Massive C*-algebras, Winter 2021, I. Farah, Lecture 11

From the last time:

Lemma *Suppose $A \leq C$, and u, v are in $\mathcal{U}(C)$. TFAE:*

1. $\operatorname{Ad} u(a) = \operatorname{Ad} v(a)$ *for all* $a \in A$ $\iff uau^* = vav^*, \forall a \in A$
2. $v^*u \in C \cap A'$. $\quad v^*ua = av^*u, \forall a \in A$
3. $u^*v \in C \cap A'$. $\quad u^*v = (v^*u)^*$

TFAE:

4. $\operatorname{Ad} u^*(a) = \operatorname{Ad} v^*(a)$ *for all* $a \in A$
5. $uv^* \in C \cap A'$.
6. $vu^* \in C \cap A'$. $\quad N(A) = \{w \mid wAw^* \subseteq A, \; w^*Aw \subseteq A\}$

If u and v are in the normalizer of A, then all of the above conditions are equivalent.

Recall from the last class: $\mathcal{U}(\ell_\infty) \subseteq \mathcal{U}(\mathcal{B}(H))$

Def 17.1.8 *Let* $\mathsf{F}_\mathsf{E} := \{x \in \mathbb{T}^\mathbb{N} : \Delta_\mathsf{E}(x,1) = 0\}$, *and* $\mathsf{G}_\mathsf{E} := \mathbb{T}^\mathbb{N} / \mathsf{F}_\mathsf{E}$, *for* $\mathsf{E} \in \mathrm{Part}_\mathbb{N}$.

$\Delta_{\underset{\sim}{E}} = \lim_{n\to\infty} \sup \Delta_{E_n \cup E_{n+1}}$

Then F_E is a subgroup of $\mathbb{T}^\mathbb{N}$ and $\mathsf{E} \leq^* \mathsf{F}$ implies $\mathsf{F}_\mathsf{E} \supseteq \mathsf{F}_\mathsf{F}$ and therefore $\mathsf{G}_\mathsf{F} = \mathsf{G}_\mathsf{E} / (\mathsf{F}_\mathsf{F} / \mathsf{F}_\mathsf{E})$. Also,

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{F}_\mathsf{F} \arrow[r] \arrow[d, "\iota_{\mathsf{E}\,\mathsf{F}}"'] & \mathbb{T}^\mathbb{N} \arrow[r] \arrow[d, "\mathrm{id}"'] & \mathsf{G}_\mathsf{F} \arrow[r] \arrow[d, "\pi_{\mathsf{E}\,\mathsf{F}}"'] & 0 \\
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$\uparrow \leq^*$

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Lemma 17.1.9 *Suppose* $\mathsf{E} \in \mathsf{Part}_\mathbb{N}$ *and* u *and* v *belong to* $\mathbb{T}^\mathbb{N}$. *Then* $u \sim_\mathsf{E} v$ *if and only if* $uv^* \in \mathsf{F}_\mathsf{E}$.

$\mathrm{Ad}\, u \restriction \mathcal{F}[E] = \mathrm{Ad}\, v \restriction \mathcal{F}[E]$

$2^{\aleph_0}$

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Prop ≈17.1.11 *If* $\mathsf{E}(\alpha)$, *for* $\alpha < \aleph_1$, *is* $\leq^*$*-cofinal in* $\mathrm{Part}_\mathbb{N}$, *then the inverse limit* $\varprojlim_\alpha \mathsf{G}_{\mathsf{E}(\alpha)}$ *has cardinality* $2^{\aleph_1}$.

Thm (Coskey–F., 2014) *If* $\mathsf{E}(\alpha)$, *for* $\alpha < \kappa$, *is* $\leq^*$*-cofinal in* $\mathsf{Part}_{\mathbb{N}}$, *then there is an injective group homomorphism from* $\varprojlim_{\alpha} \mathsf{G}_{\mathsf{E}(\alpha)}$ *into* $\mathrm{Aut}(\mathcal{Q}(H))$.

