Sums of triangular numbers and sums of squares

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Historical background

- For $s \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$, let

$$r_s(n) = \#\{(x_1, x_2, \cdots, x_s) \in \mathbb{Z}^s;\ n = x_1^2 + x_2^2 + \cdots + x_s^2\}.$$

- $5 = (\pm 2)^2 + (\pm 1)^2 = (\pm 1)^2 + (\pm 2)^2$; $r_2(5) = 8$.

- $2021 = (?)^2 + (?)^2$; $r_2(2021) =?$.

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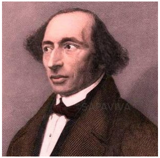
A sample problem

Problem

- Express $r_s(n)$ in terms of known (simpler) arithmetical functions of n.

Jacobi's two-square theorem



- $d_1(n) = \#$ of divisors of n in the form $4m + 1$.
- $d_3(n) = \#$ of divisors of n in the form $4m + 3$.

Theorem (Jacobi)

We have

$$r_2(n) = 4(d_1(n) - d_3(n)).$$

- $r_2(2021) = r_2(43 \times 47) = 4(2 - 2) = 0.$

The theta series $\vartheta(x)$

- $\vartheta(x) := \sum_{n=-\infty}^{\infty} x^{n^2} = 1 + 2x + 2x^4 + \cdots .$
- $\vartheta^2(x) := \left(\sum_{n=-\infty}^{\infty} x^{n^2}\right)^2 = r_2(0)+r_2(1)x+\cdots+r_2(n)x^n+\cdots .$
- Jacobi proved his theorem by showing that

$$\vartheta^2(x) = 1 + 4\left(\frac{x}{1-x} - \frac{x^3}{1-x^3} + \frac{x^5}{1-x^5} - \frac{x^7}{1-x^7} + \cdots\right).$$

The theta series $\vartheta(x)$

- Jacobi further proved that

$$\vartheta^4(x) = 1 + 8\left(\frac{x}{1-x} + \frac{2x^2}{1+x^2} + \frac{3x^3}{1-x^3} - \frac{4x^4}{1+x^4} + \cdots\right).$$

Jacobi's four-square theorem

- $\sigma(n)$= the sum of divisors of n.
- $\sigma^{\text{o}}(n)$= the sum of odd divisor of n.

Theorem (Jacobi)

If n is odd

$$r_4(n) = 8\sigma(n),$$

and if n is even

$$r_4(n) = 24\sigma^{\text{o}}(n).$$

- $r_4(2n+1) = 8\sigma(2n+1)$.
- $r_4(8n+4) = r_4(4(2n+1)) = 24\sigma^{\text{o}}(4(2n+1)) = 24\sigma(2n+1)$.
- $r_4(8n+4) = 3r_4(2n+1)$.

Digression

$$\mathbb{Z}^{\mathrm{o}} = \mathbb{Z}^{\mathrm{odd}}$$

$$r_4^*(n) = \#\{(x_1, x_2, x_3, x_4) \in (\mathbb{Z}^{\mathrm{o}})^4\,;\, x_1^2 + x_2^2 + x_3^2 + x_4^2 = n\}.$$

- The square of any integer mod 8 is congruent to 1, 0, or 4.
- $r_4(8n+4) = r_4^*(8n+4) + r_4(2n+1)$.
- $3r_4(2n+1) = r_4^*(8n+4) + r_4(2n+1)$.

$$r_4(2n+1) = \frac{1}{2} r_4^*(8n+4).$$

The number of representations of an odd number as as a sum of 4 squares is half of the number of representations of four times that number as a sum of 4 odd squares.

Back to the sums of squares

- Jacobi found formulas for $s = 6$ and $s = 8$.
- Liuoville did the case $s = 10$.
- Eisenstein found formulas for $s = 12$.

