Arithmetic (occult) periods

Jeff Achter

j.achter@colostate.edu
 Colorado State University
https://www.math.colostate.edu/~achter

April 2021 Fields Number Theory Seminar



Curves of genus 4

- N₄ non-hyperelliptic curves of genus 4.
- Complex Torelli map

$$N_4(\mathbb{C}) \xrightarrow{\tau} Sp_8(\mathbb{Z}) \backslash \mathbb{H}_4$$

identifies a curve with its periods

$$\left(\int_{\gamma_i}\omega_i\right)$$
.

- Right-hand side is 10-dimensional algebraic variety
- N₄ is 9-dimensional; image is a divisor.

Uniformization of A₄ tells us little about N₄



1/59

Ball quotients

Nonetheless:

Theorem (Kondō)

There is a holomorphic open immersion

$$\mathsf{N}_4(\mathbb{C}) \stackrel{\tau_{\mathsf{N}_4}}{-\!-\!-\!-\!-} \Gamma \backslash \mathbb{B}^9$$

where

- $\Gamma \cong SU_{(1,8)}(\mathbb{Z}[\zeta_3]);$
- $\mathbb{B}^9 \subset \mathbb{C}^9$ is the unit ball.



Complex cubic surfaces

Hodge diamond for a cubic surface:

The Hodge filtration is *trivial*; period map has no information.

3/59

Ball quotients

Nonetheless:

Theorem (Dolgachev-van Geemen-Kondō, Allcock-Carlson-Toledo)

Let Cub₂ *be the moduli space of cubic surfaces. There is a holomorphic open immersion*

$$\mathsf{Cub}_2(\mathbb{C}) \xrightarrow{\tau_{\mathsf{Cub}_2}} \Gamma \backslash \mathbb{B}^4$$

where
$$\Gamma \cong SU_{(1,4)}(\mathbb{Z}[\zeta_3])$$
.



- Period maps
- 2 Periods for cubics
 - Cubic threefolds
 - Cubic surfaces
- Occult periods
- 4 Interlude
 - K3 surfaces
 - Shimura varieties
- Arithmetic period maps
 - K3
 - Occult period maps
- 6 Twenty seven lines



Periods for complex families

Suppose $\omega: X \to S/\mathbb{C}$ smooth projective. Consider $R^j \omega_* \mathbb{Z}$ as a family of Hodge structures. Get a period map

$$S(\mathbb{C}) \longrightarrow \Gamma \backslash \mathbb{D}$$

- D period domain parametrizing appropriate Hodge structures
- Γ (arithmetic) group of automorphisms.

Example

 $X \rightarrow S$ a family of smooth projective curves of genus g. Get

$$S(\mathbb{C}) \longrightarrow \operatorname{Sp}_{2g}(\mathbb{Z}) \backslash \mathbb{H}_g$$

Torelli theorems

Sometimes, members of $X \rightarrow S$ are distinguished by their Hodge structures:

Example (Curves)

 $X \rightarrow S$ a family of distinct curves of genus g; obtain

$$S(\mathbb{C}) \hookrightarrow \operatorname{Sp}_{2g}(\mathbb{Z}) \backslash \mathbb{H}_g.$$

Torelli theorems for cubics

Example (Cubic curves)

A complex curve *X* of genus one "is" the complex structure on $H^1(X,\mathbb{Z}) \otimes \mathbb{R}$.

Non-example (Cubic surfaces)

Middle cohomology has Hodge numbers 0 7 0, thus rigid. Hodge theory doesn't distinguish cubic surfaces!

Example (Cubic threefolds)

 $Z \to T$ a family of distinct cubic threefolds; computing $H^3(Z_t, \mathbb{Q})$ gives inclusion

$$T(\mathbb{C}) \hookrightarrow \operatorname{Sp}_{10}(\mathbb{Z}) \backslash \mathbb{H}_5$$

(Clemens-Griffiths).



Algebra

- \mathbb{D} Hermitian symmetric and Γ torsion-free. Then $\Gamma \backslash \mathbb{D}$ is quasiprojective. (Baily–Borel)
- $V(\mathbb{C}) \to \Gamma \backslash \mathbb{D}$ holomorphic. Then $V \to \Gamma \backslash \mathbb{D}$ is algebraic (Borel).

Example

The classical Torelli map is a morphism of (algebraic) stacks

$$M_{g,\mathbb{C}} \longrightarrow A_{g,\mathbb{C}}$$

Arithmetic

 \mathbb{D} Hermitian symmetric \leadsto canonical model $\mathsf{S}h_{\Gamma}(\mathbb{D})$ over number field E.

Question

Given V/E, and $V_{\mathbb{C}} \to \mathsf{Sh}_{\Gamma}(\mathbb{D})_{\mathbb{C}}$, does the map descend to $V \to \mathsf{Sh}_{\Gamma}(\mathbb{D})$?

