Upper bounds on the number of rigid graph embeddings

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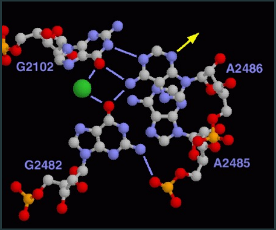


Minimally generically rigid graphs

- A graph is (generically) rigid if the number of embeddings (for generic edge lengths) is finite modulo rigid transforms.
- If G is rigid, and $G \setminus \{e\}, \forall e \in E$, is not rigid, then G is *minimally rigid*.
- A Euclidean embedding of a simple undirected weighted graph $G = (V, E)$ is a map $\rho \colon V \to \mathbb{R}^d$, respecting the edge lengths $\lambda_{u,v} = \|\rho(u) - \rho(v)\|_2$, for all $(u, v) \in E$.
- Complex embeddings: $\rho \colon V \to \mathbb{C}^d$, s.t. $\lambda_{u,v}^2 = \sum_{i=1}^{d} (\rho(u)_i - \rho(v)_i)^2$, for all $(u, v) \in E$.

Edge count

Theorem (Maxwell:1864)

If $G = (V, E)$ is generically minimally rigid in $\mathbb{R}^d$, and $|V| = n$, then

- $|E| = d \cdot n - \binom{d+1}{2}$, *and*
- $|E'| \leq d \cdot |V'| - \binom{d+1}{2}$, *for all vertex-induced subgraphs* (V', E')

[Pollaczek-Geiringer] [Laman:1970] Equivalence in $d = 2$.

Idea: $\binom{d+1}{2}$ fixed coordinates.

Upper bounding the number of embeddings

- Number of complex embeddings bounds the Euclidean ones.
- Bézout's (trivial) bound on quadratic system implies $\mathcal{O}(2^{dn})$.
- Henneberg-I steps exactly multiply embeddings by 2.
- Lower bounds on embedding numbers: $\Omega(2.507^n)$ for $\mathbb{C}^2$, $\Omega(3.067^n)$ for $\mathbb{C}^3$ [Grasegger et al.2020].
- Upper bound attempts (not improved): Determinantal variety [Borcea,Streinu'04], mixed volume [Steffens,Theopald'10]
- Recent improvement in $\mathbb{C}^d$, $d \geq 5$ [Bartzos,E,Schicho'20], using m-Bézout and graph orientations (and permanents).

Multi-homogeneous Bézout bound

m-Bezout bound for length equations

Express embedding by $X_v \in \mathbb{C}^d$, $v \in V$.

$$\sum_{i=1}^{d} X_{v_i}^2 = s_v,\ v \in V;\ s_u + s_v - 2\langle X_u, X_v\rangle = \lambda_{uv}^2,\ (u,v) \in E.$$

Let $E' = E \setminus edges(K_d)$, then:

$$\prod_{i=1}^{n-d} 2Y_i \cdot \prod_{k=1}^{|E'|} (Y_{k_1} + Y_{k_2}) = 2^{n-d} \prod_{i=1}^{n-d} Y_i \cdot \prod_{k=1}^{|E'|} (Y_{k_1} + Y_{k_2}).$$

The m-Bézout bound on the embedding number is the coefficient of $Y_1^{d+1} \cdots Y_n^{d+1}$ in the product of products multiplied by 2^{n-d}.

Graph orientations and m-Bézout

Lemma (Bartzos,E,Schicho'20)

Let L be a graph obtained after removing a "fixed" K_d from G. Let B be the number of G's orientations where each non-fixed vertex has outdegree d. Then, B equals the coefficient of $Y_1^d \cdots Y_m^d$ in

$$\prod_{k=1}^{|E'|} (Y_{k_1} + Y_{k_2}).$$

Proof: Expanding $\prod(Y_u + Y_v)$, only one Y_i per edge contributes to the monomial, and every Y_i contributes d times.

Similar idea: Pebble game.

Graph orientations and embeddings

Theorem (Bartzos,E,Schicho'20)

Let $G(V, E)$ be a minimally rigid graph containing a "fixed" clique $K_d = (v_1, \ldots v_d)$, and let $G' = (V, E \backslash edges(K_d))$. Let $B(G, K_d)$ denote the number of "valid" orientations of G', i.e. which are constrained as follows:

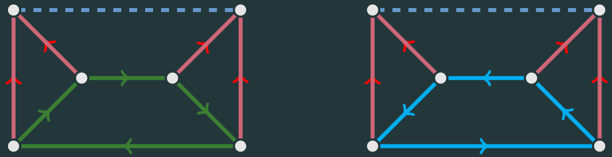
- *the outdegree of $v_1, \ldots, v_d$ is 0.*
- *the outdegree of every vertex in $V \backslash \{v_1, \ldots, v_d\}$ is d.*

By applying the m-Bézout bound, the embedding number of G in $\mathbb{C}^d$ is bounded by

$$2^{n-d} \cdot B(G, K_d).$$

Goal: Bound $B(G, K_d)$ [Bartzos,E,Vidunas'21]

Desargues



$d = 2$, fixed K_2 is the dashed edge,

$B = 2$ orientations $\Rightarrow 2 \cdot 2^{6-2} = 32$,

actually 24 real/complex embeddings [Hunt'83].