$E(\alpha) \rightsquigarrow \underbrace{\mathcal{F}[E(\alpha)]}_{u_\alpha}$

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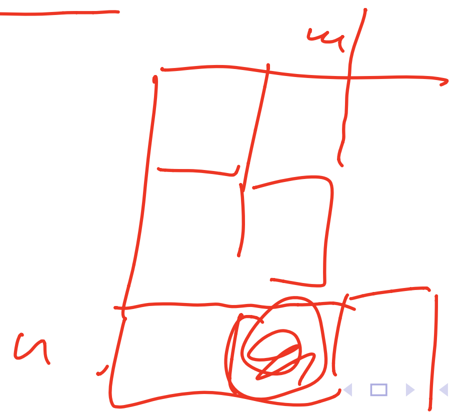
1. *A has an approximate unit e_m, $m \in \mathbb{N}$, consisting of projections and $f_n := e_n - e_{n-1}$ ($e_0 = 0$) satisfy $f_m A f_n \neq \{0\}$ for all m and n.*

$\sum f_n \to 1_{\mathcal{M}(A)}$



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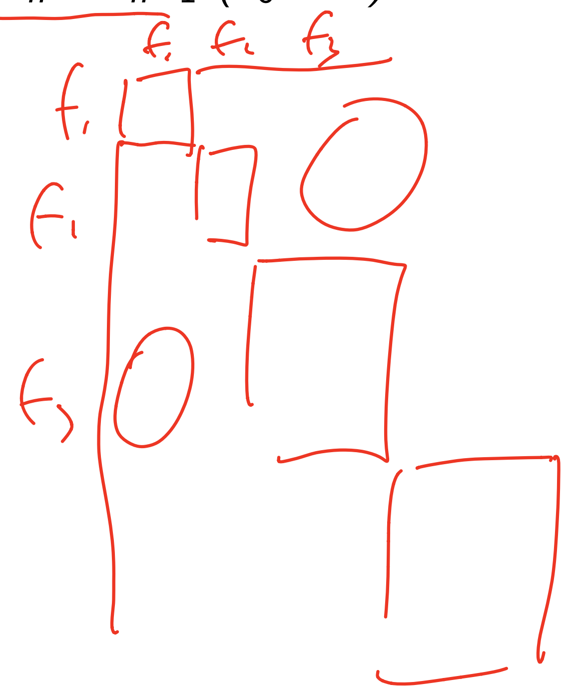
2. *A is stable, (i.e., $A \cong A \otimes \mathcal{K}$).*
3. *A is primitive (i.e., it has a faithful, nondegenerate, representation).*

(Idea for (2) and (3): A quasicentral approximate unit will satisfy the analog of the condition from (1).)

The other opposite and a curiosity

Prop *Suppose that A has an approximate unit e_n, $n \in \mathbb{N}$, consisting of projectons, such that with $f_n = e_n - e_{n-1}$ $(e_0 = 0)$ we have $f_m A f_n = \{0\}$ whenever $m \neq n$.*



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Then $A \cong \bigoplus_m f_m A f_m$, $\mathcal{M}(A) \cong \prod_m f_m A f_m$, hence $\mathcal{M}(A)/A$ is countably saturated and CH implies that it has $2^{\aleph_1}$ automorphisms.

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Exercise. There exists a σ-unital C^*-algebra with an approximate unit consisting of projections, but no such approximate unit of A can be chosen so that (with $f_n = e_n - e_{n-1}$) $f_m A f_n \neq \{0\}$ for all m and n and no approximate unit of A can be chosen so that $f_m A f_n = \{0\}$ whenever $m \neq n$.

Back to $\mathcal{Q}(H)$

The original BDF question is still open.

Question *Is it possible to find, in some model of ZFC, a K-theory-reversing automorphism Φ of $\mathcal{Q}(H)$?*

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Even the following is open.

Question *Is it possible to find, in some model of ZFC,* $\Phi \in \mathrm{Aut}(\mathcal{Q}(H))$ *such that* $\Phi \restriction A$ *is not implemented by a unitary for some separable* $A \leq \mathcal{Q}(H)$*?*

(By Woodin's theorem, this is essentially the same as trying to construct such Φ using CH.)

When does $\mathcal{M}(A)/A$ have many automorphisms, assuming CH? (the abelian case)

If $A = C_0(X)$, X locally compact metrizable, then $\mathcal{M}(A) \cong C(\beta X)$ and $\mathcal{M}(A)/A \cong C(\beta X \setminus X)$.

