Arithmetic (occult) periods

Jeff Achter

j.achter@colostate.edu
Colorado State University
https://www.math.colostate.edu/~achter

April 2021
Fields Number Theory Seminar

Curves of genus 4

- N_4 non-hyperelliptic curves of genus 4.
- Complex Torelli map

$$N_4(\mathbb{C}) \overset{\tau}{\hookrightarrow} Sp_8(\mathbb{Z})\backslash\mathbb{H}_4$$

identifies a curve with its periods

$$\left(\int_{\gamma_j} \omega_i\right).$$

- Right-hand side is 10-dimensional algebraic variety
- N_4 is 9-dimensional; image is a divisor.

Uniformization of A_4 tells us little about N_4

Ball quotients

Nonetheless:

Theorem (Kondō)

There is a holomorphic open immersion

$$\mathsf{N}_4(\mathbb{C}) \xhookrightarrow{\tau_{\mathsf{N}_4}} \Gamma\backslash\mathbb{B}^9$$

where

- $\Gamma \cong \mathrm{SU}_{(1,8)}(\mathbb{Z}[\zeta_3])$;
- $\mathbb{B}^9 \subset \mathbb{C}^9$ is the unit ball.

Complex cubic surfaces

Hodge diamond for a cubic surface:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 7 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}.$$

The Hodge filtration is *trivial*; period map has no information.

Ball quotients

Nonetheless:

Theorem (Dolgachev–van Geemen–Kondō, Allcock–Carlson–Toledo)

Let Cub_2 *be the moduli space of cubic surfaces. There is a holomorphic open immersion*

$$\mathsf{Cub}_2(\mathbb{C}) \xhookrightarrow{\tau_{\mathsf{Cub}_2}} \Gamma \backslash \mathbb{B}^4$$

where $\Gamma \cong \mathrm{SU}_{(1,4)}(\mathbb{Z}[\zeta_3])$.

Periods for complex families

Suppose $\varpi : X \to S/\mathbb{C}$ smooth projective.
Consider $R^j\varpi_*\mathbb{Z}$ as a family of Hodge structures.
Get a period map

$$S(\mathbb{C}) \longrightarrow \Gamma\backslash\mathbb{D}$$

- $\mathbb{D}$ period domain parametrizing appropriate Hodge structures
- Γ (arithmetic) group of automorphisms.

Example

$X \to S$ a family of smooth projective curves of genus g. Get

$$S(\mathbb{C}) \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathbb{H}_g$$

Torelli theorems

Sometimes, members of $X \to S$ are distinguished by their Hodge structures:

Example (Curves)

$X \to S$ a family of distinct curves of genus g; obtain

$$S(\mathbb{C}) \hookrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}) \backslash \mathbb{H}_g.$$

Torelli theorems for cubics

Example (Cubic curves)

A complex curve X of genus one "is" the complex structure on $H^1(X, \mathbb{Z}) \otimes \mathbb{R}$.

Non-example (Cubic surfaces)

Middle cohomology has Hodge numbers $0\ 7\ 0$, thus rigid. Hodge theory doesn't distinguish cubic surfaces!

Example (Cubic threefolds)

$Z \to T$ a family of distinct cubic threefolds; computing $H^3(Z_t, \mathbb{Q})$ gives inclusion

$$T(\mathbb{C}) \hookrightarrow \mathrm{Sp}_{10}(\mathbb{Z}) \backslash \mathbb{H}_5$$

(Clemens-Griffiths).

Algebra

- $\mathbb{D}$ Hermitian symmetric and Γ torsion-free.
  Then $\Gamma\backslash\mathbb{D}$ is quasiprojective. (Baily–Borel)
- $V(\mathbb{C}) \to \Gamma\backslash\mathbb{D}$ holomorphic. Then $V \to \Gamma\backslash\mathbb{D}$ is algebraic (Borel).

Example

The classical Torelli map is a morphism of (algebraic) stacks

$$\mathsf{M}_{g,\mathbb{C}} \longrightarrow \mathsf{A}_{g,\mathbb{C}}$$

Arithmetic

$\mathbb{D}$ Hermitian symmetric $\rightsquigarrow$ canonical model $\mathsf{Sh}_\Gamma(\mathbb{D})$ over number field E.

Question

Given V/E, and $V_{\mathbb{C}} \to \mathsf{Sh}_\Gamma(\mathbb{D})_{\mathbb{C}}$, does the map descend to $V \to \mathsf{Sh}_\Gamma(\mathbb{D})$?

Prototypical result

There is a morphism

$$\mathsf{M}_g \longrightarrow \mathsf{A}_g$$

of stacks over $\mathbb{Z}$ which specializes to

$$\mathsf{M}_{g,\mathbb{C}} \longrightarrow \mathsf{A}_{g,\mathbb{C}}$$

and on points gives

$$\mathsf{M}_g(\mathbb{C}) \longrightarrow \mathsf{Sp}_{2g}(\mathbb{Z})\backslash\mathbb{H}_g.$$

In praise of open arithmetic period maps

Shimura varieties support interesting arithmetic structures:

- special points;
- reciprocity law;
- Hecke operators;
- modular forms;
- ...

An *open arithmetic period map*

$$\mathsf{V} \hookrightarrow \mathsf{A}$$

lets us pull these structures back to V.

Example

What is the Hecke orbit of a cubic surface?

In praise of open arithmetic period maps

Can use modular forms to find (global) equations for V.

Example

Allcock and Freitag compute equations for Cub_2 using modular forms on $U(1,4)$:

- Explicitly realize Cub_2 as intersection of cubic 8-folds in $\mathbb{P}^9$;
- Intepret Cayley's cross-ratios as modular forms;
- Recover Coble (1917) equations for Cub_2.

