Massive C^*-algebras, Winter 2021, I. Farah, Lecture 12

Today we start the proof that i~~n~~f ZFC has a model, then it has a model in which all automorphisms of the Calkin algebra are inner. More precisely, we'll prove that this follows from a certain consequence, $\mathrm{OCA_T}$, of forcing axioms.

Massive C*-algebras, Winter 2021, I. Farah, Lecture 12

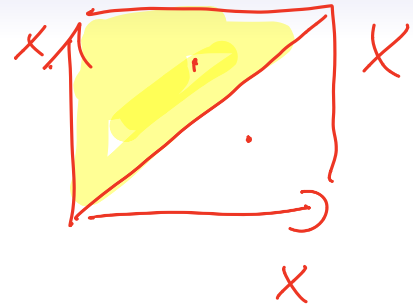
Today we start the proof that in ZFC has a model, then it has a model in which all automorphisms of the Calkin algebra are inner. More precisely, we'll prove that this follows from a certain consequence, $\mathrm{OCA_T}$, of forcing axioms.

separable

Here is the roadmap of the proof that $\Phi \in \mathrm{Aut}(\mathcal{Q}(H))$ is inner (note that the unit ball $\mathcal{B}(H)_1$ is a Polish space with respect to the strong operator topology, and that $\mathcal{F}[\mathrm{E}]$ is a SOT-closed subspace):

1. $\mathrm{OCA_T}$ implies that for every $\mathrm{E} \in \mathrm{Part}_{\mathbb{N}}$ some SOT-continuous function $f \colon \mathcal{F}[\mathrm{E}]_1 \to \mathcal{B}(H)_1$ lifts Φ.
2. The function f as in (1) can be implemented as conjugation by a unitary u_{E}. (Ulam-stability of *-homomorphisms.)
3. $\mathrm{OCA_T}$ implies that the 'coherent family' $(\mathrm{E}, u_{\mathrm{E}})$ obtained in (2) can be uniformized: a single unitary v implements the restriction of Φ to $\mathcal{F}[\mathrm{E}]_1$ for all E.

Partitions of pairs



Def 8.6.1 *For a set* X *let* $[X]^2 := \{s \subseteq X : |s| = 2\}$.

Partitions of pairs

Def 8.6.1 *For a set* X *let* $[X]^2 := \{s \subseteq X : |s| = 2\}$.

The following two theorems are included only as an illustration; they will be neither used nor proved here:

Partitions of pairs

Def 8.6.1 *For a set* X *let* $[X]^2 := \{s \subseteq X : |s| = 2\}$.

The following two theorems are included only as an illustration; they will be neither used nor proved here:

Thm (Ramsey) *For every partition* $[\mathbb{N}]^2 = L_0 \sqcup L_1$ *there is an infinite* $Y \subseteq \mathbb{N}$ *such that* $[Y]^2 \subseteq L_0$ *or* $[Y]^2 \subseteq L_1$.

$Y_0 \to (Y_0)^2_2$

Partitions of pairs

Def 8.6.1 *For a set* X *let* $[X]^2 := \{s \subseteq X : |s| = 2\}$.

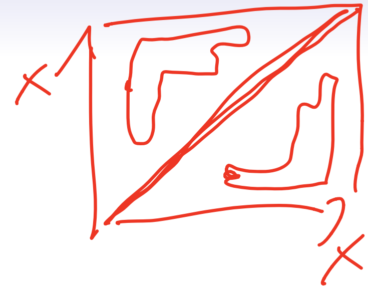
The following two theorems are included only as an illustration; they will be neither used nor proved here:

Thm (Ramsey) *For every partition* $[\mathbb{N}]^2 = L_0 \sqcup L_1$ *there is an infinite* $Y \subseteq \mathbb{N}$ *such that* $[Y]^2 \subseteq L_0$ *or* $[Y]^2 \subseteq L_1$.