Ramanujan

$\sigma_{11}^*(n) = \sigma_{11}(n) = \sum_{d|n} d^{11}$ if n is odd.
$\sigma_{11}^*(n) = \sigma_{11}^{\mathrm{e}}(n) - \sigma_{11}^{\mathrm{o}}(n)$ if n is even.

Theorem (Ramanujan)

We have

$$r_{24}(n) = \frac{16}{691}\sigma_{11}^*(n) + e_{24}(n),$$

where

$$e_{24}(n) = \frac{128}{691}\{(-1)^{n-1}259\tau(n) - 512\tau(n/2)\}.$$

Here $\tau(n)$ is the Ramanujan's tau function.

Ramanujan

Theorem (Ramanujan)

There is a divisor function $\delta_{2s}(n)$ *for which*

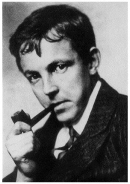
$$r_{2s}(n) = \delta_{2s}(n) + e_{2s}(n),$$

where

$$e_{2s}(n) = o(\delta_{2s}(n)),$$

as $n \to \infty$.

Singular series



Hardy (and Mordell) used the Hardy-Littlewood circle method to express $\delta_s(n)$ (for $s \geq 5$) in terms of the singularities of a certain complex function to get

$$r_s(n) = \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})} n^{\frac{s}{2}-1} \sum_{k=1}^{\infty} \sum_{\substack{h \mod k \\ (h,k)=1}} \left\{ \frac{\sum_{j=1}^{k} e^{\frac{2\pi i h j^2}{k}}}{k} \right\}^s e^{\frac{-2\pi i h m}{k}} + O(m^{\frac{s}{4}}).$$

Triangular Numbers

- For $s \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$, let

$$t_s(n) = \#\left\{(x_1, \ldots, x_s) \in \mathbb{Z}^s; \;\; n = \frac{x_1(x_1-1)}{2} + \cdots + \frac{x_s(x_s-1)}{2}\right\}.$$

- There is a one to one correspondence between the integer solutions of

$$n = \frac{x_1(x_1-1)}{2} + \cdots + \frac{x_s(x_s-1)}{2}$$

and the integer solutions of

$$8n + s = (2x_1 - 1)^2 + \cdots + (2x_s - 1)^2.$$

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Triangular Numbers

- Let $r_s^*(n)$ be the number of representations of n as a sum of s odd squares. Then,

$$t_s(n) = r_s^*(8n+s).$$

- The number of representations of integers as a sum of triangular numbers is related to the number of representations of an arithmetic progression of integers as a sum of odd squares.
- The number $t_s(n)$ also can be described as the number of lattice points on the ellipsoid

$$(2x_1-1)^2+\cdots+(2x_s-1)^2 = 8n+s.$$

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$$(2x_1-1)^2 + \cdots + (2x_s-1)^2 = 8n+s.$$

Goal

To describe a method for computing the δ part without appealing to the singular series.

A specific problem/Set up

- For non-negative integers a, b, and n, let

$$r(a,b;n) = \#\{(x_1,\ldots,x_a,y_1,\ldots,y_b) \in \mathbb{Z}^{a+b};\ n = x_1^2+\cdots+x_a^2+3y_1^2+\cdots+3y_b^2\}$$

and

$$t(a,b;n) = \#\Big\{(x_1,\ldots,x_a,y_1,\ldots,y_b) \in \mathbb{Z}^{a+b};\ n = \frac{x_1(x_1-1)}{2}+\cdots+\frac{x_a(x_a-1)}{2}+3\frac{y_1(y_1-1)}{2}+\cdots+3\frac{y_b(y_b-1)}{2}\Big\}.$$

- We also consider the following two related functions:
  $r^*(a,b;n)$ is the number of representations of n as a sum of odd squares with coefficients 1 and 3 (a of ones and b of threes).
  $\tilde{r}(a,b;n)$ is the number of representations of n as a sum of squares with coefficients 1 and 3 (a of ones and b of threes) and at least one odd component.