Intermediate Jacobians of cubic threefolds

- $X/\mathbb{C}$ cubic threefold.
- Hodge numbers of $H^3(X,\mathbb{Q})$ are 0 5 5 0.
- Intermediate Jacobian

$$J^3(X) = \mathrm{Fil}^1 H^3(X,\mathbb{C})\backslash H^3(X,\mathbb{C})/H^3(X,\mathbb{Z})$$

  is actually a principally polarized abelian variety.
- Get a period map

$$\mathsf{Cub}_{3\mathbb{C}} \hookrightarrow \mathsf{A}_{5,\mathbb{C}}$$

Descent of the Clemens–Griffiths period map

Theorem

The period map for cubic threefolds descends to a morphism

$$\mathsf{Cub}_3 \longrightarrow \mathsf{A}_5$$

over $\mathbb{Q}$.

First proof: monodromy

Proof (Deligne)

- $f : Y \to \mathsf{Cub}_3$ universal family of cubic threefolds.
- $g : J^3_{\mathbb{C}} \to \mathsf{Cub}_{3\mathbb{C}}$ family of intermediate Jacobians.
- $R^3 f_* \mathbb{Q}(1) \cong R^1 g_* \mathbb{Q}(1)$
- Big monodromy of $R^3 f_* \mathbb{Q}(1)$ means $\mathrm{Aut}_{\mathsf{Cub}_3}(J^3_{\mathbb{C}})$ trivial.
- $J_{\mathbb{C}}$ descends to $\mathbb{Q}$.

Second proof: Prym

Proof (A.-)

Beauville/Murre strategy for algebraically closed fields:

$$\begin{tikzcd} & \widetilde{X} \arrow[dl] \arrow[d] & \widetilde{\Delta} \arrow[r, hook] \arrow[d, "2:1"', "\text{étale}"] & F_X \\ X & \mathbb{P}^2 & \Delta \arrow[l, hook'] & \end{tikzcd}$$

$$\mathrm{Prym}(\widetilde{\Delta}/\Delta)_{\mathbb{C}} \cong J^3(X_{\mathbb{C}})$$

works in families.

Actually gives result over $\mathbb{Z}[1/2]$, and not just $\mathbb{Q}$.

Third proof: intermediate Jacobians

Theorem (A.–Casalaina-Martin–Vial)

X/K a smooth projective variety over a subfield of $\mathbb{C}$, $n \in \mathbb{Z}_{\geq 0}$. Then there exist an abelian variety J/K and cycle $\Gamma \in \mathrm{CH}^{\dim(J)+n}(J \times X)$ such that:

$$J_{\mathbb{C}} = J_a^{2n+1}(X_{\mathbb{C}});$$

the Abel–Jacobi map

$$\mathrm{A}^{n+1}(X_{\mathbb{C}}) \xrightarrow{\mathrm{AJ}} J(\mathbb{C})$$

is $\mathrm{Aut}(\mathbb{C}/K)$-equivariant; and

$$H^1(J_{\overline{K}}, \mathbb{Q}_\ell) \xhookrightarrow{\Gamma_*} H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$$

is a split inclusion with image $\mathrm{N}^n\, H^{2n+1}(X_{\overline{K}}, \mathbb{Q}_\ell(n))$.

Third proof: intermediate Jacobians

Corollary

$X \to T/\mathbb{Q}$ *a family of smooth projective varieties of dimension* $2n+1$, $H^{2n+1}(X_s)$ *of geometric coniveau* n. *Then the induced map*

$$T_{\mathbb{C}} \xrightarrow{\tau_{\mathbb{C}}} \mathsf{A}_{g,\mathbb{C}}$$

admits a distinguished model over $\mathbb{Q}$.

Proof.

The graph of $\tau_{\mathbb{C}}$ is stable under $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. □

Motivating statement

Theorem (Allcock–Carlson–Toledo, Dolgachev–van Geemen–Kondō)

There is an open immersion

$$\mathsf{Cub}_2(\mathbb{C}) \hookrightarrow \Gamma\backslash\mathbb{B}^4$$

where $\mathbb{B}^4$ is unit ball in $\mathbb{C}^4$, and Γ is an arithmetic group $\Gamma \cong \mathrm{SU}_{1,4}(\mathbb{Z}[\zeta_3])$

- Recall: the Hodge structure of a cubic surface is rigid.
- Kudla and Rapoport call this an occult period map.

Possibility of descent?

- Cub_2 makes sense over $\mathbb{Z}$.
- $\mathrm{SU}_{1,4}(\mathbb{Z}[\zeta_3])\backslash\mathbb{B}^4$ admits a canonical model over $\mathbb{Q}(\zeta_3)$, and even over $\mathbb{Z}[\zeta_3]$.
- Is the ACT-DvGK period map compatible with these structures?

Canonical model of ball quotient comes from

$$\mathrm{SU}_{1,4}(\mathbb{Z}[\zeta_3])\backslash\mathbb{B}^4 \cong \mathsf{A}_{\mathbb{Z}[\zeta_3],(1,4)}(\mathbb{C}),$$

where $\mathsf{A}_{\mathbb{Z}[\zeta_3],(1,4)}$ is the moduli space of principally polarized abelian fivefolds with action by $\mathbb{Z}[\zeta_3]$ of signature $(1,4)$.

Theorem (Kudla–Rapoport)

The ACT–DvGK period map descends to

$$\mathsf{Cub2}_{\mathbb{Q}(\zeta_3)} \longrightarrow \mathsf{A}_{\mathbb{Z}[\zeta_3],(1,4),\mathbb{Q}(\zeta_3)}$$

Idea

Use a Deligne-style monodromy calculation.

ACT construction

Allcock–Carlson–Toledo:

- $Y \subset \mathbb{P}^3_{\mathbb{C}}$ a complex cubic surface.
- Construct $Z \to \mathbb{P}^3$, the $\boldsymbol{\mu}_3$ cover ramified along Y.
- Their period map is

$$\begin{tikzcd} \mathsf{Cub}_2(\mathbb{C}) \arrow[r, dotted] \arrow[d] & \mathsf{A}_{\mathbb{Z}[\zeta_3],(1,4)}(\mathbb{C}) \arrow[d, hook] \\ \mathsf{Cub}_3(\mathbb{C}) \arrow[r, hook] & \mathsf{A}_5(\mathbb{C}) \end{tikzcd}$$

The arithmetic of ACT

Proposition

There is a diagram of stacks over $\mathbb{Z}[\zeta_3, 1/6]$

$$\begin{tikzcd}
\mathsf{Cub}_3 \arrow[d] & \mathsf{H}(3,3,3) \arrow[l, hook'] \arrow[r, "\kappa"] \arrow[d, hook, "\tau"] & \mathsf{Cub}_2 \arrow[dl, hook, dotted, "\text{``}\underline{\tau}\text{''}"] \\
\mathsf{A}_5 & \mathsf{A}_{\mathbb{Z}[\zeta_3],(1,4)} \arrow[l, hook']
\end{tikzcd}$$

in which $H(3,3,3)$ *is the stack of cyclic triple covers of* $\mathbb{P}^3$ *branched along a cubic surface;* τ *is an open immersion; and* κ *induces an isomorphism of coarse moduli spaces* $\underline{\mathsf{H}} \to \underline{\mathsf{Cub}_2}$.