Laman graph orientations

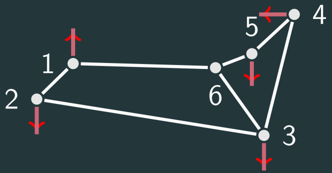
Pseudograph

Definition

A pseudograph $L(U, F, H)$ is a collection s.t.

- U is the set of vertices
- F is a set of (normal) edges (u, v)
- H is a set of *hanging (half) edges* (u)

The normal subgraph $G'(U, F)$ is connected. Each vertex has:

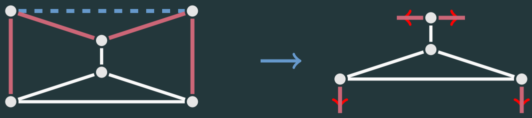


- Total degree p
- Hanging degree h
- Normal degree $r = p - h$
- Extended degree (p, h)

Construction and orientation

Let G be a Laman graph and take fixed edge $e = (v_1, v_2) \in E$.

We specify pseudograph $L_{G,e}(U, F, H)$ with $U = V \backslash \{v_1, v_2\}$, $F = \{e \in E : v_1, v_2 \notin e\}$.



Now $B(G, e)$ equals the number of "valid" orientations of $L_{G,e}$, i.e. with all vertices of outdegree 2, and hanging edges always outgoing (darts).

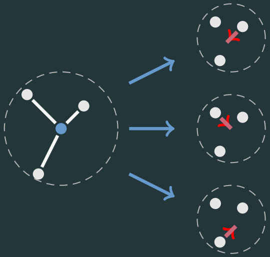
Elimination

- An elimination step removes $\ell \geq 1$ vertices, and 2ℓ adjacent edges, while keeping the pseudograph connected.
- Deleted/Hanging edges are directed outwards/towards the removed vertex.

- The possible pseudographs generated per step correspond to the valid orientations; their number is the step's "cost".
- The product of costs bounds B.

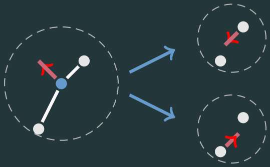
- Base case. $L(U, F, H)$ is a (connected) pseudograph such that $G'(U, F)$ is a tree. Then L has 1 or 0 valid orientations.
- Single-vertex: cost $\leq \binom{p-h}{2-h}$
- Path elimination of $(3, 1)$ vertices: cost $= 2$.

Eliminating a $(3, 0)$ vertex



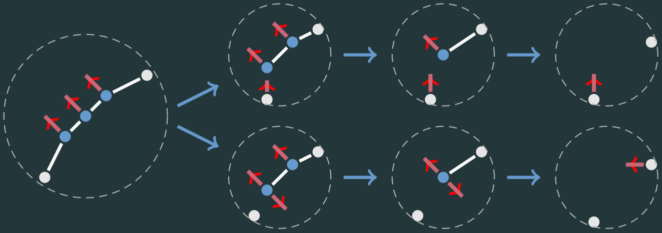
- Vertex of extended degree $(3, 0)$.
- $p - h - 2 = 1$ new hanging edge.
- At most $\binom{p-h}{2-h} = 3$ pseudographs generated.

Elimination of a $(3, 1)$ vertex



- Vertex of extended degree $(3, 1)$.
- $p - h - 2 = 0$ new hanging edges.
- At most $\binom{p-h}{2-h} = 2$ pseudographs generated.

Path elimination



- Paths of a $(3, 1)$ vertices with length $\ell \geq 2$.
- ℓ vertices, $\ell - 1$ hanging edges disappear.
- At most 2 pseudographs generated.

Asymptotic bounds

Complex embedding number $= \mathcal{O}(b^n)$, where b is as follows:

$d =$	2	3	4	5	6
[Bartzos,E,Vidunas'21]	3.776	6.840	12.69	23.90	45.53
[Bartzos,E,Schicho'20]	4.899	8.944	16.73	31.75	60.79
Bézout	4	8	16	32	64

Same results for spherical embeddings.

Bibliography

- *On the multihomogeneous Bézout bound on the number of embeddings of minimally rigid graphs*, with E. Bartzos, and J. Schicho. J. Applic. Algebra in Engineer., Communic. & Computing, 2020.
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Thank you!