Prototypical result

There is a morphism

$$M_g \longrightarrow A_g$$

of stacks over Z which specializes to

$$M_{g,\mathbb{C}} \longrightarrow A_{g,\mathbb{C}}$$

and on points gives

$$\mathsf{M}_g(\mathbb{C}) \longrightarrow \mathsf{Sp}_{2g}(\mathbb{Z}) \backslash \mathbb{H}_g.$$

In praise of open arithmetic period maps

Shimura varieties support interesting arithmetic structures:

- special points;
- reciprocity law;
- Hecke operators;
- modular forms;
- ...

An open arithmetic period map



lets us pull these structures back to V.

Example

What is the Hecke orbit of a cubic surface?



In praise of open arithmetic period maps

Can use modular forms to find (global) equations for V.

Example

Allcock and Freitag compute equations for Cub_2 using modular forms on U(1,4):

- Explicitly realize Cub_2 as intersection of cubic 8-folds in \mathbb{P}^9 ;
- Intepret Cayley's cross-ratios as modular forms;
- Recover Coble (1917) equations for Cub₂.

- Period maps
- Periods for cubics
 - Cubic threefolds
 - Cubic surfaces
- Occult periods
- 4 Interlude
 - K3 surfaces
 - Shimura varieties
- 6 Arithmetic period maps
 - K3
 - Occult period maps
- Twenty seven lines



Intermediate Jacobians of cubic threefolds

- X/\mathbb{C} cubic threefold.
- Hodge numbers of $H^3(X, \mathbb{Q})$ are 0.5.5.0.
- Intermediate Jacobian

$$J^{3}(X) = \operatorname{Fil}^{1} H^{3}(X, \mathbb{C}) \backslash H^{3}(X, \mathbb{C}) / H^{3}(X, \mathbb{Z})$$

is actually a principally polarized abelian variety.

• Get a period map

$$Cub_3\mathbb{C} \longrightarrow A_{5,\mathbb{C}}$$



Descent of the Clemens–Griffiths period map

Theorem

The period map for cubic threefolds descends to a morphism

$$Cub_3 \longrightarrow A_5$$

over Q.



First proof: monodromy

Proof (Deligne)

- $f: Y \to \mathsf{Cub}_3$ universal family of cubic threefolds.
- $g: J^3_{\mathbb{C}} \to \mathsf{Cub}_{3\mathbb{C}}$ family of intermediate Jacobians.
- $\bullet \ R^3 f_* \mathbb{Q}(1) \cong R^1 g_* \mathbb{Q}(1)$
- Big monodromy of $R^3 f_* \mathbb{Q}(1)$ means $\operatorname{Aut}_{\mathsf{Cub}_3}(J^3_{\mathbb{C}})$ trivial.
- $J_{\mathbb{C}}$ descends to \mathbb{Q} .

Second proof: Prym

Proof (A.-)

Beauville/Murre strategy for algebraically closed fields:



$$\operatorname{Prym}(\widetilde{\Delta}/\Delta)_{\mathbb{C}} \cong J^{3}(X_{\mathbb{C}})$$

works in families.

Actually gives result over $\mathbb{Z}[1/2]$, and not just \mathbb{Q} .



Third proof: intermediate Jacobians

Theorem (A.–Casalaina-Martin–Vial)

X/K a smooth projective variety over a subfield of \mathbb{C} , $n \in \mathbb{Z}_{\geq 0}$. Then there exist an abelian variety J/K and cycle $\Gamma \in CH^{\dim(J)+n}(J \times X)$ such that:

$$J_{\mathbb{C}}=J_a^{2n+1}(X_{\mathbb{C}});$$

the Abel-Jacobi map

$$A^{n+1}(X_{\mathbb{C}}) \xrightarrow{AJ} J(\mathbb{C})$$

is $Aut(\mathbb{C}/K)$ -equivariant; and

$$H^1(J_{\overline{K}}, \mathbb{Q}_{\ell}) \stackrel{\Gamma_*}{\longrightarrow} H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_{\ell}(n))$$

is a split inclusion with image $N^n H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_{\ell}(n))$.



Third proof: intermediate Jacobians

Corollary

 $X \to T/\mathbb{Q}$ a family of smooth projective varieties of dimension 2n+1, $H^{2n+1}(X_s)$ of geometric coniveau n. Then the induced map

$$T_{\mathbb{C}} \xrightarrow{\tau_{\mathbb{C}}} \mathsf{A}_{g,\mathbb{C}}$$

admits a distinguished model over Q.

Proof.

The graph of $\tau_{\mathbb{C}}$ is stable under $\operatorname{Aut}(\mathbb{C}/\mathbb{Q})$.