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  and

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Relations between r, t, r^*, and $\tilde{r}$

We have

- $r^*(a,b;n) \leq \tilde{r}(a,b;n) \leq r(a,b;n)$
- $t(a,b;n) = r^*(a,b;8n+a+3b)$
- $\tilde{r}(a,b;8n+a+3b) =$
$$\begin{cases} r(a,b;8n+a+3b) & if \;\; a+3b \not\equiv 0 \pmod 4, \\ r(a,b;8n+a+3b) - r(a,b;\frac{8n+a+3b}{4}) & if \;\; a+3b \equiv 0 \pmod 4. \end{cases}$$

Questions

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What can we say about the following proportions?

$$\frac{r^*(a,b;8n+a+3b)}{r(a,b;8n+a+3b)} \text{ and } \frac{r^*(a,b;8n+a+3b)}{\tilde{r}(a,b;8n+a+3b)}.$$

The main theorem

Theorem (A.-Aygin, 2021)

Let $a > 1$ and $b \geq 0$ be integers such that $a + b \equiv 0 \pmod 2$ and assume $a + b \geq 4$. The following assertions hold:

(i) *If $a + 3b \not\equiv 0 \pmod 8$, then*

$$\lim_{n\to\infty} \frac{r^*(a,b;8n+a+3b)}{r(a,b;8n+a+3b)} = \frac{2}{2^{a+b-2} + (-1)^b 2^{\frac{a+b-2}{2}} \cos(\frac{a+3b}{4}\pi) + 1}.$$

(ii) *If $a + 3b \equiv 0 \pmod 4$, then*

$$\lim_{n\to\infty} \frac{r^*(a,b;8n+a+3b)}{\tilde{r}(a,b;8n+a+3b)} = \frac{2}{2^{a+b-2} + (-1)^b 2^{\frac{a+b-2}{2}} \cos(\frac{a+3b}{4}\pi)}.$$

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(i)′ *If $a + 3b \equiv 0 \pmod 8$, then, for fixed non-negative integer ν,*

$$\lim_{\substack{n \to \infty \\ 2^\nu || 8n+a+3b}} \frac{r^*(a, b; 8n + a + 3b)}{r(a, b; 8n + a + 3b)} = \frac{2\delta(a, b, \nu)}{2^{a+b-2} + (-2)^{\frac{a+b-2}{2}} + 1},$$

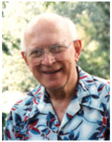
where $\delta(a, b, \nu)$ is an explicit expression depending on a, b, and ν.

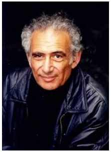
Bateman-Knopp's lemma

Lemma (Bateman-Knopp, 1998)

For $1 \leq a \leq 7$, we have

$$r(a, 0; 8n + a) = \left[1 + \frac{1}{2}\binom{a}{4}\right] r^*(a, 0; 8n + a).$$





Proof.

Note that if x is odd, then $x^2 \equiv 1 \pmod 8$, and if x is even, then $x^2 \equiv 0$ or 4 (mod 8).
If $a = 1, 2, 3$ and $8n + a$ is a sum of a squares, then all squares should be odd. Thus,

$$r(a, 0; 8n + a) = r^*(a, 0; 8n + a).$$

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(Continuation...)

If $4 \leq a \leq 7$ and $8n+a$ is a sum of a squares, then either all of the squares are odd or exactly 4 are even and the rest $a-4$ are odd. Thus,

$$r(a,0;8n+a) = r^*(a,0;8n+a) + \binom{a}{4} \sum_{\substack{k_i \ odd \\ k_1^2+\cdots+k_{a-4}^2 \leq 8n+a}} r\left(4,0;\frac{8n+a-(k_1^2+\cdots+k_{a-4}^2)}{4}\right).$$

Note that $8n+a-(k_1^2+\cdots+k_{a-4}^2) \equiv 4 \pmod 8$. Thus,

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This is the same as

$$r(a,0;8n+a) = \left[1 + \frac{1}{2}\binom{a}{4}\right] r^*(a,0;8n+a).$$

The case $a \geq 8$

Theorem (Bateman-Datskovsky-Knopp, 2001)

(i) For $a \geq 8$, the ratio

$$\frac{r(a,0;8n+a)}{r^*(a,0;8n+a)}$$

is not constant.