Thm (Sierpiński) *There is a partition* $[\mathbb{R}]^2 = L_0 \sqcup L_1$ *such that for every uncountable* $Y \subseteq \mathbb{R}$ *we have* $\underline{[Y]^2 \not\subseteq L_0}$ *and* $\underline{[Y]^2 \not\subseteq L_1}$,

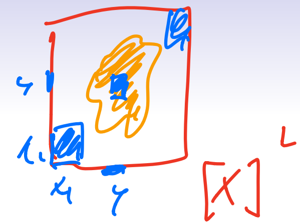
$2^{\aleph_0} \not\to (\aleph_1)^2_2$

Some terminology



1. If X is a topological space, then $L \subseteq [X]^2$ is called *open* if
$$\{(x, y) | \{x, y\} \in L\}$$
is an open subset of X^2.
2. A partition $[X]^2 = L_0 \cup L_1$ is *open* if L_0 is open.
3. A set Y such that $[Y]^2 \subseteq L$ is called *L-homogeneous*.

OCA$_T$



OCA$_T$ Whenever X is a separable metrizable space and $[X]^2 = L_0 \sqcup L_1$ is an open colouring, one of the following alternatives applies.

1. There exists an uncountable L_0-homogeneous $Y \subseteq X$.
2. There are L_1-homogeneous sets X_n, for $n \in \mathbb{N}$, such that $\bigcup_n X_n = X$.

Remark: In (2), each X_n can be replaced by its closure, hence we may assume all X_n are closed.

$[X_n]^2 \subseteq L_1$

$\{x, x'\} \in [\overline{X_n}]^2 \Rightarrow \{x, x'\} \in L_1$

$\mathrm{OCA_T}$

$\mathrm{OCA_T}$ Whenever X is a separable metrizable space and $[X]^2 = L_0 \sqcup L_1$ is an open colouring, one of the following alternatives applies.

1. There exists an uncountable L_0-homogeneous $Y \subseteq X$.
2. There are L_1-homogeneous sets X_n, for $n \in \mathbb{N}$, such that $\bigcup_n X_n = X$.

Remark: In (2), each X_n can be replaced by its closure, hence we may assume all X_n are closed.

A proof of the following requires Cohen's method of forcing, and it will be omitted.

PFA $\Rightarrow$ $\mathrm{OCA_T}$

Thm *The axiom* $\mathrm{OCA_T}$ *is relatively consistent with ZFC.*

ZFC + $\mathrm{OCA_T}$ $\Rightarrow$ ______

$\mathrm{OCA_T}$

$\mathrm{OCA_T}$ Whenever X is a separable metrizable space and $[\mathrm{X}]^2 = L_0 \sqcup L_1$ is an open colouring, one of the following alternatives applies.

1. There exists an uncountable L_0-homogeneous $\mathrm{Y} \subseteq \mathrm{X}$.
2. There are L_1-homogeneous sets X_n, for $n \in \mathbb{N}$, such that $\bigcup_n \mathrm{X}_n = \mathrm{X}$.

Remark: In (2), each X_n can be replaced by its closure, hence we may assume all X_n are closed.
A proof of the following requires Cohen's method of forcing, and it will be omitted.

Thm *The axiom* $\mathrm{OCA_T}$ *is relatively consistent with ZFC.*

We'll prove two consequences of $\mathrm{OCA_T}$ as a warmup, but first, let's see examples of relevant separable metric topologies.

Relevant separable metric spaces

(0) $\mathbb{R}$, any separable C^*-algebra/Hilbert space/Banach space.

Relevant separable metric spaces

(0) $\mathbb{R}$, any separable C^*-algebra/Hilbert space/Banach space.
(1) SOT on $\mathcal{B}(H)_1$: Fix dense ξ_n, for $n \in \mathbb{N}$, in H_1, and let

$$d(a, b) := \sum_n 2^{-n} \|(a - b)\xi_n\|_2.$$

Relevant separable metric spaces

(0) $\mathbb{R}$, any separable C^*-algebra/Hilbert space/Banach space.