DvGK construction

Dolgachev–van Geemen–Kondō:

- $Y/\mathbb{C}$ a complex cubic surface.
- Choose a line L on Y.
- For generic hyperplane $H \supset L$, $H \cap Y$ is smooth quadric.
- Discriminant D is a (singular) plane curve of degree five.
- Define a double cover $C \to D$ of degree six.
- Let $Z \to \mathbb{P}^2$ be the double cover ramified along C.
- Then Z is a polarized K3 surface with $\boldsymbol{\mu}_3$-action.
- Compute the periods of Z.

Occult periods (d'apres Kudla and Rapoport)

Given a variety $Y/\mathbb{C}$, try to understand it by:

- Constructing a new variety $Z/\mathbb{C}$; and
- computing periods of Z.

Several examples due to Kondō, Dolgachev, Looijenga, ...

Kudla and Rapoport

In many cases, target of period map is a Shimura variety; occult period map descends to reflex field.

(Lattice) Polarizations

Fix $d \geq 1$.

- R_{2d} moduli of K3 surfaces with primitive polarization, degree $2d$.
-

$$\mathsf{R}_{2d}(S) = \{ (Z \to S, \lambda) : Z \to S \text{ a K3 space}, \\ \lambda \in \mathrm{Pic}_{Z/S}(S) \text{ primitive}, (\lambda, \lambda) = 2d \} .$$

(Lattice) Polarizations

Fix $d \geq 1$.

- R_{2d} moduli of K3 surfaces with primitive polarization, degree $2d$.
- $$\mathsf{R}_{2d}(S) = \{ (Z \to S, \lambda) : Z \to S \text{ a K3 space}, \\ \lambda \in \mathrm{Pic}_{Z/S}(S) \text{ primitive}, (\lambda, \lambda) = 2d \} .$$

Choice of λ is the same as primitive inclusion of lattices:

$$\langle 2d \rangle \overset{\alpha}{\hookrightarrow} \mathrm{Pic}_{Z/S}(S)$$

where $\langle 2d \rangle$ is the lattice of rank 1 with pairing $(2d)$.

Positive cone condition ignored here.

(Lattice) Polarizations

Fix $d \geq 1$.

- R_{2d} moduli of K3 surfaces with primitive polarization, degree $2d$.
-

$$\mathsf{R}_{2d}(S) = \{ (Z \to S, \lambda) : Z \to S \text{ a K3 space}, \\ \lambda \in \mathrm{Pic}_{Z/S}(S) \text{ primitive}, (\lambda, \lambda) = 2d \} .$$

Fix $L \subset L_{K3} := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$ a primitive sublattice of signature $(1, r-1)$.

- R_L the moduli space of L-polarized K3 surfaces:

$$\mathsf{R}_L(S) = \{(Z \to S, \alpha) : \alpha : L \hookrightarrow \mathrm{Pic}_{Z/S}(S) \text{ primitive}\} .$$

Group action

- $\boldsymbol{\mu}_n$ group (scheme) of n^{th} roots of unity.
- $\mathsf{R}^*_{L,\boldsymbol{\mu}_n}$ the space of L-polarized K3 surfaces with action by $\boldsymbol{\mu}_n$:

$$\begin{aligned}\mathsf{R}^*_{L,\boldsymbol{\mu}_n}(S) = \{(Z \to S, \alpha, \rho) : &(Z \to S, \alpha) \in \mathsf{R}_L(S) \\ &\rho : \boldsymbol{\mu}_n \hookrightarrow \mathrm{Aut}_S(Z \to S, \alpha)\ \}\end{aligned}$$

- Data $\underline{\chi} = (\chi, \chi^\omega)$ describes component $\mathsf{R}_{L,\underline{\chi}} \subset \mathsf{R}^*_{L,\boldsymbol{\mu}_n}$ where:
  - $\boldsymbol{\mu}_n$ acts on $H^0(Z, \Omega^2_Z)$ via χ^ω;
  - $\boldsymbol{\mu}_n$ acts on $H^2(Z)$ via χ;
  - $\mathrm{mult}_\chi(\chi_{\mathrm{triv}}) = \mathrm{rank}(L)$.

K3 surfaces have good moduli spaces

Proposition

a. *Let* L *be a lattice of signature* $(1, r-1)$ *and discriminant* Δ_L*. Then* R_L *is a smooth Deligne-Mumford stack over* $\operatorname{Spec} \mathbb{Z}[1/2\Delta_L]$ *of relative dimension* $r-1$*.*
b. *For* $\underline{\chi} = (\chi, \chi^\omega)$, $\mathsf{R}_{L,\underline{\chi}}$ *is a smooth Deligne-Mumford stack over* $\mathbb{Z}[\zeta_n, 1/2\Delta_L n]$ *of relative dimension* $\operatorname{mult}(\chi^\omega) - 1$*.*

We always assume L, $\underline{\chi}$ chosen so that R_L, $\mathsf{R}_{L,\underline{\chi}}$ are nonempty.