Motivating statement

Theorem (Allcock–Carlson–Toledo, Dolgachev–van Geemen–Kondō)

There is an open immersion

$$\mathsf{Cub}_2(\mathbb{C}) \hookrightarrow \Gamma \backslash \mathbb{B}^4$$

where \mathbb{B}^4 is unit ball in \mathbb{C}^4 , and Γ is an arithmetic group $\Gamma \cong SU_{1,4}(\mathbb{Z}[\zeta_3])$

- Recall: the Hodge structure of a cubic surface is rigid .
- Kudla and Rapoport call this an occult period map.

Possibility of descent?

- Cub_2 makes sense over \mathbb{Z} .
- $SU_{1,4}(\mathbb{Z}[\zeta_3])\setminus \mathbb{B}^4$ admits a canonical model over $\mathbb{Q}(\zeta_3)$, and even over $\mathbb{Z}[\zeta_3]$.
- Is the ACT-DvGK period map compatible with these structures?

Canonical model of ball quotient comes from

$$SU_{1,4}(\mathbb{Z}[\zeta_3])\backslash \mathbb{B}^4 \cong A_{\mathbb{Z}[\zeta_3],(1,4)}(\mathbb{C}),$$

where $A_{\mathbb{Z}[\zeta_3],(1,4)}$ is the moduli space of principally polarized abelian fivefolds with action by $\mathbb{Z}[\zeta_3]$ of signature (1,4).



Theorem (Kudla–Rapoport)

The ACT-DvGK period map descends to

$$\mathsf{Cub}_{2\mathbb{Q}(\zeta_3)} {\:\longrightarrow\:} \mathsf{A}_{\mathbb{Z}[\zeta_3],(1,4),\mathbb{Q}(\zeta_3)}$$

Idea

Use a Deligne-style monodromy calculation.

ACT construction

Allcock-Carlson-Toledo:

- $Y \subset \mathbb{P}^3_{\mathbb{C}}$ a complex cubic surface.
- Construct $Z \to \mathbb{P}^3$, the μ_3 cover ramified along Y.
- Their period map is



The arithmetic of ACT

Proposition

There is a diagram of stacks over $\mathbb{Z}[\zeta_3, 1/6]$

in which H(3,3,3) is the stack of cyclic triple covers of \mathbb{P}^3 branched along a cubic surface; τ is an open immersion; and κ induces an isomorphism of coarse moduli spaces $\underline{H} \to \mathsf{Cub}_2$.

DvGK construction

Dolgachev-van Geemen-Kondō:

- Y/\mathbb{C} a complex cubic surface.
- Choose a line L on Y.
- For generic hyperplane $H \supset L$, $H \cap Y$ is smooth quadric.
- Discriminant *D* is a (singular) plane curve of degree five.
- Define a double cover $C \rightarrow D$ of degree six.
- Let $Z \to \mathbb{P}^2$ be the double cover ramified along C.
- Then Z is a polarized K3 surface with μ_3 -action.
- Compute the periods of Z.



- Period maps
- Periods for cubics
 - Cubic threefolds
 - Cubic surfaces
- Occult periods
- 4 Interlude
 - K3 surfaces
 - Shimura varieties
- Arithmetic period maps
 - K3
 - Occult period maps
- 6 Twenty seven lines



Occult periods (d'apres Kudla and Rapoport)

Given a variety Y/\mathbb{C} , try to understand it by:

- Constructing a new variety \mathbb{Z}/\mathbb{C} ; and
- computing periods of Z.

Several examples due to Kondō, Dolgachev, Looijenga, ...

Kudla and Rapoport

In many cases, target of period map is a Shimura variety; occult period map descends to reflex field.

- Period maps
- Periods for cubics
 - Cubic threefolds
 - Cubic surfaces
- Occult periods
- 4 Interlude
 - K3 surfaces
 - Shimura varieties
- 6 Arithmetic period maps
 - K3
 - Occult period maps
- 6 Twenty seven lines



(Lattice) Polarizations

Fix $d \ge 1$.

• R_{2d} moduli of K3 surfaces with primitive polarization, degree 2d.

•

$$\mathsf{R}_{2d}(S) = \{ (Z \to S, \lambda) : Z \to S \text{ a K3 space,}$$

 $\lambda \in \mathsf{Pic}_{Z/S}(S) \text{ primitive, } (\lambda, \lambda) = 2d \}.$

(Lattice) Polarizations

Fix $d \geq 1$.

• R_{2d} moduli of K3 surfaces with primitive polarization, degree 2d.

•

$$R_{2d}(S) = \{ (Z \to S, \lambda) : Z \to S \text{ a K3 space,}$$

 $\lambda \in \text{Pic}_{Z/S}(S) \text{ primitive, } (\lambda, \lambda) = 2d \}.$

Choice of λ is the same as primitive inclusion of lattices:

$$\langle 2d \rangle \stackrel{\alpha}{\longrightarrow} \operatorname{Pic}_{Z/S}(S)$$

where $\langle 2d \rangle$ is the lattice of rank 1 with pairing (2d). *Positive cone* condition ignored here.



(Lattice) Polarizations

Fix $d \ge 1$.