(1) SOT on $\mathcal{B}(H)_1$: Fix dense ξ_n, for $n \in \mathbb{N}$, in H_1, and let

$$d(a,b) := \sum_n 2^{-n} \|(a-b)\xi_n\|_2.$$

span ξ_j, $j \leq n$

(2) WOT on $\mathcal{B}(H)_1$: Fix dense ξ_n, for $n \in \mathbb{N}$, in H_1, and let

ap

$$d(a,b) := \sum_{m,n} 2^{-m-n} |((a-b)\xi_m|\xi_n)|$$

$M_n(\mathbb{C})$

ξ_k a a n

ε

Relevant separable metric spaces

(0) $\mathbb{R}$, any separable C^*-algebra/Hilbert space/Banach space.
(1) SOT on $\mathcal{B}(H)_1$: Fix dense ξ_n, for $n \in \mathbb{N}$, in H_1, and let

$$d(a, b) := \sum_n 2^{-n} \|(a - b)\xi_n\|_2.$$

(2) WOT on $\mathcal{B}(H)_1$: Fix dense ξ_n, for $n \in \mathbb{N}$, in H_1, and let

$$d(a, b) := \sum_{m,n} 2^{-m-n} |((a - b)\xi_m | \xi_n)|$$

(3) $\mathbb{T}^{\mathbb{N}}$: $d(u, v) := \max_n \frac{1}{n+1} |u(n) - v(n)|$.
(A similar metric is used on the product of any sequence of bounded metric spaces, or any sequence of metric spaces.)

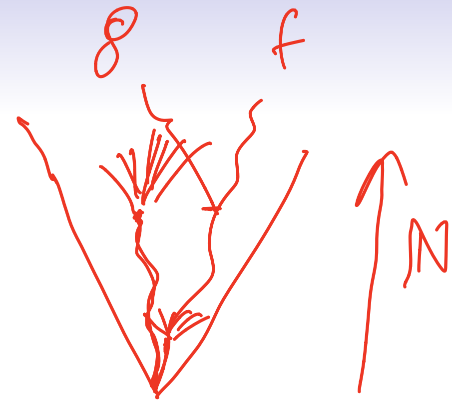
(4) The Baire space, $\mathbb{N}^{\mathbb{N}}$. Let

$$\Delta(f,g) := \inf\{j \mid f(j) \neq g(j)\}$$

and let

$$d(f,g) := 1/(\Delta(f,g)+1).$$

$(d(f,f)=0,\ \Delta(f,f)=\infty.)$



$d(f,h) \leq \max(d(f,g), d(g,h))$

(4) The Baire space, $\mathbb{N}^{\mathbb{N}}$. Let

$$\Delta(f,g) := \inf\{j | f(j) \neq g(j)\}$$

and let

$$d(f,g) := 1/(\Delta(f,g)+1).$$

$(d(f,f) = 0,\ \Delta(f,f) = \infty.)$

(5) $\mathrm{Part}_{\mathbb{N}}$: Identify $\mathrm{Part}_{\mathbb{N}}$ with a subspace of $\mathbb{N}^{\mathbb{N}}$, by

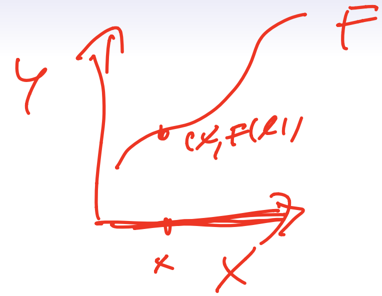
$f(j) < f(j+1)$

$\forall j$

$$\mathrm{Part}_{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}\colon\ \mathsf{E} \mapsto f_{\mathsf{E}},$$

where for $\mathsf{E} = \langle E_j | j \in \mathbb{N}\rangle$ we let $f_{\mathsf{E}}(j) := \min E_j$, for $j \in \mathbb{N}$.

