth domains and admissible complexes !

Heuristic reasoning

For conformal mapping $\mathbb{D} \to G$ both conditions $f(z_0) = 0$, $f'(z_0) > 0$ are imposed at the same point. The second condition eliminates "rotations".

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In the continuous case this is similar to the requirement to map a point on some contour line $C_r := \{z : f(z) = r\}$ to the positive real axis.

If r is sufficiently small, C_r is starlike (even convex), there is only one such point and the solution is unique. If r is too large, this need not be so; we may get several solutions.

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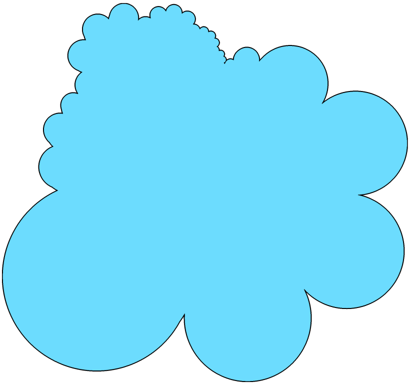
The problem in the construction of a concrete example is to find a packing where the circles involved in the normalization are sufficiently large (to keep their centers apart) and sufficiently close to the boundary (to preserve the effect of non-convexity).

Numerically computed counterexample

For a concrete counterexample, the domain must be highly non-convex.



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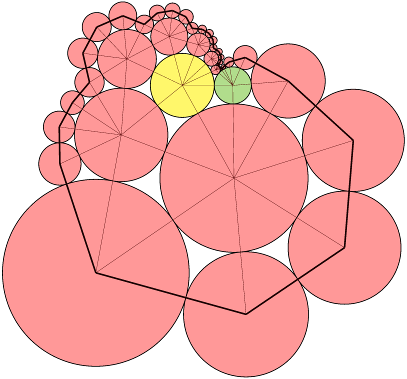

The first packing

It is constructed as the union of the carrier of two packings.

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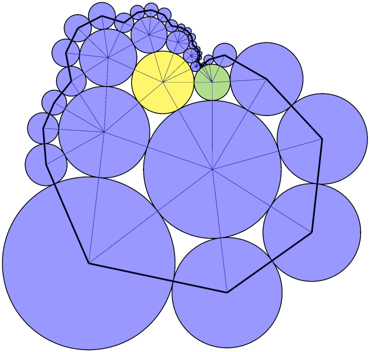

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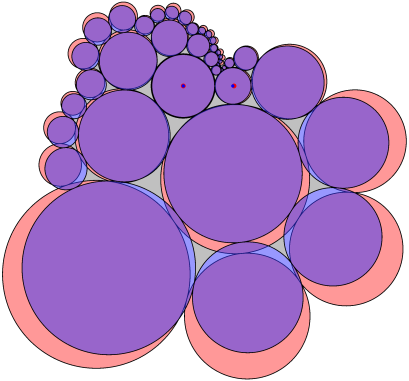

Overlay of both packings

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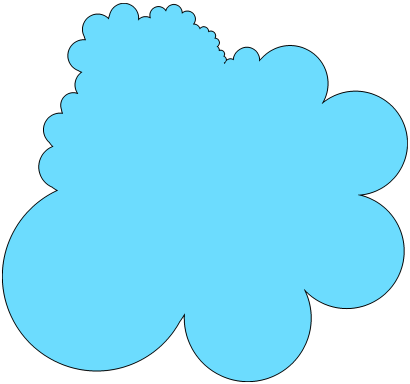

Union of both carriers

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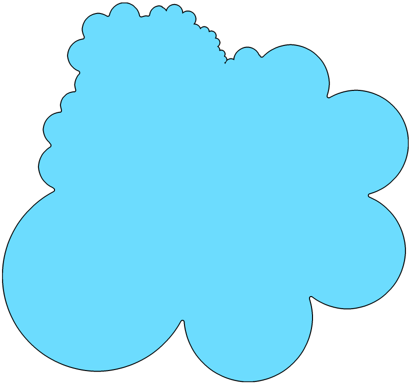

It is constructed as the union of the carrier of two packings. Rounding the corners makes the domain smooth.

