Bracing frameworks using edge colorings

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Workshop on Progress and Open Problems in Rigidity Theory
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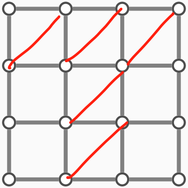
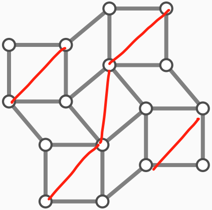
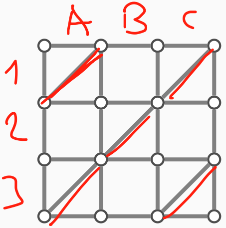


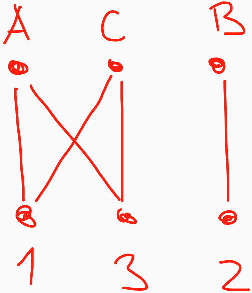
rigid

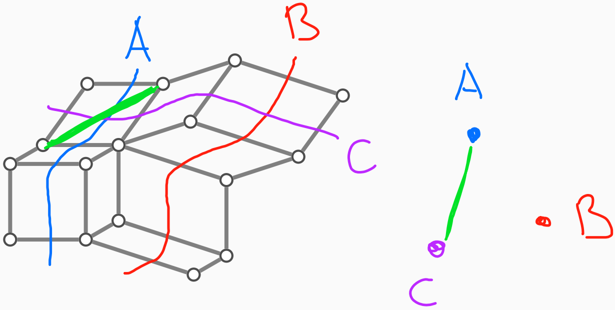


flexible

Theorem (Bolker, Crapo)
A braced grid of squares is infinitesimally rigid if and only if the corresponding bracing graph is connected.





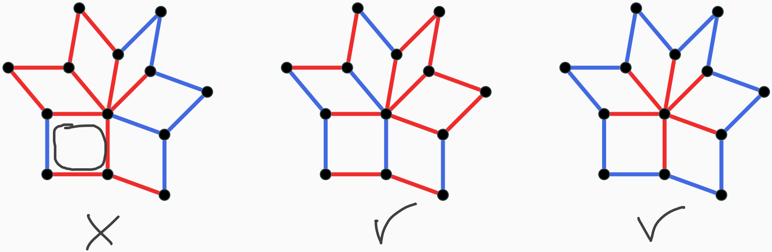


Theorem (Duarte, Francis)
If the bracing graph of a braced rhombic carpet is connected, then the carpet is rigid. $\Leftarrow$ is also true

No Almost Cycle

Definition

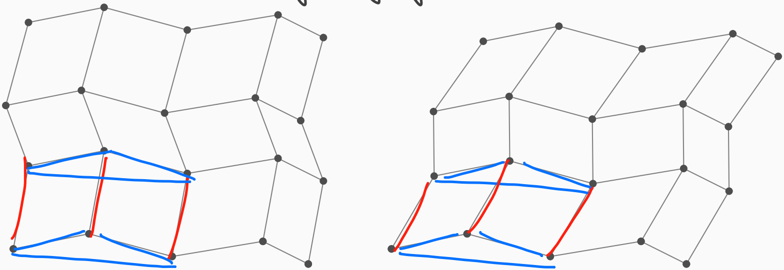
A coloring of edges $\delta : E \to \{\text{blue}, \text{red}\}$ is called a *NAC-coloring*, if it is surjective and for every cycle in G, either all edges in the cycle have the same color, or there are at least two blue and two red edges in the cycle.

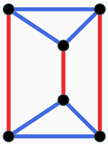


Theorem (G., L., Schicho)

A connected graph with at least one edge admits a flexible framework in $\mathbb{R}^2$ if and only if it has a NAC-coloring.

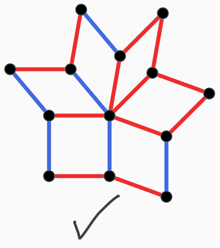
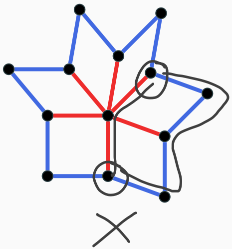
$\Leftarrow$ using "zigzag grid"



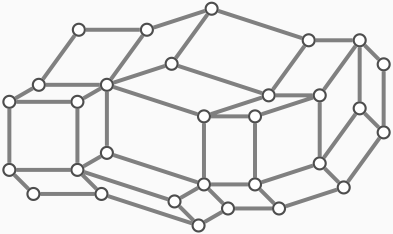


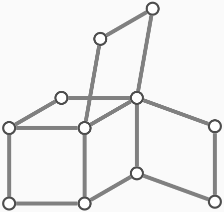
Definition

A NAC-coloring δ of a graph G is called *cartesian* if no two distinct vertices are connected by a red and blue path simultaneously.