$E_j = [f_E(j), f_E(j+1))$



(6) If X and Y are separable metric and $F\colon X \to Y$, refine the topology on X by identifying $x \in X$ with $(x, F(x)) \in X \times Y$.

$(\mathbb{R}, <)$

If X has a natural linear ordering $<$, then $\{x, y\}_<$ stands for $\{x, y\} \in [X]^2$, and it is understood that $x < y$.

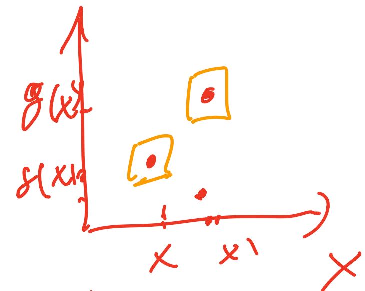
Prop 8.6.3 *Assume* $\mathrm{OCA_T}$. *If* $X \subseteq \mathbb{R}$ *is uncountable and* $g\colon X \to \mathbb{R}$, *then there exists an uncountable* $Y \subseteq X$ *such that the restriction of* g *to* Y *is continuous.*

If X has a natural linear ordering $<$, then $\{x, y\}_<$ stands for $\{x, y\} \in [X]^2$, and it is understood that $x < y$.

Prop 8.6.3 *Assume* $\mathrm{OCA_T}$. *If $X \subseteq \mathbb{R}$ is uncountable and $g : X \to \mathbb{R}$, then there exists an uncountable $Y \subseteq X$ such that the restriction of g to Y is continuous.*

Proof: As in (6), refine the topology on X by identifying $x \in X$ with $(x, g(x)) \in X \times Y$. Let

$$\{x, x'\}_< \in L_0 \text{ if } g(x) < g(x')$$



hence $\{x, x'\}_< \in L_1$ if $g(x) \geq g(x')$.

Case 1 $\exists Y \subseteq X$, $|Y| \geq \aleph_1$ $[Y]^2 \subseteq L_0$.

$g \restriction Y$ is $\nearrow$

has $\leq \aleph_0$ discontinuities,

so g

and $g \restriction (Y \setminus Z)$ is cts for some

z, $|z| \le \aleph_0$

Case 2: $X = \bigcup_n X_n$, $[X_n]^2 \subseteq L_1$, $\forall n$.

Then $\exists n$, $|X_n| \ge \aleph_1$, $g \restriction X_n$ is non-increasing, hence it has $\le \aleph_0$ discontinuities, and $g \restriction [X_n \setminus z]$ is cts for some z, $|z| \le \aleph_0$.

If X has a natural linear ordering $<$, then $\{x, y\}_<$ stands for $\{x, y\} \in [X]^2$, and it is understood that $x < y$.

Prop 8.6.3 *Assume* $\mathrm{OCA_T}$. *If $X \subseteq \mathbb{R}$ is uncountable and $g\colon X \to \mathbb{R}$, then there exists an uncountable $Y \subseteq X$ such that the restriction of g to Y is continuous.*

Proof: As in (6), refine the topology on X by identifying $x \in X$ with $(x, g(x)) \in X \times Y$. Let

$$\{x, x'\}_< \in L_0 \text{ if } g(x) < g(x')$$

hence $\{x, x'\}_< \in L_1$ if $g(x) \geq g(x')$.

Exercise. Prove that CH implies there is $g\colon \mathbb{R} \to \mathbb{R}$ such that the restriction of g to Y is discontinuous for any uncountable Y.

Coro 8.6.4 $\mathrm{OCA_T}$ *and the Continuum Hypothesis are incompatible.*

(Y, g)

G_δ

If X has a natural linear ordering $<$, then $\{x, y\}_<$ stands for $\{x, y\} \in [X]^2$, and it is understood that $x < y$.