(ii) If $a \not\equiv 0 \pmod 8$, then

$$\lim_{n\to\infty} \frac{r^*(a,0;8n+a)}{r(a,0;8n+a)} = \frac{2}{2^{a-2} + 2^{\frac{a-2}{2}} \cos(\frac{a}{4}\pi) + 1}.$$

(iii) If $a \equiv 0 \pmod 8$, then the limit in (ii) does not exist.

The case $1 \leq a + 3b \leq 7$

Theorem (Adiga-Cooper-Han, 2005)

For non-negative integers a, b *with* $1 \leq a + 3b \leq 7$ *we have*

$$r^*(a, b; 8n + a + 3b) = \frac{2}{2 + \binom{a}{4} + ab} r(a, b; 8n + a + 3b).$$

Note

Their theorem is more general and treats all the partitions of integers between 1 and 7.

The case $a + 3b = 8$

Theorem (Baruah-Cooper-Hirschhorn, 2008)

For non-negative integers a, b with $a + 3b = 8$ we have

$$r^*(a, b; 8n + a + 3b) = \frac{2}{2 + \binom{a}{4} + ab}\tilde{r}(a, b; 8n + a + 3b).$$

Note

Their theorem is more general and treats all the partitions of 8.

Two identities

For non-negative integers with a, b with $1 \leq a+3b \leq 7$ we have

$$\frac{2}{2+\binom{a}{4}+ab} = \frac{2}{2^{a+b-2}+(-1)^b 2^{\frac{a+b-2}{2}}\cos(\frac{a+3b}{4}\pi)+1}.$$

For non-negative integers with a, b with $a+3b=8$ we have

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Back to the sum of squares

- Recall the theta function $\vartheta(x)$ and set $x = q = e^{2\pi iz}$, where $z \in \mathcal{H} = \{z \in \mathbb{C};\ \Im(z) > 0\}$. Thus,

$$\vartheta(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2q + 2q^4 + \cdots .$$

- Let $\Gamma_0(4) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right);\ 4 \mid c \text{ and } ad - bc = 1\}$.
- Let $\chi_{-4} = \left(\frac{-4}{*}\right)$ be the Kronecker symbol attached to discriminant -4.

For $s = 4k + 2$ we have

$$\vartheta^s(q) \in M_{2k+1}(\Gamma_0(4), \chi_{-4}),$$

i.e., $\vartheta^s(q)$ is a modular form of weight $2k + 1$, level 4, and character χ_{-4}.

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Back to the sum of squares

For $s = 4k + 2$ let

$$\psi_s(q) = 1 + \sum_{n=1}^{\infty} \delta_s(n) q^n,$$

where $\delta_s(n)$ is the singular series at n. Then

$$\psi_s(q) \in M_{2k+1}(\Gamma_0(4), \chi_{-4}),$$

and moreover $\vartheta^s(q) - \psi_s(q)$ vanishes at all cusps of $\Gamma_0(4)$.

Back to the sum of squares

The delta part of $r_s(n)$ is the n-the coefficient of the Fourier expansion of the Eisenstein component of $\vartheta^s(q)$.

Modular forms

- A modular form f of weight k, level N, and character χ is a holomorphic function defined on the upper half plane that satisfies a certain transformation property (under $\Gamma_0(N)$) and it is holomorphic at the cusps of $\Gamma_0(N)$.
- If f vanishes at all the cusps, it is called a cusp form.