• R_{2d} moduli of K3 surfaces with primitive polarization, degree 2d.

•

$$R_{2d}(S) = \{ (Z \to S, \lambda) : Z \to S \text{ a K3 space,}$$

 $\lambda \in \text{Pic}_{Z/S}(S) \text{ primitive, } (\lambda, \lambda) = 2d \}.$

Fix $L \subset L_{K3} := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$ a primitive sublattice of signature (1, r-1).

• R_L the moduli space of L-polarized K3 surfaces:

$$R_L(S) = \{(Z \to S, \alpha) : \alpha : L \hookrightarrow Pic_{Z/S}(S) \text{ primitive}\}.$$



Group action

- μ_n group (scheme) of n^{th} roots of unity.
- R_{L,μ_n}^* the space of *L*-polarized K3 surfaces with action by μ_n :

$$\mathsf{R}_{L,\boldsymbol{\mu}_n}^*(S) = \{ (Z \to S, \alpha, \rho) : (Z \to S, \alpha) \in \mathsf{R}_L(S) \\ \rho : \boldsymbol{\mu}_n \hookrightarrow \mathsf{Aut}_S(Z \to S, \alpha) \}$$

- Data $\underline{\chi} = (\chi, \chi^{\omega})$ describes component $\mathsf{R}_{L,\underline{\chi}} \subset \mathsf{R}_{L,\pmb{\mu}_n}^*$ where:
 - μ_n acts on $H^0(Z, \Omega_Z^2)$ via χ^{ω} ;
 - μ_n acts on $H^2(Z)$ via χ ;
 - $\operatorname{mult}_{\chi}(\chi_{\operatorname{triv}}) = \operatorname{rank}(L)$.

K3 surfaces have good moduli spaces

Proposition

- **1** Let L be a lattice of signature (1, r-1) and discriminant Δ_L . Then R_L is a smooth Deligne-Mumford stack over $\operatorname{Spec} \mathbb{Z}[1/2\Delta_L]$ of relative dimension r-1.
- For $\underline{\chi} = (\chi, \chi^{\omega})$, $R_{L,\underline{\chi}}$ is a smooth Deligne-Mumford stack over $\mathbb{Z}[\zeta_n, 1/2\Delta_L n]$ of relative dimension $\mathrm{mult}(\chi^{\omega}) 1$.

We always assume L, χ chosen so that R_L , $R_{L,\chi}$ are nonempty.

Periods for complex K3 surfaces

- \mathbb{Z}/\mathbb{C} a K3 surface; middle Hodge numbers 1 20 1.
- $H^2(Z, \mathbb{Z}) \cong L_{K3} := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$ signature (3, 19)
- Choose marking $\phi: H^2(Z,\mathbb{Z}) \stackrel{\sim}{\to} L_{K3}$, get

$$\phi_{\mathbb{C}}(H^{2,0}(Z)) \in \mathbb{X}_{L_{K3}} = \{ [\sigma] \in \mathbb{P}(L_{K3} \otimes \mathbb{C}) : (\sigma,\sigma) = 0, (\sigma,\overline{\sigma}) > 0 \}.$$

Period maps

$$\{\text{marked K3's/}\mathbb{C}\}\longrightarrow \mathbb{X}_{L_{K3}}$$

$$\{K3/\mathbb{C}\}$$
 \longrightarrow $O_{L_{K3}}(\mathbb{Z})\backslash \mathbb{X}_{L_{K3}}$

32 / 59

where RHS parametrizes polarized Hodge structures on L_{K3} with Hodge numbers (1, 20, 1).

• Problem: Right-hand side is a terrible space.

Periods for polarized K3 surfaces

- $(Z/\mathbb{C},\lambda) \in \mathsf{R}_{2d}(\mathbb{C})$, ϕ a marking.
- λ gives $\langle 2d \rangle \hookrightarrow H^2(Z, \mathbb{Z}) \stackrel{\phi}{\to} L_{K3}$.
- Then $H^{20}(Z) \perp c_1(\lambda)$.
- So

$$\phi_{\mathbb{C}}(H^{20}(Z)) \subset \langle 2d \rangle^{\perp} \subset L_{K3}.$$

• Get period map

$$\mathsf{R}_{2d}(\mathbb{C}) {\:\longrightarrow\:} \widetilde{O}^{\langle 2d \rangle}(\mathbb{Z}) \backslash \mathbb{X}^{\langle 2d \rangle} \subset O_{L_{K3}}(\mathbb{Z}) \backslash \mathbb{X}_{L_{K3}}$$

where:

 $ightharpoonup \mathbb{X}^{\langle 2d \rangle}$ means $\mathbb{X}_{\langle 2d \rangle^{\perp}}$, etc.

$$1 \longrightarrow \widetilde{\mathcal{O}}_L \longrightarrow \mathcal{O}_L \longrightarrow \operatorname{Aut}(L^{\vee}/L)$$



Torelli for $K3/\mathbb{C}$

Theorem (Piateskii-Shapiro, Shafarevich)

The period map

$$\mathsf{R}_{2d,\mathbb{C}} \overset{\tau_{2d,\mathbb{C}}}{\hookrightarrow} \widetilde{\mathsf{O}}^{\langle 2d \rangle}(\mathbb{Z}) \backslash \mathbb{X}^{\langle 2d \rangle}.$$

is an open immersion (of complex orbifolds).