Prop 8.6.3 *Assume* $\mathrm{OCA_T}$. *If $X \subseteq \mathbb{R}$ is uncountable and $g\colon X \to \mathbb{R}$, then there exists an uncountable $Y \subseteq X$ such that the restriction of g to Y is continuous.*

Proof: As in (6), refine the topology on X by identifying $x \in X$ with $(x, g(x)) \in X \times Y$. Let

$$\{x, x'\}_< \in L_0 \text{ if } g(x) < g(x')$$

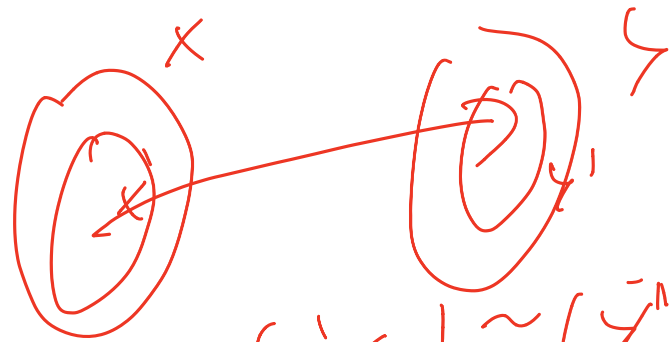
hence $\{x, x'\}_< \in L_1$ if $g(x) \geq g(x')$.

Exercise. Prove that CH implies there is $g\colon \mathbb{R} \to \mathbb{R}$ such that the restriction of g to Y is discontinuous for any uncountable Y.

Coro 8.6.4 $\mathrm{OCA_T}$ *and the Continuum Hypothesis are incompatible.*

(Remark: The exercise has little to do with the CH. To see this, prove (in ZFC) that there is $g\colon \mathbb{R} \to \mathbb{R}$ such that the restriction of g to Y is discontinuous, for any Y of cardinality $2^{\aleph_0}$.)

Prop 8.6.5 *Assume* $\mathrm{OCA_T}$. *If* X *and* Y *are uncountable subsets of* $\mathbb{R}$, *then there exists an uncountable* $X' \subseteq X$ *and an increasing* $f : X' \to Y$.



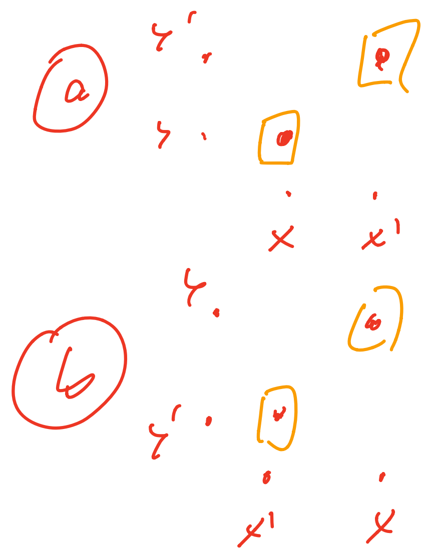
$(X', <) \cong (Y', <)$

Prop 8.6.5 *Assume* $\mathrm{OCA_T}$. *If* X *and* Y *are uncountable subsets of* $\mathbb{R}$, *then there exists an uncountable* $X' \subseteq X$ *and an increasing* $f : X' \to Y$.

$[X \times Y]^2$

Proof: Let $\{(x, y), (x', y')\} \in L_0$ if (a) $x < x'$ and $y < y'$ or (b) $x > x'$ and $y > y'$.
From an uncountable L_0-homogeneous $Z \subseteq X \times Y$ we can define f as required.