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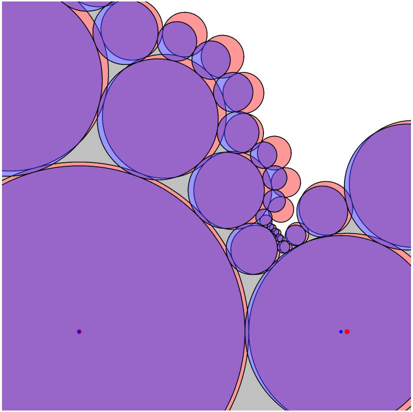

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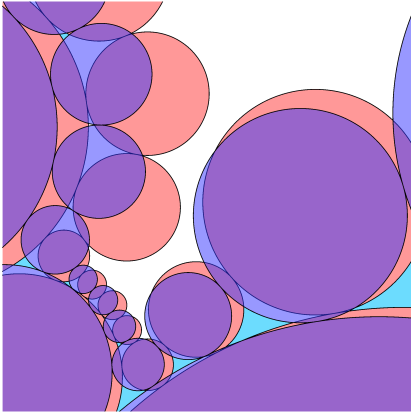

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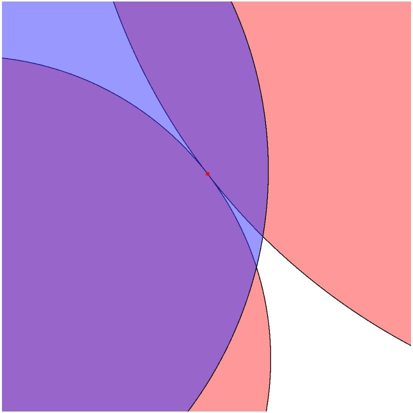

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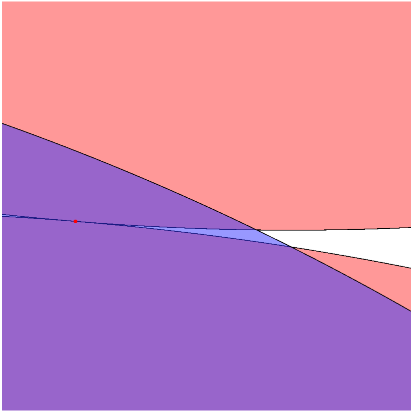
Verification of numerical results

Though we have no strict proof, the numerical results are verified carefully:



closeups of the relevant boundary parts

- Error in contact conditions less than 10^{-12}

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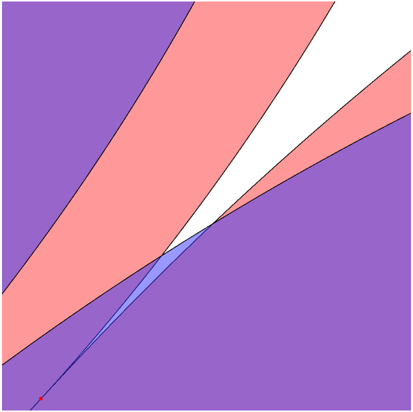

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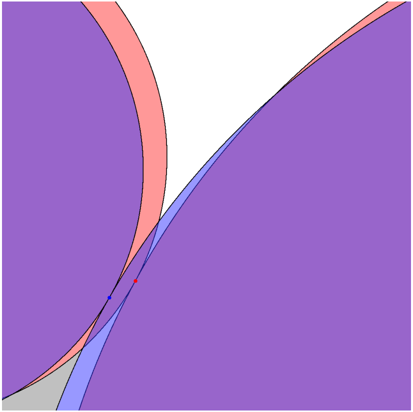

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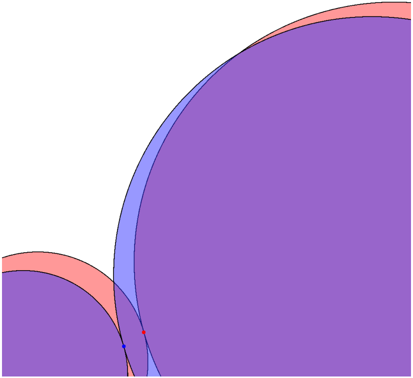

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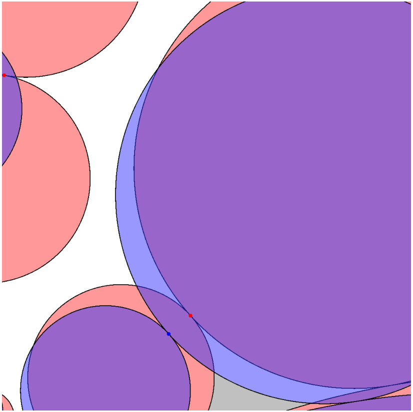