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Modular forms

- $M_k(\Gamma_0(N), \chi) = E_k(\Gamma_0(N), \chi) \oplus S_k(\Gamma_0(N), \chi)$.
- Any $f \in M_k(\Gamma_0(N), \chi)$ can be written in a unique was as $f = E_f + C_f$, where $E_f \in E_k(\Gamma_0(N), \chi)$ and $C_f \in S_k(\Gamma_0(N), \chi)$.

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Modular forms

The δ part

For $f(z)=\sum_{n=0}^{\infty} a_f(n)q^n$, $E_f(z)=\sum_{n=0}^{\infty} e_f(n)q^n$, and $C_f(z)=\sum_{n=0}^{\infty} c_f(n)q^n$, we have

$$a_f(n)=e_f(n)+c_f(n).$$

Moreover,

$$c_f(n)=O(n^{\frac{k}{2}}).$$

A basis for $E_k(\Gamma_0(N), \chi)$

- The dimension of the space $E_k(\Gamma_0(N), \chi)$ is equal to the number of certain cusps of $\Gamma_0(N)$.
- Thus,

$$\dim(E_k(\Gamma_0(N), \chi)) \leq \# \text{ of cusps of } \Gamma_0(N).$$

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- The dimension of the space $E_k(\Gamma_0(N), \chi)$ is equal to the number of certain cusps of $\Gamma_0(N)$.
- Thus,

$$\dim(E_k(\Gamma_0(N), \chi)) \leq \sum_{d|N} \phi\left(\gcd(d, \frac{n}{d})\right).$$

A basis for $E_k(\Gamma_0(N), \chi)$

The weight k Eisenstein series associated with the Dirichlet characters ϵ and ψ is defined as

$$E_k(z; \epsilon, \psi) := \epsilon(0) - \frac{2k}{B_{k,\chi}} \sum_{n=1}^{\infty} \sigma_{k-1}(n; \epsilon, \psi) e^{2\pi i n z},$$

where $\chi = \epsilon \cdot \psi$, $B_{k,\chi}$ is the k-th Bernoulli number associated with characters χ, and

$$\sigma_{k-1}(n; \epsilon, \psi) = \sum_{1 \leq d | n} \epsilon(n/d) \psi(d) d^{k-1}.$$

A basis for $E_k(\Gamma_0(N), \chi)$

Theorem (Weisinger, 1977)

For integer $k \geq 3$

$$\mathcal{E}_k(\Gamma_0(N), \chi) = \{E_k(dz; \epsilon, \psi);\ \epsilon \cdot \psi = \chi \text{ and } \mathrm{cond}(\epsilon)\mathrm{cond}(\psi) d \mid N\}$$

forms a basis for the space $E_k(\Gamma_0(N), \chi)$.

Note

Explicit basis for $k = 1$ and $k = 2$ are also known.

Notation

Notation

$[0]_{a/c} f(z)$= the constant in the Fourier expansion of $f(z)$ at the cusp a/c.

A procedure for finding the Eisenstein component of $f \in M_k(\Gamma_0(N), \chi)$

- We like to find the coefficients $c_{d,\epsilon,\psi}$ such that

$$f(z) = \sum_{E_k(dz;\epsilon,\psi) \in \mathcal{E}_k(\Gamma_0(N),\chi)} c_{d,\epsilon,\psi} E_k(dz;\epsilon,\psi) + C(z),$$

where $C(z) \in S_k(\Gamma_0(N))$.

1. Compute $[0]_{a/c} f(z)$ at all the cusps a/c.
2. Compute $[0]_{a/c} E_k(dz;\epsilon,\psi)$ at all the cusps a/c.
3. Form and solve the system

$$[0]_{a/c} f(z) = \sum_{E_k(dz;\epsilon,\psi) \in \mathcal{E}_k(\Gamma_0(N),\chi)} c_{d,\epsilon,\psi} \left([0]_{a/c} E_k(dz;\epsilon,\psi) \right),$$

for all regular cusps a/c.