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(1) If X is an increasing union of clopen, compact subsets K_n then $A = \bigoplus_{\mathbb{N}} C(K_n \setminus K_{n-1})$ and $\mathcal{M}(A)/A$ is countably saturated. Thus CH implies $\mathcal{M}(A)/A$ has $2^{\aleph_1}$ automorphisms.

$\bigoplus_{\mathbb{N}} [0,1]$

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(2b) (F.–Shelah) The corona of $C_0(\mathbb{R})$ is countably saturated.

$C_0(\mathbb{R}^n)$

Let's consider the abelian case, $A = C_0(X)$.

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The simplest nontrivial case, $A = C_0(\mathbb{N})$:

C^*-algebra	topological space	Boolean algebra
c_0	$\mathbb{N}$	
ℓ_∞	$\beta\mathbb{N}$	$\mathcal{P}(\mathbb{N})$
ℓ_∞/c_0	$\beta\mathbb{N} \setminus \mathbb{N}$	$\mathcal{P}(\mathbb{N})/\mathrm{Fin}$

X

A topological space X is *homogeneous* if its autohomeomorphism group acts transitively on X.

Thm (W. Rudin, 1956) *CH implies the following:*

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$\mathcal{P}(\mathbb{N})/\mathrm{Fin}$

$x \subseteq^* y$

$\iff x \setminus y \in \mathrm{Fin}$

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Kunen (1972): $\beta\mathbb{N} \setminus \mathbb{N}$ is not homogeneous. (Notably, Kunen's construction was extended by Shelah, and this form the basis of non-structure theory for ultrapowers, including the result that CH implies there are $2^{\aleph_1}$ nonisomorphic ultrapowers of every separable, infinite-dimensional C^*-algebra.)

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Exercise. The group $\prod_{\mathbb{N}}(\mathbb{Z}/2\mathbb{Z})/\bigoplus_{\mathbb{N}}(\mathbb{Z}/2\mathbb{Z})$ has $2^{2^{\aleph_0}}$ automorphisms (in ZFC).

F_2

$(\mathcal{P}(\mathbb{N}), \Delta)$

$$\frac{\text{"}(\mathcal{P}(\mathbb{N})/\mathrm{Fin}, \Delta)\text{"}}{F_2\text{-v.s.}}$$

$\dim = 2^{\aleph_0}$

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An *almost permutation* of $\mathbb{N}$ is a bijection between cofinite subsets of $\mathbb{N}$.

6

Thm (Alperin–Covington–McPherson) *Let G be the quotient of the semigroup of almost permutations of $\mathbb{N}$ modulo the finitely supported permutations of $\mathbb{N}$. This is a group, and every automorphism of G is inner (in ZFC).*

$f^{-1}(n+1) = n$

$f \in S_\infty \; (\forall^\infty n \; f(n) = n$

$\mathrm{Out}\left(S_\infty / \mathrm{Fin}\text{-}S_\infty\right) \cong \mathbb{Z}$

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Lifting a homomorphism Φ between quotient structures

$$\begin{tikzcd} M \arrow[r, dash, "?"] \arrow[d, dash, "\pi"'] & N \arrow[d, dash, "\pi"] \\ M/I \arrow[r, "\Phi"] & N/J \end{tikzcd}$$

Algebraically trivial automorphisms

S_∞: The group of permutations of $\mathbb{N}$.

Lemma *Every automorphism Φ of the Boolean algebra $\mathcal{P}(\mathbb{N})$ is of the form $x \mapsto f^{-1}(x)$, for $f \in S_\infty$.*

Pf If $\phi(\{m\}) = n$ let $f(n) = m$

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Proof: Take $x \mapsto \{n-1 | n \in x\}$.

$\phi([x]_{Fin}) = [\{n-1 | n \in x\}]_{Fin}$

If ϕ were lifted to, $x \to f^{-1}(x)$, for some $f \in S_\infty$, then

[illegible]

$(\forall n)\ f(n) = n+1$

If $X = \{n \mid f(n) \neq n+1\}$ is ∞,

Then find $Y \subseteq X$, ∞, so that

$\forall n \quad n \in Y \Rightarrow n+1 \notin Y$, $f(n+1) \notin Y$

Then $Z = \{n+1 \mid n \in Y\}$

$\phi([Z]) = [Y]$, but

$f^{-1}(Z) \cap Y = \emptyset$

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1978

Thm (Shelah, 1970s) *(If ZFC is consistent then) there is a model of ZFC in which all automorphisms of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ are trivial.*

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Proof: Take $x \mapsto \{n-1 | n \in x\}$.