For *L*-polarized K3, period point is in $L^{\perp} \subset L_{K3}$.

Proposition (Dolgachev, Kondō)

The period map gives an open immersion

$$R_{L,\mathbb{C}} \xrightarrow{\tau_{L,\mathbb{C}}} \widetilde{O}^L(\mathbb{Z}) \backslash \mathbb{X}^L.$$

Setup

- A Shimura datum is (G, \mathbb{X}) :
 - G/\mathbb{Q} a reductive group;
 - ▶ \mathbb{X} a $G(\mathbb{R})$ -conjugacy of homomorphisms $\mathbf{R}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_{\mathbb{R}}$ (subject to certain axioms)
- If $\mathbb{K} \subset G(\mathbb{A}_f)$ compact open, quotient stack

$$\mathsf{S}h_{\mathbb{K}}[G,\mathbb{X}] := [G(\mathbb{Q}) \setminus (\mathbb{X} \times G(\mathbb{A}_f)/\mathbb{K})]$$

algebraizes to complex quasiprojective variety.

Canonical models

- $Sh_{\mathbb{K}}[G, \mathbb{X}]$ descends to reflex field $E(G, \mathbb{X})$.
- Let

$$M(\mathbb{K}) = \prod_{p:\mathbb{K}_p \text{ is not hyperspecial}} p$$

Then $Sh_{\mathbb{K}}[G, \mathbb{X}]$ admits canonical integral model over $\mathcal{O}_{E(G, \mathbb{X})}[1/M(\mathbb{K})]$.

Orthogonal Shimura varieties

- L a lattice of signature (2, n)
- $G_L = SO_{L \otimes \mathbb{O}}$
- $\mathbb{K}_L = \ker G_L(\hat{\mathbb{Z}}) \to \operatorname{Aut}(\operatorname{disc}(L))(\hat{\mathbb{Z}})$

Set

$$\mathsf{S}h_L = \mathsf{S}h_{\mathbb{K}_L}[G_L, \mathbb{X}_L]$$

over $\mathbb{Z}[1/2\Delta_L]$.

Lemma

A primitive embedding $L_1 \hookrightarrow L_2$ induces

$$\mathsf{S}h_{L_1} \xrightarrow{\psi_{L_1L_2}} \mathsf{S}h_{L_2}$$

over $\mathbb{Z}[1/2\Delta_{L_1}\Delta_{L_2}]$, with generic fiber a closed embedding.



Unitary Shimura varieties

- K a quadratic imaginary field
- *L* a free \mathcal{O}_K -module with Hermitian form $h(\cdot, \dot{)}$ of signature (1, r 1).
- \bullet G = U(L,h)
- $\mathbb{X}_{\mathcal{O}_{\nu},L} \cong \mathbb{B}^{r-1}$
- $\mathbb{K}_{\mathcal{O}_K,L}$ the stabilizer in $G_{\mathcal{O}_K,L}(\mathbb{A}_f)$ of L

Set

$$\mathsf{S}h_{\mathcal{O}_K,L} = \mathsf{S}h_{\mathbb{K}_{\mathcal{O}_K,L}}[G_{\mathcal{O}_K,L},\mathbb{X}_{\mathcal{O}_K,L}].$$

Lemma

 $\mathsf{Sh}_{\mathcal{O}_K,L}$ is the moduli space of abelian varieties of dimension r equipped with an action by \mathcal{O}_K of signature (1,r-1), and a polarization λ with $\ker(\lambda) \cong \operatorname{disc}(L)$.

Also variants for K/\mathbb{Q} CM, arbitrary degree.



- Period maps
- 2 Periods for cubics
 - Cubic threefolds
 - Cubic surfaces
- Occult periods
- 4 Interlude
 - K3 surfaces
 - Shimura varieties
- 5 Arithmetic period maps
 - K3
 - Occult period maps
- Twenty seven lines



Period maps reconsidered

Period map for polarized K3 surfaces is

$$\mathsf{R}_{2d}(\mathbb{C}) \longrightarrow \mathsf{S}h^{\langle 2d \rangle}(\mathbb{C}) = \mathsf{S}h_{\langle 2d \rangle^{\perp}}(\mathbb{C}) \ .$$

- $L \hookrightarrow L_{K3}$ primitive of signature (1, r-1)
- Set

$$\mathsf{S}h^L = \mathsf{S}h_{\mathbb{K}^L}[G^L, \mathbb{X}^L] = \mathsf{S}h_{\mathbb{K}_{L^\perp}}[G_{L^\perp}, \mathbb{X}_{L^\perp}].$$

Period map becomes

$$\mathsf{R}_L(\mathbb{C}) \stackrel{\tau_{L,\mathbb{C}}}{\longrightarrow} \mathsf{S}h^L(\mathbb{C}).$$



Period maps reconsidered

- Fix $\chi = (\boldsymbol{\mu}_n, \chi^\omega, \chi)$.
- Let $E(\chi) = \mathbb{Q}(\zeta_n)$.

