MAXIMIZING THE NUMBER OF PLANAR LIFTS (AND WHY STRUCTURAL ENGINEERS CARE)

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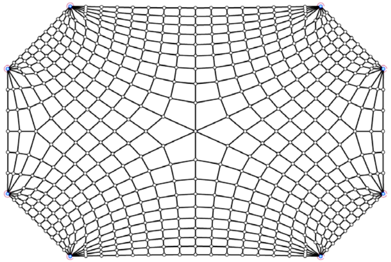
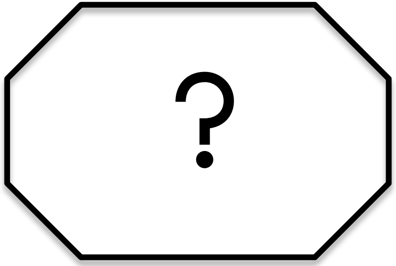
PROBLEM STATEMENTS

Problem Statements

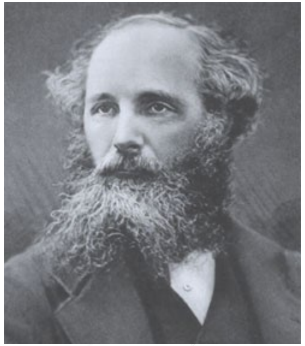
1. For a given planar graph (often quad-dominant), what vertex positions in the plane achieve the maximum number of planar lifts under the conditions of non-degeneracy (no faces collapsed to lines, no lines collapsed to points…)?
   - This is equivalent to maximizing the number of states of self-stress of the planar graph or maximising the number of mechanisms/flexes.
2. How would one design a quad-dominant planar graph that maximizes the number of lifts for a given convex, polygonal boundary that is also held in a plane in the lift?





COMMON ANCESTORS

Common Ancestors



XLV. *On Reciprocal Figures and Diagrams of Forces.* By J. Clerk Maxwell, *F.R.S., Professor of Natural Philosophy in King's College, London*‡.

Reciprocal figures are such that the properties of the first relative to the second are the same as those of the second relative to the first. Thus inverse figures and polar reciprocals are instances of two different kinds of reciprocity.

The kind of reciprocity which we have here to do with has reference to figures consisting of straight lines joining a system of points, and forming closed rectilinear figures; and it consists in the directions of all lines in the one figure having a constant relation to those of the lines in the other figure which correspond to them.

XXXIX. *On Reciprocal Figures, Frames, and Diagrams of Forces.*

(Received 17th Dec. 1869; read 7th Feb. 1870.)

Two figures are reciprocal when the properties of the first relative to the second are the same as those of the second relative to the first. Several kinds of reciprocity are known to mathematicians, and the theories of Inverse Figures and of Polar Reciprocals have been developed at great length, and have led to remarkable results. I propose to investigate a different kind of geometrical reciprocity, which is also capable of considerable development, and can be applied to the solution of mechanical problems.

A Frame may be defined geometrically as a system of straight lines connecting a number of points. In actual structures these lines are material pieces,

GRAPHICAL STATICS

TWO TREATISES

ON THE

GRAPHICAL CALCULUS

AND

RECIPROCAL FIGURES IN GRAPHICAL STATICS

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Oxford
AT THE CLARENDON PRESS
1890

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Int. J. Solids Structures, 1978, Vol. 14, pp. 161–172. Pergamon Press. Printed in Great Britain

BUCKMINSTER FULLER'S "TENSEGRITY" STRUCTURES AND CLERK MAXWELL'S RULES FOR THE CONSTRUCTION OF STIFF FRAMES

$$2v - e - 3 = m - s$$

necessary to satisfy Maxwell's rule, and yet are not "mechanisms" as one might expect, but are actually stiff structures. Maxwell anticipates special cases of this sort, and states that their stiffness will "be of a low order". In fact, the conditions under which Maxwell's exceptional cases occur also permit at least one state of "self-stress" in the frame.

Linear algebra enables us to find the number of "incipient" modes of low-order stiffness of the frame in terms of the numbers of bars, joints and independent states of self-stress. Self-stress in the frame has the effect of imparting first-order stiffness to the frame, and it seems from experiments that a single state of self-stress can stiffen a large number of modes. It is this factor which Fuller exploits to make satisfactory structures.

Maxwell J.C. (1864) *On reciprocal figures and diagrams of forces*
Maxwell J.C. (1870) *On reciprocal figures, frames, and diagrams of forces*