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Thm (Shelah, 1970s) *(If ZFC is consistent then) there is a model of ZFC in which all automorphisms of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ are trivial.*

Shelah–Steprāns: (If ZFC is consistent then) there is a model of ZFC in which CH fails but $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ has nontrivial automorphisms.

We will prove the following lemma later on:

Lemma *For an automorphism Φ of $\mathcal{Q}(H)$ the following are equivalent.*

1. *Φ is inner.*
2. *There is a Borel-measurable $f : \mathcal{B}(H)_1 \to \mathcal{B}(H)_1$ (where $\mathcal{B}(H)_1$ is considered with respect to the strong operator topology) lift of Φ.*
3. *There is a continuous $f : \mathcal{B}(H)_1 \to \mathcal{B}(H)_1$ (where $\mathcal{B}(H)_1$ is considered with respect to the strong operator topology) lift of Φ.*

Trivial?

$a \to \|ha\|$

$\to \|ah\|$

$\|\cdot\|$

Lemma *If A is separable, then the strict topology on $\mathcal{M}(A)_1$ is Polish (i.e., separable, completely metrizable).*

$\sum 2^{-n}$

Suppose A is separable, non-unital. An automorphism Φ of $\mathcal{M}(A)/A$ is *topologically trivial* if

$$\{(a,b) \in \mathcal{M}(A)_1^2 \mid \Phi(a+A) = b+A\}$$

is Borel in the strict topology.

Shoenfield's Absoluteness Thm

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Conjecture *(Coskey–F.) If A is a separable, non-unital C^*-algebra, then CH implies that $\mathcal{M}(A)/A$ has topologially nontrivial automorphisms.*

(weaker than "has $2^{\aleph_1}$ automorphisms"

Lemma *If A is separable, then the strict topology on $\mathcal{M}(A)_1$ is Polish (i.e., separable, completely metrizable).*

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There are partial positive answers by Coskey–F., F.–Shelah, Vignati.

Forcing axioms (Baire Category Theorem on steriods)

Auto → tri. trivial

The conclusion of Shelah's theorem ('all automorphisms of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ are trivial') is true in a class of canonical models of ZFC.

Def *Suppose that Ω is a class of compact Hausdorff spaces. Then $FA(\Omega)$ is the statement*

> *If $K \in \Omega$, then the intersection of any family of $\aleph_1$ dense open subsets of K is dense in K.*

Example

If $[0,1] \in \Omega$, then $\mathrm{FA}(\Omega)$ contradicts CH.

$\bigcap [0,1] \setminus \{x\}$

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Some forcing axioms: Martin's Axiom (MA), Proper Forcing Axiom (PFA), Martin's Maximum (MM).

Thm (Shelah–Steprāns) *PFA implies that all automorphisms of* $\mathcal{P}(\mathbb{N})/\,\mathrm{Fin}$ *are trivial.*

Thm (Veličković, 1992) *A consequence of PFA, MA+OCA, implies that all automorphisms of* $\mathcal{P}(\mathbb{N})/\,\mathrm{Fin}$ *are trivial.*

Thm (Veličković, 1992) *MA does not imply that all automorphisms of* $\mathcal{P}(\mathbb{N})/\,\mathrm{Fin}$ *are trivial (unless ZFC implies* $0 = 1$*).*

$$M(A)/A \to M(B)/B$$

Isomorphisms of coronas

Here is another motivation for studying isomorphisms of coronas.

Question *(Sakai, 1971) If A and B are separable and simple C^*-algebras, does $\mathcal{M}(A)/A \cong \mathcal{M}(B)/B$ imply $A \cong B$?*

$$\mathcal{M}(A \oplus \mathbb{C})/(A \oplus \mathbb{C}) \cong \mathcal{M}(A)/A$$

unital

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G. Elliott, Derivations of Matroid C^*-algebras, II (1973): A positive answer for the matroid (aka AM) algebras.

$\underline{M_n(\mathbb{C})}$ $\quad$ $\underline{\mathcal{K}}(H) \otimes M_{2^\infty}$