Periods for complex K3 surfaces

- $Z/\mathbb{C}$ a K3 surface; middle Hodge numbers $1\ 20\ 1$.
- $H^2(Z,\mathbb{Z}) \cong L_{K3} := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$ signature $(3,19)$
- Choose marking $\phi : H^2(Z,\mathbb{Z}) \xrightarrow{\sim} L_{K3}$, get

$$\phi_{\mathbb{C}}(H^{2,0}(Z)) \in \mathbb{X}_{L_{K3}} = \{[\sigma] \in \mathbb{P}(L_{K3} \otimes \mathbb{C}) : (\sigma,\sigma) = 0, (\sigma,\overline{\sigma}) > 0\}.$$

- Period maps

$$\{\text{marked K3's}/\mathbb{C}\} \longrightarrow \mathbb{X}_{L_{K3}}$$

$$\{\text{K3}/\mathbb{C}\} \longrightarrow \mathrm{O}_{L_{K3}}(\mathbb{Z})\backslash\mathbb{X}_{L_{K3}}$$

where RHS parametrizes polarized Hodge structures on L_{K3} with Hodge numbers $(1,20,1)$.

- Problem: Right-hand side is a terrible space.

Periods for polarized K3 surfaces

- $(Z/\mathbb{C},\lambda) \in \mathsf{R}_{2d}(\mathbb{C})$, ϕ a marking.
- λ gives $\langle 2d\rangle \hookrightarrow H^2(Z,\mathbb{Z}) \xrightarrow{\phi} L_{K3}$.
- Then $H^{20}(Z) \perp c_1(\lambda)$.
- So
$$\phi_{\mathbb{C}}(H^{20}(Z)) \subset \langle 2d\rangle^{\perp} \subset L_{K3}.$$
- Get period map
$$\mathsf{R}_{2d}(\mathbb{C}) \longrightarrow \widetilde{\mathrm{O}}^{\langle 2d\rangle}(\mathbb{Z})\backslash\mathbb{X}^{\langle 2d\rangle} \subset \mathrm{O}_{L_{K3}}(\mathbb{Z})\backslash\mathbb{X}_{L_{K3}}$$

where :
  - $\mathbb{X}^{\langle 2d\rangle}$ means $\mathbb{X}_{\langle 2d\rangle^{\perp}}$, etc.
  - $$1 \longrightarrow \widetilde{\mathrm{O}}_L \longrightarrow \mathrm{O}_L \longrightarrow \mathrm{Aut}(L^{\vee}/L)$$

Torelli for K3 / $\mathbb{C}$

Theorem (Piateskii-Shapiro,Shafarevich)

The period map

$$\mathsf{R}_{2d,\mathbb{C}} \xhookrightarrow{\tau_{2d,\mathbb{C}}} \widetilde{\mathrm{O}}^{\langle 2d\rangle}(\mathbb{Z})\backslash\mathbb{X}^{\langle 2d\rangle}.$$

is an open immersion (of complex orbifolds).

For L-polarized K3, period point is in $L^{\perp} \subset L_{K3}$.

Proposition (Dolgachev, Kondō)

The period map gives an open immersion

$$\mathsf{R}_{L,\mathbb{C}} \xhookrightarrow{\tau_{L,\mathbb{C}}} \widetilde{\mathrm{O}}^{L}(\mathbb{Z})\backslash\mathbb{X}^{L}.$$

Setup

- A Shimura datum is $(G, \mathbb{X})$:
  - $G/\mathbb{Q}$ a reductive group;
  - $\mathbb{X}$ a $G(\mathbb{R})$-conjugacy of homomorphisms $\mathbf{R}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_{\mathbb{R}}$

  (subject to certain axioms)
- If $\mathbb{K} \subset G(\mathbb{A}_f)$ compact open, quotient stack

$$\mathsf{Sh}_{\mathbb{K}}[G, \mathbb{X}] := [G(\mathbb{Q})\backslash(\mathbb{X} \times G(\mathbb{A}_f)/\mathbb{K})]$$

  algebraizes to complex quasiprojective variety.

Canonical models

- $Sh_{\mathbb{K}}[G, \mathbb{X}]$ descends to reflex field $E(G, \mathbb{X})$.
- Let
$$M(\mathbb{K}) = \prod_{p:\mathbb{K}_p \text{is not hyperspecial}} p.$$
Then $Sh_{\mathbb{K}}[G, \mathbb{X}]$ admits canonical integral model over $\mathcal{O}_{E(G,\mathbb{X})}[1/M(\mathbb{K})]$.

Orthogonal Shimura varieties

- L a lattice of signature $(2, n)$
- $G_L = \mathrm{SO}_{L\otimes\mathbb{Q}}$
- $\mathbb{K}_L = \ker G_L(\hat{\mathbb{Z}}) \to \mathrm{Aut}(\mathrm{disc}(L))(\hat{\mathbb{Z}})$

Set

$$\mathsf{Sh}_L = \mathsf{Sh}_{\mathbb{K}_L}[G_L, \mathbb{X}_L]$$

over $\mathbb{Z}[1/2\Delta_L]$.

Lemma

A primitive embedding $L_1 \hookrightarrow L_2$ induces

$$\mathsf{Sh}_{L_1} \xrightarrow{\psi_{L_1 L_2}} \mathsf{Sh}_{L_2}$$

over $\mathbb{Z}[1/2\Delta_{L_1}\Delta_{L_2}]$, with generic fiber a closed embedding.

Unitary Shimura varieties

- K a quadratic imaginary field
- L a free $\mathcal{O}_K$-module with Hermitian form $h(\cdot, \dot{)}$ of signature $(1, r-1)$.
- $G = \mathrm{U}(L, h)$
- $\mathbb{X}_{\mathcal{O}_K,L} \cong \mathbb{B}^{r-1}$
- $\mathbb{K}_{\mathcal{O}_K,L}$ the stabilizer in $G_{\mathcal{O}_K,L}(\mathbb{A}_f)$ of L

Set

$$\mathsf{Sh}_{\mathcal{O}_K,L} = \mathsf{Sh}_{\mathbb{K}_{\mathcal{O}_K,L}}[G_{\mathcal{O}_K,L}, \mathbb{X}_{\mathcal{O}_K,L}].$$

Lemma

$\mathsf{Sh}_{\mathcal{O}_K,L}$ *is the moduli space of abelian varieties of dimension* r *equipped with an action by* $\mathcal{O}_K$ *of signature* $(1, r-1)$*, and a polarization* λ *with* $\ker(\lambda) \cong \mathrm{disc}(L)$.