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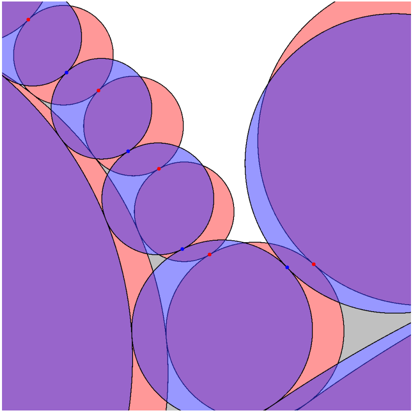

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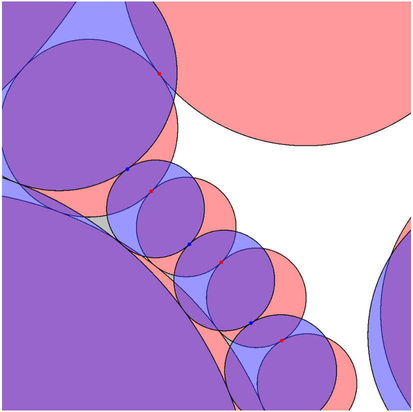

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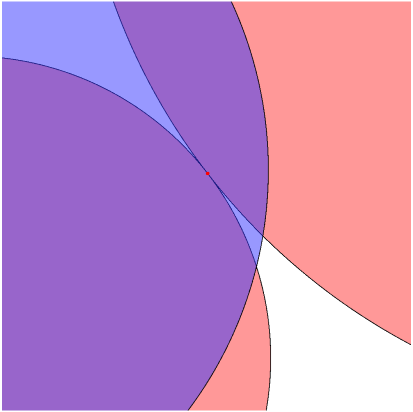

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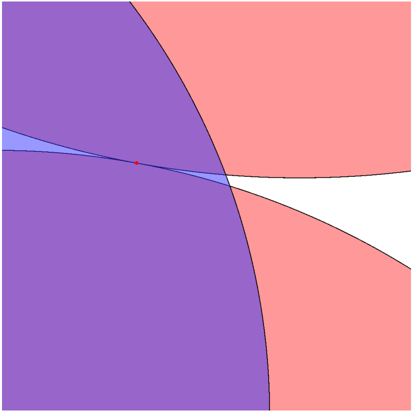

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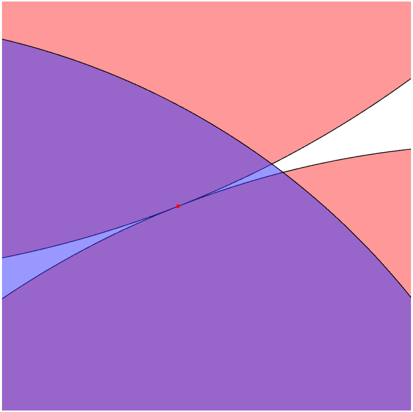

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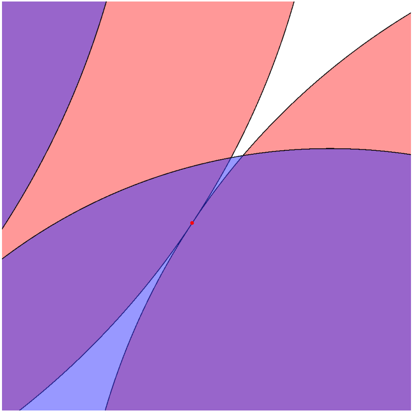

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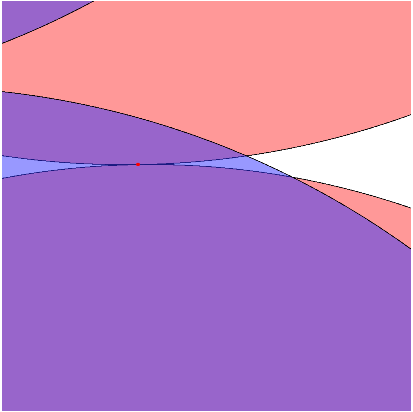