The Eisenstein component can be explicitly computed

Theorem (Aygin, 2021)

The system in Step 3 of the procedure has a unique solution that can be expressed in terms of the data in Step 1.

Proof.

See Theorem 3 of arXiv.2102.04278. □

The main theorem

Theorem (A.-Aygin, 2021)

Let $a > 1$ *and* $b \geq 0$ *be integers such that* $a + b \equiv 0 \pmod 2$ *and assume* $a + b \geq 4$. *The following assertions hold:*

(i) *If* $a + 3b \not\equiv 0 \pmod 8$, *then*

$$\lim_{n\to\infty} \frac{r^*(a,b;8n+a+3b)}{r(a,b;8n+a+3b)} = \frac{2}{2^{a+b-2} + (-1)^b 2^{\frac{a+b-2}{2}} \cos(\frac{a+3b}{4}\pi) + 1}.$$

(ii) *If* $a + 3b \equiv 0 \pmod 4$, *then*

$$\lim_{n\to\infty} \frac{r^*(a,b;8n+a+3b)}{\tilde{r}(a,b;8n+a+3b)} = \frac{2}{2^{a+b-2} + (-1)^b 2^{\frac{a+b-2}{2}} \cos(\frac{a+3b}{4}\pi)}.$$

The main theorem

Theorem (A.-Aygin, 2021)

Let $a > 1$ and $b \geq 0$ be integers such that $a + b \equiv 0 \pmod 2$ and assume $a + b \geq 4$. The following assertion hold:

(i)′ *If $a + 3b \equiv 0 \pmod 8$, then, for fixed non-negative integer ν,*

$$\lim_{\substack{n\to\infty \\ 2^{\nu} || 8n+a+3b}} \frac{r^*(a,b;8n+a+3b)}{r(a,b;8n+a+3b)} = \frac{2\delta(a,b,\nu)}{2^{a+b-2} + (-2)^{\frac{a+b-2}{2}} + 1},$$

where $\delta(a,b,\nu)$ is an explicit expression depending on a, b, and ν.

Steps of the proof (for r^* and r)

1. Forming the ϑ-series of r^* and r in appropriate space of modular forms.
2. Applying Aygin's theorem to explicitly compute the Eisenstein components of the theta series of r^* and r.
3. Use Step 2 in finding a relation between the coefficients of the Eisenstein components.
4. Proving that the Eisenstein components are dominant and establishing the asymptotic result.

Step 1: Forming the ϑ series

If $\vartheta(z) = \sum_{m=-\infty}^{\infty} q^{m^2}$, then

$$\sum_{n=0}^{\infty} r(a,b;n)q^n = \vartheta^a(z)\vartheta^b(3z).$$

If $\Psi(z) = \sum_{m=1}^{\infty} q^{m(m-1)/2+1/8}$ and $\Psi_8(z) = \Psi(8z), then$

$$\sum_{n=0}^{\infty} r^*(a,b;8n+a+3b)q^{8n+a+3b} = 2^{a+b}\Psi_8^a(z)\Psi_8^b(3z).$$

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Step 1: Forming the ϑ series

Let $a+b$ be even and $k = \lfloor \frac{a+b}{4} \rfloor$.

$$\Psi_8^a(z)\Psi_8^b(3z) \in \begin{cases} M_{2k}(\Gamma_0(48), \chi_1) & \text{if } a,b \equiv 0 \pmod 2 \text{ and } a+b \equiv 0 \pmod 4, \\ M_{2k+1}(\Gamma_0(48), \chi_{-3}) & \text{if } a,b \equiv 1 \pmod 2 \text{ and } a+b \equiv 2 \pmod 4, \\ M_{2k+1}(\Gamma_0(48), \chi_{-4}) & \text{if } a,b \equiv 0 \pmod 2 \text{ and } a+b \equiv 2 \pmod 4, \\ M_{2k}(\Gamma_0(48), \chi_{12}) & \text{if } a,b \equiv 1 \pmod 2 \text{ and } a+b \equiv 0 \pmod 4. \end{cases}$$