Also variants for $K/\mathbb{Q}$ CM, arbitrary degree.

Period maps reconsidered

Period map for polarized K3 surfaces is

$$\mathsf{R}_{2d}(\mathbb{C}) \longrightarrow \mathsf{Sh}^{\langle 2d \rangle}(\mathbb{C}) = \mathsf{Sh}_{\langle 2d \rangle^{\perp}}(\mathbb{C}) \ .$$

- $L \hookrightarrow L_{K3}$ primitive of signature $(1, r-1)$
- Set

$$\mathsf{Sh}^{L} = \mathsf{Sh}_{\mathbb{K}^L}[G^L, \mathbb{X}^L] = \mathsf{Sh}_{\mathbb{K}_{L^{\perp}}}[G_{L^{\perp}}, \mathbb{X}_{L^{\perp}}].$$

- Period map becomes

$$\mathsf{R}_L(\mathbb{C}) \xrightarrow{\tau_{L,\mathbb{C}}} \mathsf{Sh}^L(\mathbb{C}).$$

Period maps reconsidered

- Fix $\underline{\chi} = (\boldsymbol{\mu}_n, \chi^\omega, \chi)$.
- Let $E(\underline{\chi}) = \mathbb{Q}(\zeta_n)$.
- $$\mathsf{Sh}^{(L,\underline{\chi})} = \begin{cases} \mathsf{Sh}_{\mathcal{O}_{E(\underline{\chi})}, L^\perp} & n \geq 3 \\ \mathsf{Sh}_{L^\perp} & n = 2 \end{cases}$$
- Period map is
$$\mathsf{R}_{L,\underline{\chi}}(\mathbb{C}) \xrightarrow{\tau_{(L,\underline{\chi}),\mathbb{C}}} \mathsf{Sh}^{(L,\underline{\chi})}(\mathbb{C}) \, .$$

Period maps are arithmetic

Theorem (Rizov, Madapusi Pera, Taelman)

The Piateskii-Shapiro/Shafarevich period map descends to a morphism

$$\mathsf{R}_{2d} \xrightarrow{\tau_{2d}} \mathsf{S}h^{\langle 2d \rangle}$$

over $\mathbb{Z}[1/6d]$.

The transcendental map preserves integral structures.

Periods for structured K3 surfaces

Proposition

For L and $\underline{\chi}$ as before:

- (a) $\tau_{L,\mathbb{C}}$ *is the fiber over* $\mathbb{C}$ *of a morphism*

$$\mathsf{R}_L \xrightarrow{\tau_L} \mathsf{Sh}^L$$

of stacks over $\mathbb{Z}[1/2\Delta(L)]$.

- (b) $\tau_{(L,\underline{\chi}),\mathbb{C}}$ *is the fiber over* $\mathbb{C}$ *of a morphism*

$$\mathsf{R}_{(L,\underline{\chi}),\mathbb{C}} \xrightarrow{\tau_{(L,\underline{\chi})}} \mathsf{Sh}^{(L,\underline{\chi})}$$

of stacks over $\mathcal{O}_{E(\chi)}[1/2n\Delta(L)]$.

Strategy of proof

Descent to $\mathbb{Q}$:

Fix $\langle 2d\rangle \hookrightarrow L$ and level N, and show $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$-equivariance in:

$$\begin{tikzcd} \mathsf{R}_{L,N}(\mathbb{C}) \arrow[r, hook, "\tau_{L,N,\mathbb{C}}"] \arrow[d, hook, "\phi_{L,2d,\mathbb{C}}"] & \mathsf{Sh}^{L}_{N}(\mathbb{C}) \arrow[d, hook, "\psi_{G^L, G^{\langle 2d\rangle}}"] \\ \mathsf{R}_{2d,N}(\mathbb{C}) \arrow[r, hook] & \mathsf{Sh}^{\langle 2d\rangle}_{N}(\mathbb{C}). \end{tikzcd}$$

Spread to $\mathbb{Z}[1/2\Delta(L)]$

Use smoothness of R_L and the extension property of integral canonical models.

Curves of genus 4

Recall from the beginning:

Theorem (Kondō)

There is a holomorphic open immersion

$$\mathsf{N}_4(\mathbb{C}) \xhookrightarrow{\tau_{\mathsf{N}_4}} \Gamma\backslash\mathbb{B}^9$$

Kondō's construction

- $C \in \mathsf{N}_4(\mathbb{C})$
- Canonical model is $C = Q \cap S \subset \mathbb{P}^3$, intersection of quadric and cubic.
- $\varpi : Z \to Q$ triple cover branched along C.
- M_1, M_2 on Q represent two rulings
- $N_i = \varpi^{-1}(M_i)$ are cycle (classes) on Z.
- Each N_i an elliptic curve, and $(N_1, N_2) = 3$.

Then:

- Z is a K3 surface.
- $\mathbb{Z}N_1 + \mathbb{Z}N_2 \hookrightarrow \mathrm{NS}(Z)$ gives

$$L_4 := U(3) \xhookrightarrow{\alpha} \mathrm{Pic}(Z) \subset L_{K3}.$$

- Z has $\boldsymbol{\mu}_3$ action ρ fixing each N_i.

$\boxed{C \rightsquigarrow (Z, \alpha, \rho) \in \mathsf{R}_{(L, \underline{\chi}_4)}}$

Kondō's occult period map is arithmetic

Theorem

There is a diagram of stacks over $\mathcal{O}_{\mathbb{Z}[\zeta_3,1/6]}$

$$\begin{tikzcd} \mathsf{R}_{(L_4,\underline{\chi}_4)} \arrow[r, "\kappa_4"] \arrow[d, "\tau_4"] & \mathsf{N}_4 \\ \mathsf{S}h^{(L_4,\underline{\chi}_4)} & \end{tikzcd}$$

where κ_4 *induces an isomorphism on coarse moduli spaces, and* τ_4 *induces an open immersion* $\mathsf{R}_{(L_4,\underline{\chi}_4)}(\mathbb{C}) \hookrightarrow \mathsf{S}h^{(L_4,\underline{\chi}_4)}(\mathbb{C})$.