0

$$\mathsf{S}h^{(\mathsf{L},\underline{\chi})} = egin{cases} \mathsf{S}h_{\mathcal{O}_{\mathsf{E}(\underline{\chi})},\mathsf{L}^{\perp}} & n \geq 3 \\ \mathsf{S}h_{\mathsf{L}^{\perp}} & n = 2 \end{cases}$$

• Period map is

$$\mathsf{R}_{L,\chi}(\mathbb{C}) \overset{\tau_{(L,\underline{\chi}),\mathbb{C}}}{\longrightarrow} \mathsf{S}h^{(L,\underline{\chi})}(\mathbb{C}) \;.$$



Period maps are arithmetic

Theorem (Rizov, Madapusi Pera, Taelman)

The Piateskii-Shapiro/Shafarevich period map descends to a morphism

$$R_{2d} \xrightarrow{\tau_{2d}} Sh^{\langle 2d \rangle}$$

over $\mathbb{Z}[1/6d]$.

The transcendental map preserves integral structures.

Periods for structured K3 surfaces

Proposition

For L and χ as before:

1 $\tau_{L,\mathbb{C}}$ is the fiber over \mathbb{C} of a morphism

$$R_L \xrightarrow{\tau_L} Sh^L$$

of stacks over $\mathbb{Z}[1/2\Delta(L)]$.

1 $\tau_{(L,\chi),\mathbb{C}}$ is the fiber over \mathbb{C} of a morphism

$$\mathsf{R}_{(L,\chi),\mathbb{C}} \xrightarrow{\tau_{(L,\underline{\chi})}} \mathsf{S}h^{(L,\underline{\chi})}$$

of stacks over $\mathcal{O}_{E(\chi)}[1/2n\Delta(L)]$.



Strategy of proof

Descent to \mathbb{Q} :

Fix $\langle 2d \rangle \hookrightarrow L$ and level N, and show $\operatorname{Aut}(\mathbb{C}/\mathbb{Q})$ -equivariance in:

$$\begin{split} \mathsf{R}_{L,N}(\mathbb{C}) & \stackrel{\tau_{L,N,\mathbb{C}}}{\longrightarrow} \mathsf{S}h_N^L(\mathbb{C}) \\ & & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & &$$

Spread to $\mathbb{Z}[1/2\Delta(L)]$

Use smoothness of R_L and the extension property of integral canonical models.



Curves of genus 4

Recall from the beginning:

Theorem (Kondō)

There is a holomorphic open immersion

$$N_4(\mathbb{C}) \stackrel{\tau_{N_4}}{\longrightarrow} \Gamma \backslash \mathbb{B}^9$$

Kondō's construction

- $C \in N_4(\mathbb{C})$
- Canonical model is $C = Q \cap S \subset \mathbb{P}^3$, intersection of quadric and cubic.
- ω : $Z \to Q$ triple cover branched along C.
- M_1 , M_2 on Q represent two rulings
- $N_i = \omega^{-1}(M_i)$ are cycle (classes) on Z.
- Each N_i an elliptic curve, and $(N_1, N_2) = 3$.

Then:

- Z is a K3 surface.
- $\mathbb{Z}N_1 + \mathbb{Z}N_2 \hookrightarrow NS(Z)$ gives

$$L_4 := U(3) \stackrel{\alpha}{\longrightarrow} \operatorname{Pic}(Z) \subset L_{K3}.$$

• Z has μ_3 action ρ fixing each N_i .

$$C \leadsto (Z, \alpha, \rho) \in \mathsf{R}_{(L, \underline{\chi}_4)}$$



Kondō's occult period map is arithmetic

Theorem

There is a diagram of stacks over $\mathcal{O}_{\mathbb{Z}[\zeta_3,1/6]}$

$$\begin{array}{c} \mathsf{R}_{(L_4, \underline{\chi}_4)} \xrightarrow{\kappa_4} \mathsf{N}_4 \\ \downarrow^{\tau_4} \\ \mathsf{S}h^{(L_4, \underline{\chi}_4)} \end{array}$$

where κ_4 induces an isomorphism on coarse moduli spaces, and τ_4 induces an open immersion $\mathsf{R}_{(L_4,\chi_4)}(\mathbb{C}) \hookrightarrow \mathsf{S}h^{(L_4,\chi_4)}(\mathbb{C})$.