The grid construction for a cartesian NAC-coloring gives an injective placement.



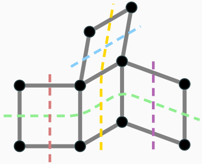


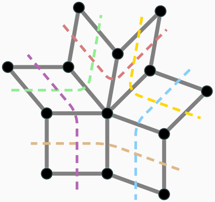
Definition

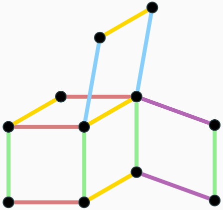
An injective placement $\rho : V_G \to \mathbb{R}^2$ of a graph G such that each 4-cycle in G forms a parallelogram in ρ is called a *parallelogram placement*.

Definition

Consider the relation on the set of edges, where two edges are in relation if they are opposite edges of a 4-cycle subgraph of G. An equivalence class of the reflexive-transitive closure of the relation is called a *ribbon*.

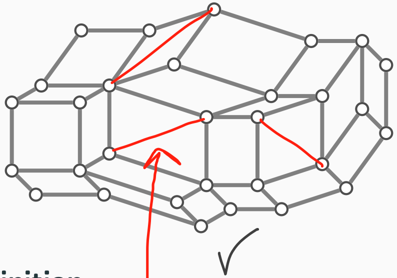


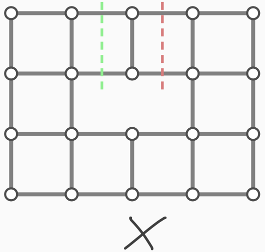




Definition

A graph G is called a *ribbon-cutting graph* if it is connected and every ribbon is an edge cut. If ρ is a parallelogram placement of G, we call the framework (G, ρ) a *P-framework*.





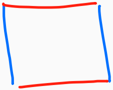
Definition

A *braced ribbon-cutting graph* is a ribbon-cutting graph G together with a set of braces, which are diagonals of some 4-cycles of G.

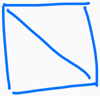
Lemma

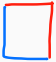
Let (G, ρ) be a braced P-framework. A NAC-coloring of G is cartesian if and only if each ribbon of G is monochromatic.

Opposite edges have
the same color in 4-cycle:



or





is not cartesian

Theorem

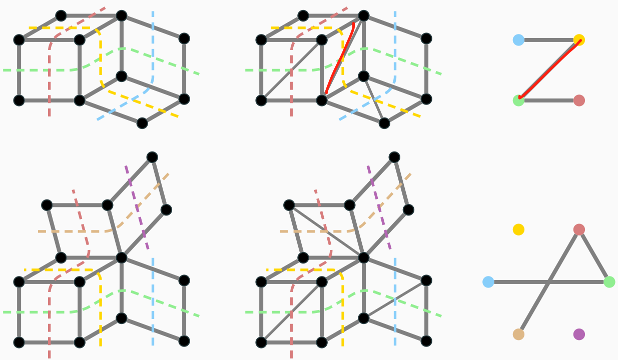
A braced P-framework (G, ρ) is flexible if and only if G has a cartesian NAC-coloring.

$\Leftarrow$ Choose the zigzag grid so that the flex starts at ρ.

$\Rightarrow$ Parallel edges get the same color, hence, the ribbons are monochromatic.

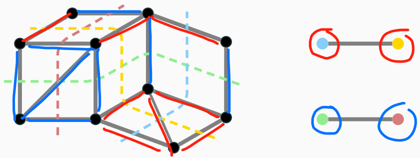
Definition

Let G be a braced ribbon-cutting graph. The graph with the vertices being the ribbons of G and two ribbons being adjacent if they meet at a braced 4-cycle is called the *bracing graph*.



Theorem

Let (G, ρ) be a braced P-framework. The bracing graph of G is disconnected if and only if G has a cartesian NAC-coloring.



Theorem

For a braced P-framework (G, ρ), the following statements are equivalent:

1. (G, ρ) *is rigid,*
2. G *has no cartesian NAC-coloring, and*
3. *the bracing graph of* G *is connected.*

In particular, the minimum number of braces making a framework rigid is one less than the number of ribbons of its underlying graph.

Corollary

If G is a braced ribbon-cutting graph admitting a parallelogram placement, then either

(i) (G, ρ) *is rigid for all parallelogram placements* ρ *of* G, *or*

(ii) (G, ρ) *is flexible for all parallelogram placements* ρ *of* G.

Corollary

A braced grid of squares is rigid if and only if it is infinitesimally rigid.

Conjecture

The number of degrees of freedom of a braced P-framework is one less than the number of connected components of its bracing graph.

Thank you

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jan.legersky.cz/bracingFrameworks

for an interactive
Jupyter notebook