Idea

The *inverse* to the occult period map is algebraic:

$$\kappa_4(Z \to S, \alpha, \rho) = Z^{\boldsymbol{\mu}_3} \to S.$$

Curves of genus 3

Theorem

For certain data $(L_3, \underline{\chi}_3)$*, there is a diagram of stacks over* $\mathcal{O}_{\mathbb{Z}[\sqrt{-1},1/2]}$

$$\begin{tikzcd} \mathsf{R}_{(L_3,\underline{\chi}_3)} \arrow[r, "\kappa_3"] \arrow[d, "\tau_3"] & \mathsf{N}_3 \\ \mathsf{S}h^{(L_3,\underline{\chi}_3)} & \end{tikzcd}$$

where κ_3 *induces an isomorphism on coarse moduli spaces, and* τ_3 *induces an open immersion* $\mathsf{R}_{(L_3,\underline{\chi}_3)}(\mathbb{C}) \hookrightarrow \mathsf{S}h^{(L_3,\underline{\chi}_3)}(\mathbb{C})$.

Idea

If $C \in \mathsf{N}_3(\mathbb{C})$, canonical model is a smooth plane quartic curve; $\varpi : Z \to \mathbb{P}^2$ the quartic cover branched along C is an L_3-polarized K3 with $\boldsymbol{\mu}_4$-action.

Five points on a line

- $\widetilde{\mathsf{M}}_{0,5}$ the moduli space of five distinct, ordered points in $\mathbb{P}^1$
- $(P_1, \cdots, P_5) \in \widetilde{\mathsf{M}}_{0,5}(\mathbb{C})$. Kondō constructs
  - $C \to \mathbb{P}^1$ $\boldsymbol{\mu}_5$-cover branched along $(P_1, \cdots, P_5)$;
  - $X \to \mathbb{P}^2$ $\boldsymbol{\mu}_2$-cover branched along C and a $\mathbb{P}^1$
- X is polarized by $L_5 \cong V \oplus A_4(-1) \oplus A_4(-1)$

Five points on a line

Proposition

There is a diagram of stacks over $\mathbb{Z}[\zeta_5, 1/10]$*:*

$$\begin{tikzcd}
\mathsf{R}_{L_5,\underline{\chi}_5} \arrow[r] \arrow[rr, bend left, "\kappa_5"] \arrow[dd, "\tau_{L_5,\underline{\chi}_5}"] & \widetilde{\mathsf{H}}(5,1,5) \arrow[r] \arrow[d, "{[S_5]}"] & \widetilde{\mathsf{M}}_{0,5} \arrow[d, "{[S_5]}"] \\
 & \mathsf{H}(5,1,5) \arrow[r] \arrow[d, "{[\boldsymbol{\mu}_2]}"] & \mathsf{M}_{0,5} \\
\mathsf{Sh}^{(L_5,\underline{\chi}_5)} \arrow[r, "{[\mathbb{K}_0/\mathbb{K}^{(L_5,\underline{\chi}_5)}]}"] & \mathsf{A}_{\mathbb{Z}[\zeta_5],\Sigma} &
\end{tikzcd}$$

Five points on a line

Proposition

There is a diagram of stacks over $\mathbb{Z}[\zeta_5, 1/10]$*:*

$$
\begin{tikzcd}
\mathsf{R}_{L_5,\underline{\chi}_5} \arrow[r] \arrow[rr, bend left, "\kappa_5"] \arrow[dd, "\tau_{L_5,\underline{\chi}_5}"] & \widetilde{\mathsf{H}}(5,1,5) \arrow[r] \arrow[d, "{[S_5]}"] & \widetilde{\mathsf{M}}_{0,5} \arrow[d, "{[S_5]}"] \\
 & \mathsf{H}(5,1,5) \arrow[r] \arrow[d, "{[\boldsymbol{\mu}_2]}"] & \color{red}{\mathsf{M}_{0,5}} \\
\mathsf{Sh}^{(L_5,\underline{\chi}_5)} \arrow[r, "{[\mathbb{K}_0/\mathbb{K}^{(L_5,\underline{\chi}_5)}]}"] & \mathsf{A}_{\mathbb{Z}[\zeta_5],\Sigma} &
\end{tikzcd}
$$

Five points on a rational curve

Five points on a line

Proposition

There is a diagram of stacks over $\mathbb{Z}[\zeta_5, 1/10]$*:*

$$
\begin{tikzcd}
\mathsf{R}_{L_5,\underline{\chi}_5} \arrow[r] \arrow[rr, bend left, "\kappa_5"] \arrow[dd, "\tau_{L_5,\underline{\chi}_5}"] & \widetilde{\mathsf{H}}(5,1,5) \arrow[r] \arrow[d, "{[S_5]}"] & \widetilde{\mathsf{M}}_{0,5} \arrow[d, "{[S_5]}"] \\
 & \mathsf{H}(5,1,5) \arrow[r] \arrow[d, "{[\boldsymbol{\mu}_2]}"] & \mathsf{M}_{0,5} \\
\mathsf{Sh}^{(L_5,\underline{\chi}_5)} \arrow[r, "{[\mathbb{K}_0/\mathbb{K}^{(L_5,\underline{\chi}_5)}]}"] & \mathsf{A}_{\mathbb{Z}[\zeta_5],\Sigma} &
\end{tikzcd}
$$

Five labelled points on $\mathbb{P}^1$

Five points on a line

Proposition

There is a diagram of stacks over $\mathbb{Z}[\zeta_5, 1/10]$*:*

$$
\begin{tikzcd}
\mathsf{R}_{L_5,\underline{\chi}_5} \arrow[r] \arrow[rr, bend left, "\kappa_5"] \arrow[dd, "\tau_{L_5,\underline{\chi}_5}"] & \widetilde{\mathsf{H}}(5,1,5) \arrow[r] \arrow[d, "{[S_5]}"] & \widetilde{\mathsf{M}}_{0,5} \arrow[d, "{[S_5]}"] \\
 & \mathsf{H}(5,1,5) \arrow[r] \arrow[d, "{[\boldsymbol{\mu}_2]}"] & \mathsf{M}_{0,5} \\
\mathsf{Sh}^{(L_5,\underline{\chi}_5)} \arrow[r, "{[\mathbb{K}_0/\mathbb{K}^{(L_5,\underline{\chi}_5)}]}"] & \mathsf{A}_{\mathbb{Z}[\zeta_5],\Sigma} &
\end{tikzcd}
$$