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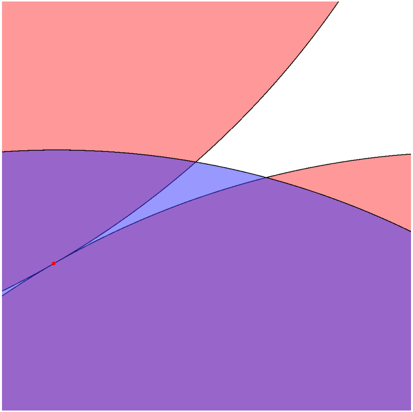

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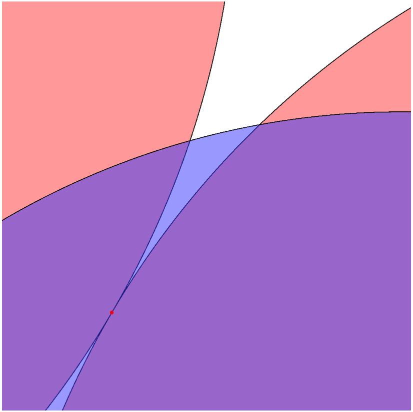

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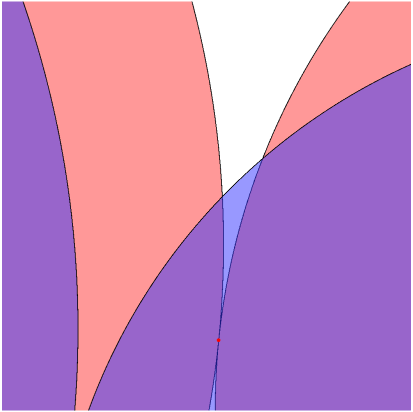

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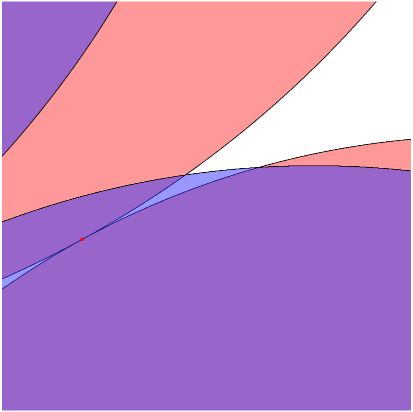

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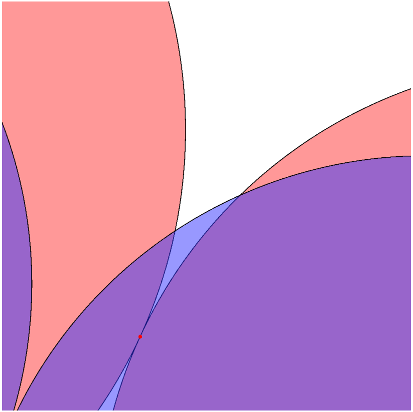

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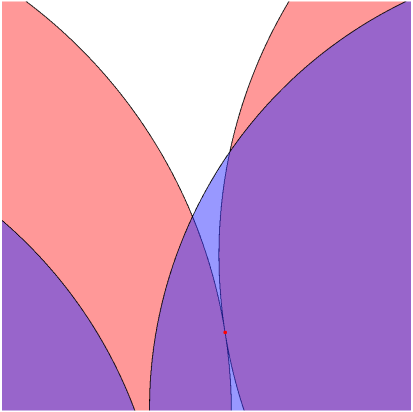

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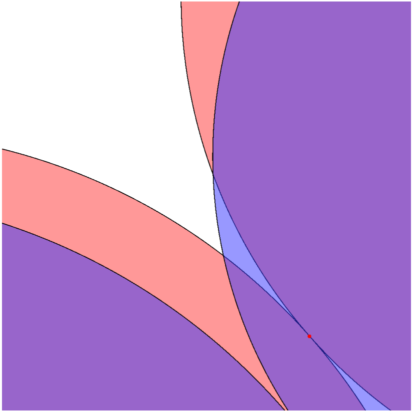

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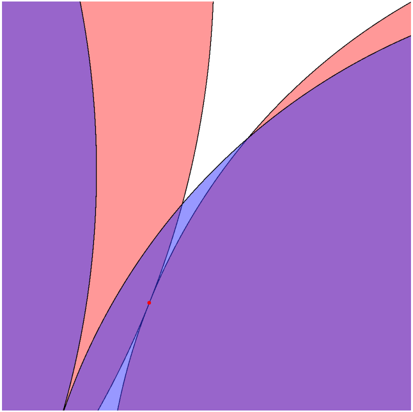