Idea

The *inverse* to the occult period map is algebraic:

$$\kappa_4(Z \to S, \alpha, \rho) = Z^{\mu_3} \to S.$$

Curves of genus 3

Theorem

For certain data $(L_3, \underline{\chi}_3)$, there is a diagram of stacks over $\mathcal{O}_{\mathbb{Z}[\sqrt{-1},1/2]}$

$$\mathsf{R}_{(L_3,\underline{\chi}_3)} \xrightarrow{\kappa_3} \mathsf{N}_3$$

$$\downarrow^{\tau_3}$$

$$\mathsf{S}h^{(L_3,\underline{\chi}_3)}$$

where κ_3 induces an isomorphism on coarse moduli spaces, and τ_3 induces an open immersion $\mathsf{R}_{(L_3,\chi_2)}(\mathbb{C}) \hookrightarrow \mathsf{S}h^{(L_3,\chi_3)}(\mathbb{C})$.

Idea

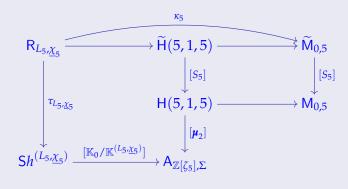
If $C \in N_3(\mathbb{C})$, canonical model is a smooth plane quartic curve; $\omega : Z \to \mathbb{P}^2$ the quartic cover branched along C is an L_3 -polarized K3 with μ_4 -action.

- $\widetilde{\mathsf{M}}_{0,5}$ the moduli space of five distinct, ordered points in \mathbb{P}^1
- $(P_1, \dots, P_5) \in \widetilde{\mathsf{M}}_{0,5}(\mathbb{C})$. Kondō constructs
 - $C \to \mathbb{P}^1 \mu_5$ -cover branched along (P_1, \dots, P_5) ;
 - ► $X \to \mathbb{P}^2 \mu_2$ -cover branched along C and a \mathbb{P}^1
- *X* is polarized by $L_5 \cong V \oplus A_4(-1) \oplus A_4(-1)$



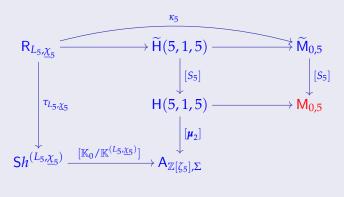
Proposition

There is a diagram of stacks over $\mathbb{Z}[\zeta_5, 1/10]$:



Proposition

There is a diagram of stacks over $\mathbb{Z}[\zeta_5, 1/10]$ *:*

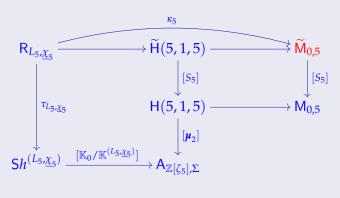


Five points on a rational curve



Proposition

There is a diagram of stacks over $\mathbb{Z}[\zeta_5, 1/10]$ *:*

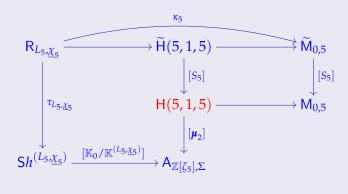


Five labelled points on \mathbb{P}^1



Proposition

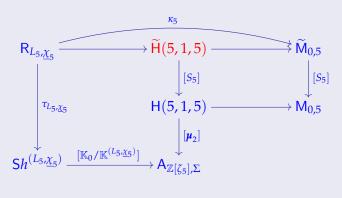
There is a diagram of stacks over $\mathbb{Z}[\zeta_5, 1/10]$ *:*



Degree 5 cyclic covers of Brauer-Severi scheme of dimension one with branch locus of degree 5

Proposition

There is a diagram of stacks over $\mathbb{Z}[\zeta_5, 1/10]$ *:*

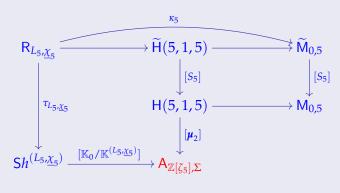


Degree 5 cyclic covers with labelled branch locus



Proposition

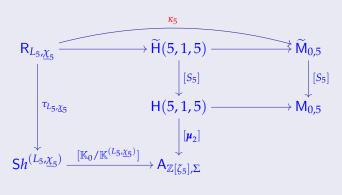
There is a diagram of stacks over $\mathbb{Z}[\zeta_5, 1/10]$ *:*



Principally polarized abelian varieties of dimension 6 with an action by $\mathbb{Z}[\zeta_5]$ of signature $\Sigma = \{(2,1),(0,3)\}$