$Z = \mathrm{graph}(f)$



Prop 8.6.5 *Assume* $\mathrm{OCA_T}$. *If* X *and* Y *are uncountable subsets of* $\mathbb{R}$, *then there exists an uncountable* $X' \subseteq X$ *and an increasing* $f : X' \to Y$. $\{(x,y),(x',y')\} \in L_1$ if $(x \geq x'$ or $y \geq y')$ and $(x \leq x'$ or $y \leq y')$

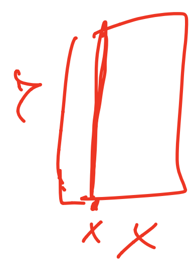
Proof: Let $\{(x,y),(x',y')\} \in L_0$ if (a) $x < x'$ and $y < y'$ or (b) $x > x'$ and $y > y'$.

From an uncountable L_0-homogeneous $Z \subseteq X \times Y$ we can define f as required.

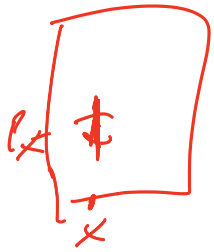
Claim. $X \times Y$ *cannot be covered by countably many* L_1*-homogeneous sets.*

Assume $X \times Y = \bigcup_n Z_n$, $[Z_n]^2 \subseteq L_1$, $\forall n$.

Fix $x \in X$. $\{x\} \times Y \subseteq \bigcup_n Z_n$



Fix $n(x)$ such that $(\{x\} \times Y) \cap Z_{n(x)}$ is unctbl. Fix $p_x \in \mathbb{Q}$ such that



$(\exists y, y')$ $(x, y) \in Z_{u(x)}$, $(x, y') \in Z_{u(x)}$,

$y < p_x < y'$

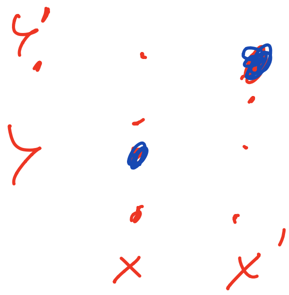
Fix u, p so that

$X_1 = \{x \mid u(x) = u,\ p_x = p\}$ is unctble

$\{(x,y),(x',y')\} \in L$, if $(\underline{x \geq x'}$ or $y \geq y')$ and $(x \leq x'$ or $y \leq y')$

Fix $x < x'$ in X_1.

$p_{x'} = p_x$



Fix $y' > p_x$ $\quad (x, y) \in Z_{u(x)}$

$y < p_x$ $\quad (x', y') \in Z_{\underline{u(x)}}$

so $\{(x,y),(x',y')\} \in L_0$

Prop 8.6.5 *Assume* $\mathrm{OCA_T}$. *If* X *and* Y *are uncountable subsets of* $\mathbb{R}$, *then there exists an uncountable* $X' \subseteq X$ *and an increasing* $f\colon X' \to Y$.

Proof: Let $\{(x,y),(x',y')\} \in L_0$ if (a) $x < x'$ and $y < y'$ or (b) $x > x'$ and $y > y'$.
From an uncountable L_0-homogeneous $Z \subseteq X \times Y$ we can define f as required.

Claim. $X \times Y$ *cannot be covered by countably many* L_1*-homogeneous sets.*

Exercise. Prove that CH implies there are uncountable subsets X and Y of $\mathbb{R}$ such that there are no uncountable X' $\subseteq X$ and an increasing $f\colon X' \to Y$ (or a decreasing $f\colon X' \to Y$).

(Again, this has little to do with the CH: drop CH and replace 'uncountable' with 'of cardinality $2^{\aleph_0}$'.)

$OCA_T \Rightarrow \mathfrak{b} = \aleph_2$

Prop $\approx$9.5.7 $\mathrm{OCA_T}$ *implies that every* $\mathcal{E} \subseteq \mathrm{Part}_{\mathbb{N}}$ *of cardinality* $\aleph_1$ *is* $\leq^*$*-bounded.*

The proof of this Proposition will require some preparations.

$E \leq^* F \Rightarrow \mathcal{J}[E] \subseteq \mathcal{J}[F] + K(H)$