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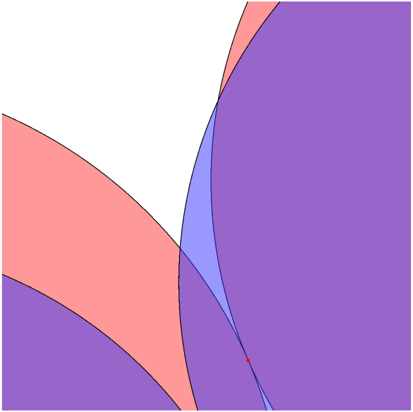

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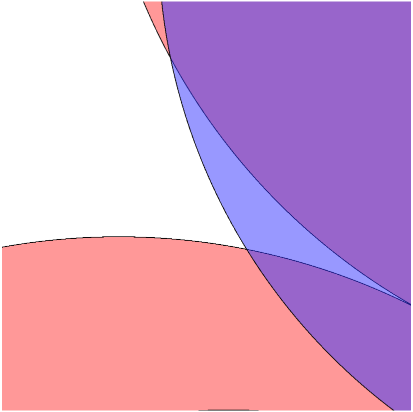

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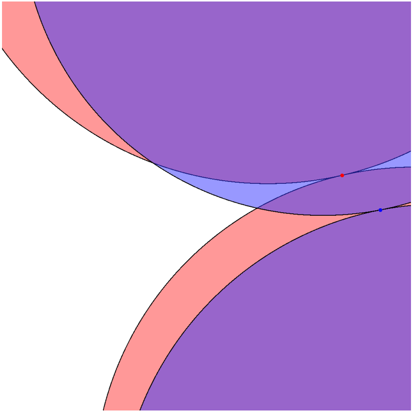

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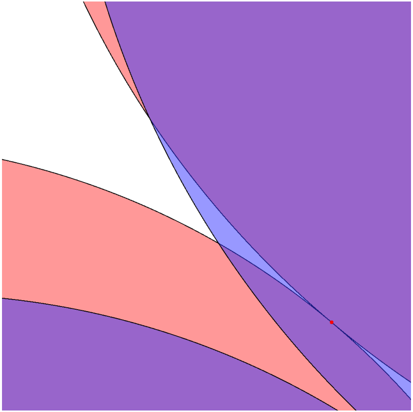

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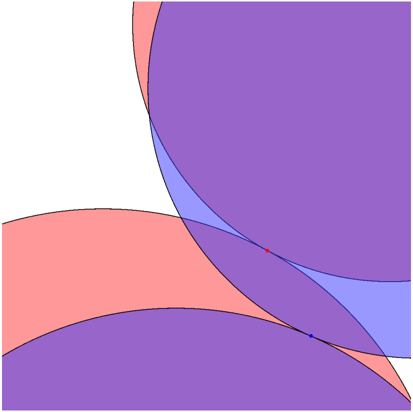

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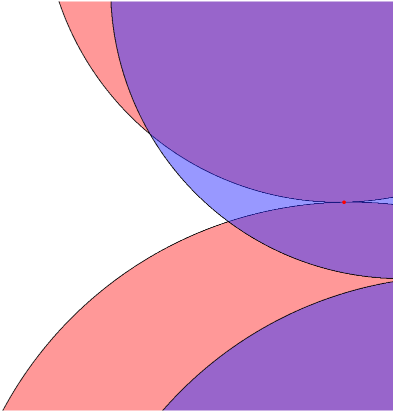

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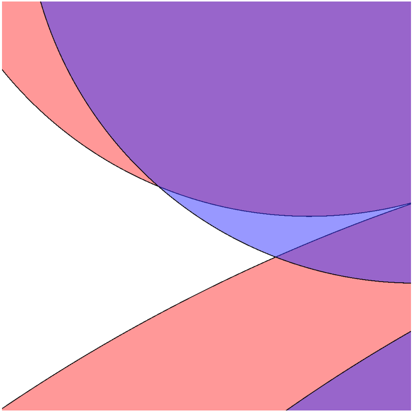