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Step 2: Applying Aygin's theorem

Proposition

Let $a, b \in \mathbb{N}_0$ be such that $a + b = 4k \geq 4$ and both a, b are even. Then, for some cusp form $C_1(z) \in S_{2k}(\Gamma_0(48), \chi_1)$, we have

$$\begin{aligned}\Psi_8^a(z)\Psi_8^b(3z) = {} & a_{1,4}\left(((-3)^{a/2} - 1)E_{2k}(4z; \chi_1, \chi_1) + (3^{2k} - (-3)^{a/2})E_{2k}(12z; \chi_1, \chi_1)\right) \\ & + a_{1,8}\left(((-3)^{a/2} - 1)E_{2k}(8z; \chi_1, \chi_1) + (3^{2k} - (-3)^{a/2})E_{2k}(24z; \chi_1, \chi_1)\right) \\ & + a_{1,16}\left(((-3)^{a/2} - 1)E_{2k}(16z; \chi_1, \chi_1) + (3^{2k} - (-3)^{a/2})E_{2k}(48z; \chi_1, \chi_1)\right) \\ & + C_1(z),\end{aligned}$$

where

$$\begin{aligned} a_{1,4} &= \frac{((-1)^{(a+3b)/4} - 1)}{2^{4k}(2^{2k} - 1)(3^{2k} - 1)}, \\ a_{1,8} &= \frac{(2^{2k} + 1 - (-1)^{(a+3b)/4})}{2^{4k}(2^{2k} - 1)(3^{2k} - 1)}, \\ a_{1,16} &= -\frac{2^{2k}}{2^{4k}(2^{2k} - 1)(3^{2k} - 1)}. \end{aligned}$$

Step 3: A relation between Eisenstein components

We write

$$\Psi_8^a(z)\Psi_8^b(3z) = \sum_{n=0}^{\infty} \alpha_n q^n + \sum_{n=0}^{\infty} \gamma_n q^n$$

and

$$\varphi^a(z)\varphi^b(3z) = \sum_{n=0}^{\infty} \beta_n q^n + \sum_{n=0}^{\infty} \gamma'_n q^n,$$

where α_n and β_n are given explicitly in terms of the generalized divisor functions and

$$\gamma_n = O(n^{(a+b)/4}) \text{ and } \gamma'_n = O(n^{(a+b)/4}).$$

Step 3: A relation between Eisenstein components

Theorem

Let $a, b \in \mathbb{N}_0$ be such that $a+b \equiv 0 \pmod 2$ and let $a+b \geq 4$. Then for all $n \in \mathbb{N}$, if $a+3b \not\equiv 0 \pmod 8$, we have

$$\alpha_{8n+a+3b} = \frac{2\beta_{8n+a+3b}}{2^{a+b}(2^{a+b-2} + (-1)^b 2^{\frac{a+b-2}{2}} \cos(\frac{a+3b}{4}\pi) + 1)}.$$

Step 4: Establishing the asymptotic results

Proposition

We have $n^{\frac{a+b}{4}} = o(|\alpha_{8n+a+3b}|)$, *as* $n \to \infty$.

Step 3 together with this proposition imply the result.

Theorem (A.-Aygin, 2021)

Let $a > 1$ *and* $b \geq 1$ *be integers such that* $a + b \equiv 0 \pmod 2$ *and assume* $a + b \geq 4$. *If* $a + 3b \not\equiv 0 \pmod 8$, *then*

$$\lim_{n\to\infty} \frac{r^*(a,b;8n+a+3b)}{r(a,b;8n+a+3b)} = \frac{2}{2^{a+b-2} + (-1)^b 2^{\frac{a+b-2}{2}} \cos(\frac{a+3b}{4}\pi) + 1}.$$

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