Proposition

There is a diagram of stacks over $\mathbb{Z}[\zeta_5, 1/10]$ *:*

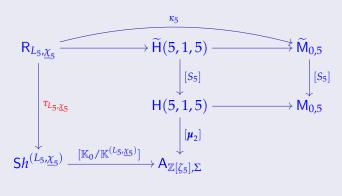


An isomorphism of coarse moduli spaces



Proposition

There is a diagram of stacks over $\mathbb{Z}[\zeta_5, 1/10]$ *:*

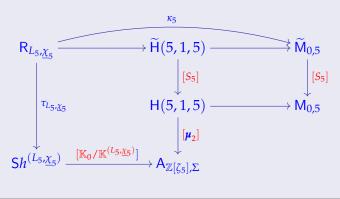


Fiber over C is an open immersion



Proposition

There is a diagram of stacks over $\mathbb{Z}[\zeta_5, 1/10]$ *:*



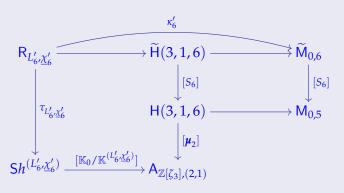
Quotient by finite group



Six points on a line

Proposition

There is a diagram of stacks over $\mathbb{Z}[\zeta_3, 1/6]$ *:*



- Period maps
- 2 Periods for cubics
 - Cubic threefolds
 - Cubic surfaces
- Occult periods
- 4 Interlude
 - K3 surfaces
 - Shimura varieties
- Arithmetic period maps
 - K3
 - Occult period maps
- Twenty seven lines



Whole books have been devoted to the configuration of the 27 lines on a smooth cubic surface (Henderson [1]; Segre [2]). Their elegant symmetry both enthrals and at the same time irritates; what use is it to know, for instance, the number of coplanar triples of such lines (forty five) or the number of double Schläffli sixfolds (thirty six)? The answer to this rhetorical Yu. Manin, Cubic forms: algebra, geometry, arithmetic, 1986.

- Z/k a cubic surface over an algebraically closed field.
- Z contains exactly 27 lines.
- What happens over arithmetic fields?

Early results

- \mathbb{Z}/\mathbb{C} has exactly 27 lines Cayley-Salmon 1849
- Automorphism group of 27 lines is a group of order 51,840 (isomorphic to $W(E_6)$) Jordan 1869
- Jordan interprets this as the Galois group of the 27 lines on a general complex cubic surface.

Modern reformulation

- Cub₂^m the moduli space of cubic surfaces with a marking of the 27 lines.
- $Cub_2^m \rightarrow Cub_2$ the forgetful map.

Theorem (Jordan 1870)

 Cub_{2}^{m} is irreducible, and

$$\mathsf{Cub}_{2\mathbb{C}}^{\ m} {\:\longrightarrow\:} \mathsf{Cub}_{2\mathbb{C}}$$

is Galois with group $W(E_6)$.



Modern reformulation

- Cub₂^m the moduli space of cubic surfaces with a marking of the 27 lines.
- $Cub_2^m \rightarrow Cub_2$ the forgetful map.

Theorem (Jordan 1870)

 Cub_{2}^{m} is irreducible, and

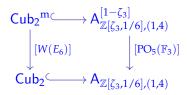
$$\mathsf{Cub}_{2\mathbb{C}}^{\ m} {\:\longrightarrow\:} \mathsf{Cub}_{2\mathbb{C}}$$

is Galois with group $W(E_6)$.



Lines and torsion points

Abel–Jacobi map identifies differences of lines on Z with $1 - \zeta_3$ torsion on J_Z^3 .



Beyond C: Hilbertian fields

Proposition

K Hilbertian, $char(K) \neq 2,3, Z/K$ a sufficiently general cubic surface. Then

$$Gal(K(lines(Z))/K) \cong W(E_6).$$

Proof.

- $Cub_{2\mathbb{C}}^{m}$ is irreducible Jordan
- $A_{\mathbb{Z}[\zeta_3],(1,4)}^{[1-\zeta_3]}$ admits toroidal compactification Lan
- Each $A_{\mathbb{Z}[\zeta_3],(1,4),\kappa(\mathfrak{p})}$ irreducible (Zariski connectedness)
- Now use Hilbert irreducibility.



Beyond \mathbb{C} : finite fields

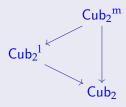
Lemma

 \mathbb{F}_q is a large finite field of characteristic ≥ 5 .

The expected number of lines on a random cubic surface over \mathbb{F}_q *is* ≈ 1 .

Proof.

Cub₂¹: space of cubic surfaces equipped with a choice of line. All geometric fibers are irreducible:



Beyond \mathbb{C} : finite fields

Lemma

 \mathbb{F}_q is a large finite field of characteristic ≥ 5 .

The expected number of lines on a random cubic surface over \mathbb{F}_q *is* ≈ 1 .

What about the other direction?

- The moduli space of cubic surfaces is well-known to be rational.
- R. Das has explicit, precise line counts over finite fields.

What do facts like these tell us about the Shimura variety?

Thanks!