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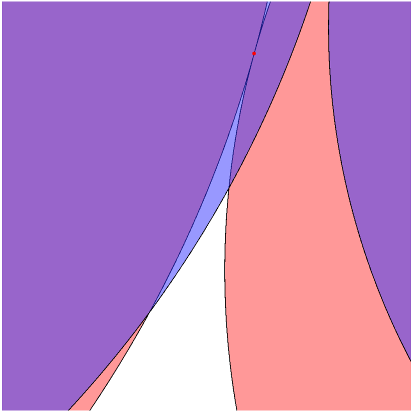

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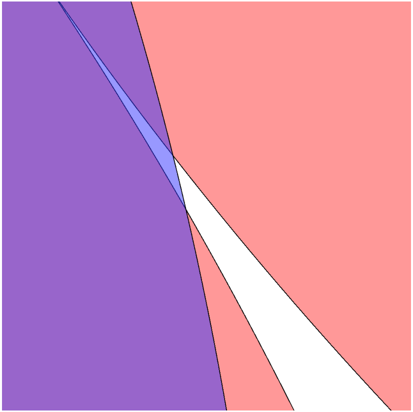

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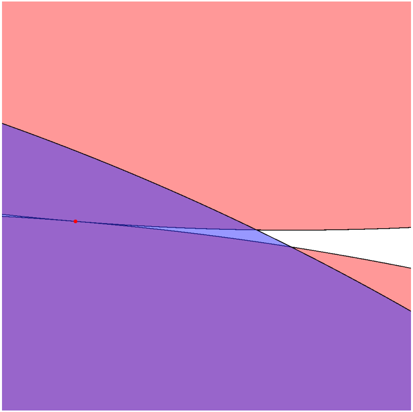

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A natural setting ?

Is there a "natural setting" for the normalizing side conditions?

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Circle centers are not invariant with respect to the chosen geometry (euclidian or hyperbolic) and should better be avoided.

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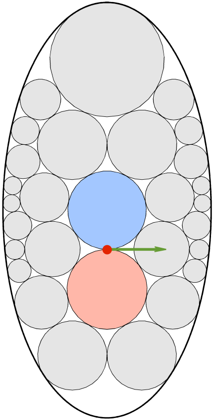

We better use the contact points between neighboring circles, and the direction of their common tangent.

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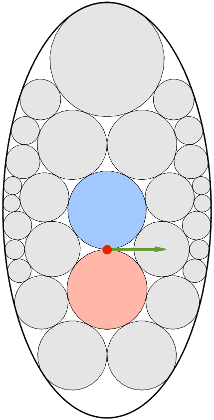

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Conjecture.

Let K be an admissible complex with two neighboring vertices u and v, and let G be a bounded simply connected domain with $0 \in G$. Then there exists a unique (non-degenerate) circle packing associated with K that fills G, normalized such that 0 is the contact point of D_u and D_v and their common (oriented) tangent is the positive real line.

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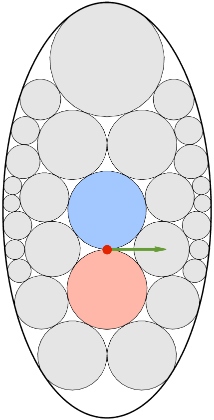

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It is my hope that someone proves it !

Domain-Filling Circle Agglomerations: References

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- Stephenson, K.: Introduction to Circle Packing. Cambridge Univ. Press, Cambridge (2005)
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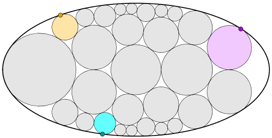
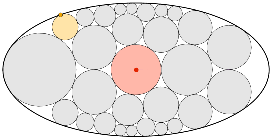
Uniqueness of generalized circle agglomerations

Existence of a domain-filling generalized circle agglomeration is guaranteed in all cases studied. In the other cases the problem is not well posed (results from David Krieg's thesis).