Degree 5 cyclic covers of Brauer-Severi scheme of dimension one with branch locus of degree 5

Five points on a line

Proposition

There is a diagram of stacks over $\mathbb{Z}[\zeta_5, 1/10]$*:*

$$\begin{tikzcd}
\mathsf{R}_{L_5,\underline{\chi}_5} \arrow[r] \arrow[rr, bend left, "\kappa_5"] \arrow[dd, "\tau_{L_5,\underline{\chi}_5}"] & \widetilde{\mathsf{H}}(5,1,5) \arrow[r] \arrow[d, "{[S_5]}"] & \widetilde{\mathsf{M}}_{0,5} \arrow[d, "{[S_5]}"] \\
 & \mathsf{H}(5,1,5) \arrow[r] \arrow[d, "{[\boldsymbol{\mu}_2]}"] & \mathsf{M}_{0,5} \\
\mathsf{Sh}^{(L_5,\underline{\chi}_5)} \arrow[r, "{[\mathbb{K}_0/\mathbb{K}^{(L_5,\underline{\chi}_5)}]}"] & \mathsf{A}_{\mathbb{Z}[\zeta_5],\Sigma} &
\end{tikzcd}$$

Degree 5 cyclic covers with labelled branch locus

Five points on a line

Proposition

There is a diagram of stacks over $\mathbb{Z}[\zeta_5, 1/10]$*:*

$$\begin{tikzcd}
\mathsf{R}_{L_5,\underline{\chi}_5} \arrow[r] \arrow[rr, bend left, "\kappa_5"] \arrow[dd, "\tau_{L_5,\underline{\chi}_5}"] & \widetilde{\mathsf{H}}(5,1,5) \arrow[r] \arrow[d, "{[S_5]}"] & \widetilde{\mathsf{M}}_{0,5} \arrow[d, "{[S_5]}"] \\
 & \mathsf{H}(5,1,5) \arrow[r] \arrow[d, "{[\boldsymbol{\mu}_2]}"] & \mathsf{M}_{0,5} \\
\mathsf{Sh}^{(L_5,\underline{\chi}_5)} \arrow[r, "{[\mathbb{K}_0/\mathbb{K}^{(L_5,\underline{\chi}_5)}]}"] & \mathsf{A}_{\mathbb{Z}[\zeta_5],\Sigma} &
\end{tikzcd}$$

Principally polarized abelian varieties of dimension 6 with an action by $\mathbb{Z}[\zeta_5]$ of signature $\Sigma = \{(2,1),(0,3)\}$

Five points on a line

Proposition

There is a diagram of stacks over $\mathbb{Z}[\zeta_5, 1/10]$*:*

$$\begin{tikzcd}
\mathsf{R}_{L_5,\underline{\chi}_5} \arrow[r] \arrow[rr, bend left, "\kappa_5"] \arrow[dd, "\tau_{L_5,\underline{\chi}_5}"] & \widetilde{\mathsf{H}}(5,1,5) \arrow[r] \arrow[d, "{[S_5]}"] & \widetilde{\mathsf{M}}_{0,5} \arrow[d, "{[S_5]}"] \\
 & \mathsf{H}(5,1,5) \arrow[r] \arrow[d, "{[\boldsymbol{\mu}_2]}"] & \mathsf{M}_{0,5} \\
\mathsf{Sh}^{(L_5,\underline{\chi}_5)} \arrow[r, "{[\mathbb{K}_0/\mathbb{K}^{(L_5,\underline{\chi}_5)}]}"] & \mathsf{A}_{\mathbb{Z}[\zeta_5],\Sigma} &
\end{tikzcd}$$

An isomorphism of coarse moduli spaces

Five points on a line

Proposition

There is a diagram of stacks over $\mathbb{Z}[\zeta_5, 1/10]$*:*

$$\begin{tikzcd} \mathsf{R}_{L_5,\underline{\chi}_5} \arrow[r] \arrow[rr, bend left, "\kappa_5"] \arrow[dd, "\tau_{L_5,\underline{\chi}_5}"] & \widetilde{\mathsf{H}}(5,1,5) \arrow[r] \arrow[d, "{[S_5]}"] & \widetilde{\mathsf{M}}_{0,5} \arrow[d, "{[S_5]}"] \\ & \mathsf{H}(5,1,5) \arrow[r] \arrow[d, "{[\boldsymbol{\mu}_2]}"] & \mathsf{M}_{0,5} \\ \mathsf{Sh}^{(L_5,\underline{\chi}_5)} \arrow[r, "{[\mathbb{K}_0/\mathbb{K}^{(L_5,\underline{\chi}_5)}]}"] & \mathsf{A}_{\mathbb{Z}[\zeta_5],\Sigma} & \end{tikzcd}$$

Fiber over $\mathbb{C}$ is an open immersion

Five points on a line

Proposition

There is a diagram of stacks over $\mathbb{Z}[\zeta_5, 1/10]$*:*

$$
\begin{tikzcd}
\mathsf{R}_{L_5,\underline{\chi}_5} \arrow[r] \arrow[rr, bend left, "\kappa_5"] \arrow[dd, "\tau_{L_5,\underline{\chi}_5}"] & \widetilde{\mathsf{H}}(5,1,5) \arrow[r] \arrow[d, "{[S_5]}"] & \widetilde{\mathsf{M}}_{0,5} \arrow[d, "{[S_5]}"] \\
 & \mathsf{H}(5,1,5) \arrow[r] \arrow[d, "{[\boldsymbol{\mu}_2]}"] & \mathsf{M}_{0,5} \\
\mathsf{Sh}^{(L_5,\underline{\chi}_5)} \arrow[r, "{[\mathbb{K}_0/\mathbb{K}^{(L_5,\underline{\chi}_5)}]}"] & \mathsf{A}_{\mathbb{Z}[\zeta_5],\Sigma} &
\end{tikzcd}
$$