Normalization	**Generalized Circle Agglomeration**	**Generalized Circle Packing**
Alpha-Beta-Gamma	$G(\alpha, \beta, \gamma)$ tame	$G(\alpha, \beta, \gamma)$ tame
Alpha-Gamma	not studied	vertex C is regular
Alpha-Beta	not studied	in general not unique even for smooth domains



Alpha-Beta-Gamma



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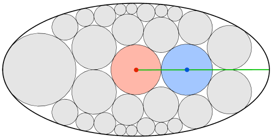
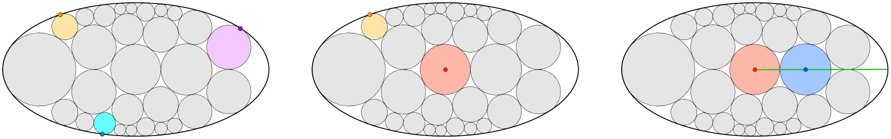


Alpha-Beta

Existence of non-degenerate circle packings (K admissible)

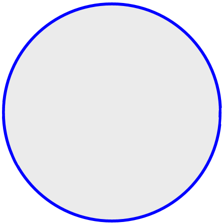
Existence of non-degenerate circle packings is not always guaranteed.

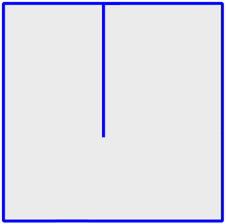
		G has no inward spikes, vertices are untouchable	G has no inward spikes	G not spiky	G general
Alpha-Beta-Gamma	degree $= 3$	yes	yes	yes	no
	degree < 3	yes	no	no	no
Alpha-Gamma	$G(A, C)$ not dubious	yes	yes	yes	yes
Alpha-Beta	$\partial D_A \cap \Gamma$ in G	yes	yes	yes	yes



Some words about prime ends

Conformal mapping $f : \mathbb{D} \to G$ extends to homeomorphism $f^* : \overline{\mathbb{D}} \to G^*$





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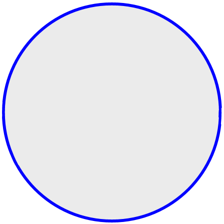

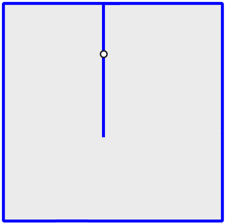

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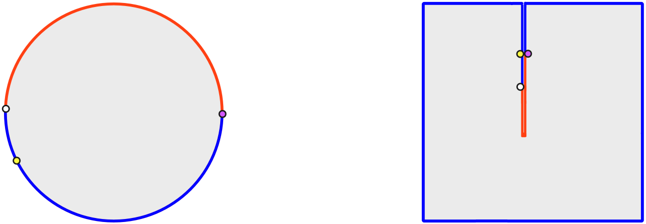

G^* compactification of G in "intrinsic geometry" of G
Elements of "intrinsic boundary" ∂G^* are prime ends of G

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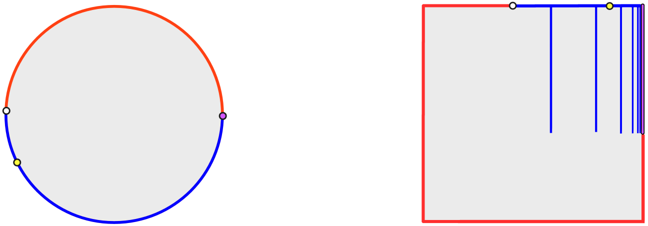

G^* compactification of G in “intrinsic geometry” of G
Elements of “intrinsic boundary” ∂G^* are prime ends of G

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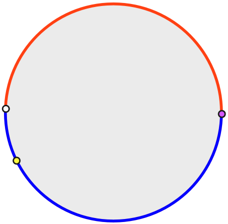

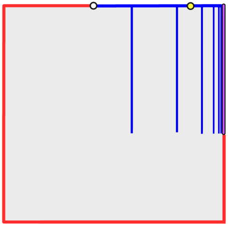

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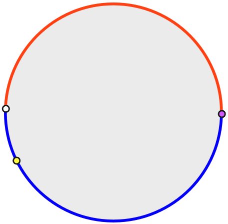

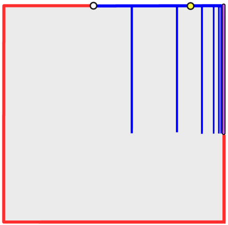

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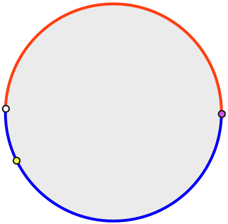

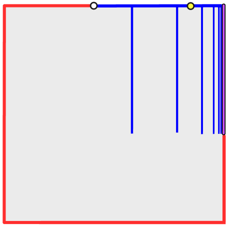

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