Quotient by finite group

Six points on a line

Proposition

There is a diagram of stacks over $\mathbb{Z}[\zeta_3, 1/6]$*:*

$$\begin{tikzcd}
\mathsf{R}_{L'_6,\underline{\chi}'_6} \arrow[r] \arrow[rr, bend left, "\kappa'_6"] \arrow[dd, "\tau_{L'_6,\underline{\chi}'_6}"] & \widetilde{\mathsf{H}}(3,1,6) \arrow[r] \arrow[d, "{[S_6]}"] & \widetilde{\mathsf{M}}_{0,6} \arrow[d, "{[S_6]}"] \\
 & \mathsf{H}(3,1,6) \arrow[r] \arrow[d, "{[\boldsymbol{\mu}_2]}"] & \mathsf{M}_{0,5} \\
\mathsf{Sh}^{(L'_6,\underline{\chi}'_6)} \arrow[r, "{[\mathbb{K}_0/\mathbb{K}^{(L'_6,\underline{\chi}'_6)}]}"] & \mathsf{A}_{\mathbb{Z}[\zeta_3],(2,1)} &
\end{tikzcd}$$

Whole books have been devoted to the configuration of the 27 lines on a smooth cubic surface (Henderson [1]; Segre [2]). Their elegant symmetry both enthrals and at the same time irritates; what use is it to know, for instance, the number of coplanar triples of such lines (forty five) or the number of double Schläffli sixfolds (thirty six)? The answer to this rhetorical

Yu. Manin, Cubic forms: algebra, geometry, arithmetic, 1986.

- Z/k a cubic surface over an algebraically closed field.
- Z contains exactly 27 lines.
- What happens over arithmetic fields?

Early results

- $Z/\mathbb{C}$ has exactly 27 lines Cayley-Salmon 1849
- Automorphism group of 27 lines is a group of order $51,840$ (isomorphic to $W(E_6)$) Jordan 1869
- Jordan interprets this as the Galois group of the 27 lines on a general complex cubic surface.

Modern reformulation

- $\mathsf{Cub}_2{}^{\mathrm{m}}$ the moduli space of cubic surfaces with a marking of the 27 lines.
- $\mathsf{Cub}_2{}^{\mathrm{m}} \to \mathsf{Cub}_2$ the forgetful map.

Theorem (Jordan 1870)

$\mathsf{Cub}_{2\mathbb{C}}^{\mathrm{m}}$ *is irreducible, and*

$$\mathsf{Cub}_{2\mathbb{C}}^{\mathrm{m}} \longrightarrow \mathsf{Cub}_{2\mathbb{C}}$$

is Galois with group $W(E_6)$.

Modern reformulation

- $\mathsf{Cub2}^{\mathrm{m}}$ the moduli space of cubic surfaces with a marking of the 27 lines.
- $\mathsf{Cub2}^{\mathrm{m}} \to \mathsf{Cub2}$ the forgetful map.

Theorem (Jordan 1870)

$\mathsf{Cub2}^{\mathrm{m}}_{\mathbb{C}}$ *is irreducible, and*

$$\mathsf{Cub2}^{\mathrm{m}}_{\mathbb{C}} \longrightarrow \mathsf{Cub2}_{\mathbb{C}}$$

is Galois with group $W(E_6)$.

Lines and torsion points

Abel–Jacobi map identifies differences of lines on Z with $1-\zeta_3$ torsion on J^3_Z.

$$\begin{tikzcd}
\mathsf{Cub_2}^{\mathrm{m}} \arrow[r, hook] \arrow[d, "{[W(E_6)]}"] & \mathsf{A}^{[1-\zeta_3]}_{\mathbb{Z}[\zeta_3,1/6],(1,4)} \arrow[d, "{[\mathrm{PO}_5(\mathbb{F}_3)]}"] \\
\mathsf{Cub_2} \arrow[r, hook] & \mathsf{A}_{\mathbb{Z}[\zeta_3,1/6],(1,4)}
\end{tikzcd}$$

Beyond $\mathbb{C}$: Hilbertian fields

Proposition

K Hilbertian, $\mathrm{char}(K) \neq 2, 3$, Z/K a sufficiently general cubic surface. Then

$$\mathrm{Gal}(K(\mathrm{lines}(Z))/K) \cong W(E_6).$$

Proof.

- $\mathsf{Cub2}^{\mathrm{m}}_{2\mathbb{C}}$ is irreducible Jordan
- $\mathsf{A}^{[1-\zeta_3]}_{\mathbb{Z}[\zeta_3],(1,4)}$ admits toroidal compactification Lan
- Each $\mathsf{A}_{\mathbb{Z}[\zeta_3],(1,4),\kappa(\mathfrak{p})}$ irreducible (Zariski connectedness)
- Now use Hilbert irreducibility.

$\square$

Beyond $\mathbb{C}$: finite fields

Lemma

$\mathbb{F}_q$ is a large finite field of characteristic ≥ 5.
The expected number of lines on a random cubic surface over $\mathbb{F}_q$ is ≈ 1.

Proof.

$\mathsf{Cub}_2{}^{\mathrm{l}}$: space of cubic surfaces equipped with a choice of line. All geometric fibers are irreducible:

$$\begin{tikzcd}
& \mathsf{Cub}_2{}^{\mathrm{m}} \arrow[dl] \arrow[dd] \\
\mathsf{Cub}_2{}^{\mathrm{l}} \arrow[dr] & \\
& \mathsf{Cub}_2
\end{tikzcd}$$

Now use Lang-Weil. □

Beyond $\mathbb{C}$: finite fields

Lemma

$\mathbb{F}_q$ is a large finite field of characteristic ≥ 5.
The expected number of lines on a random cubic surface over $\mathbb{F}_q$ is ≈ 1.

What about the other direction?

- The moduli space of cubic surfaces is well-known to be rational.
- R. Das has explicit, precise line counts over finite fields.

What do facts like these tell us about the Shimura variety?

Thanks!