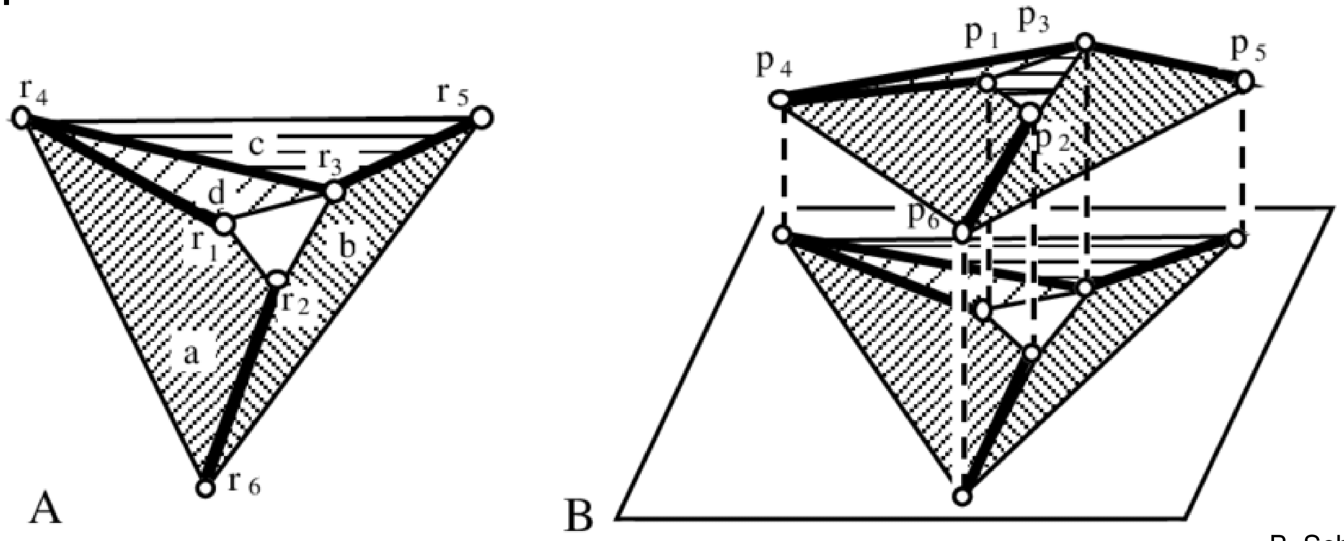
Cremona L. (1872/1890) *Graphical statics: two treatises on the graphical calculus and reciprocal figures in graphical statics*

Calladine C.R. (1978) *Buckminster Fuller's "Tensegrity" structures and Clerk Maxwell's rules for the construction of stiff frames*

Self-Stress and Planar Liftings

If a planar graph has a state of self-stress, one can *lift* the graph to form a 3D plane-faced polyhedron. This polyhedron is a *discrete Airy stress function*.

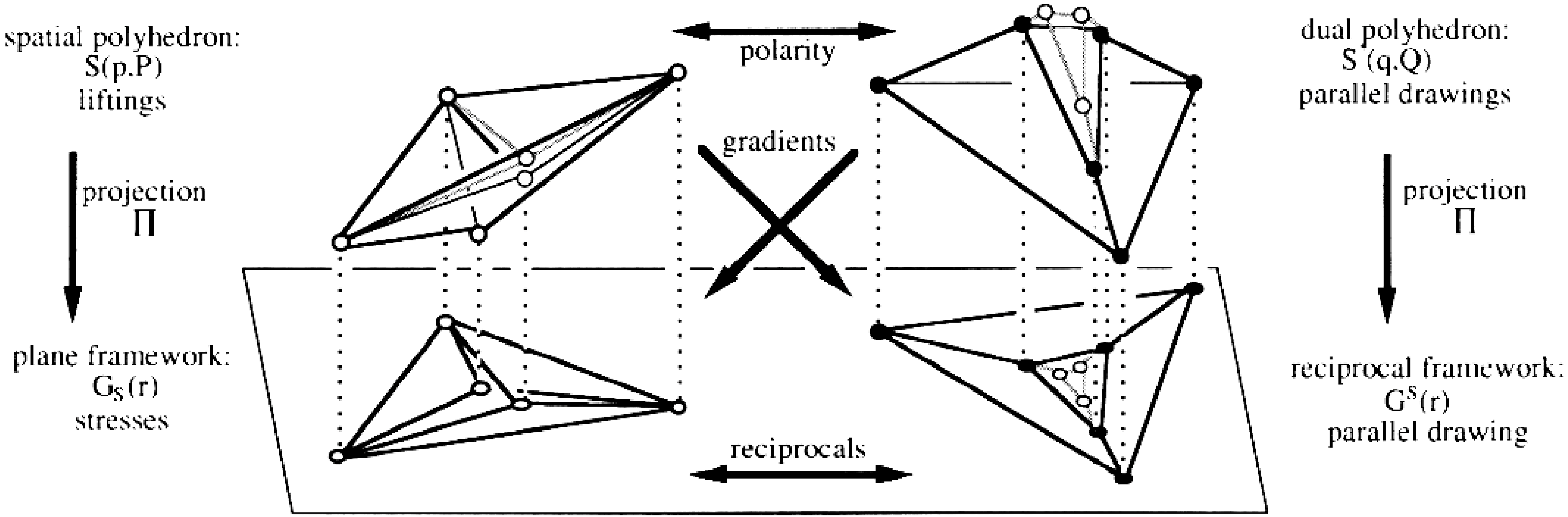
Each linearly independent lifting represents a linearly independent state of self-stress.



B. Schulze and W. Whiteley – *Rigidity and Scene Analysis*

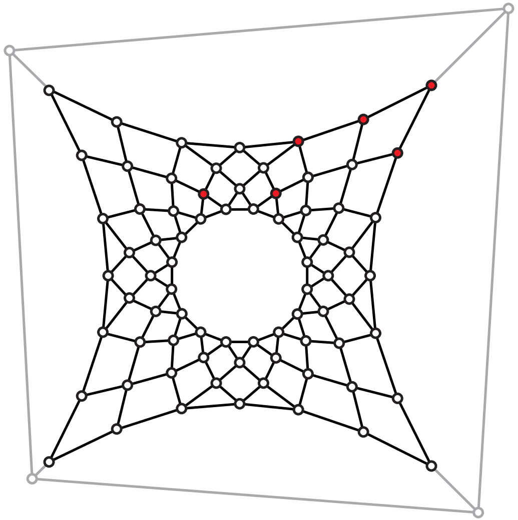
Self-Stress, Planar Liftings and Reciprocal Graphs

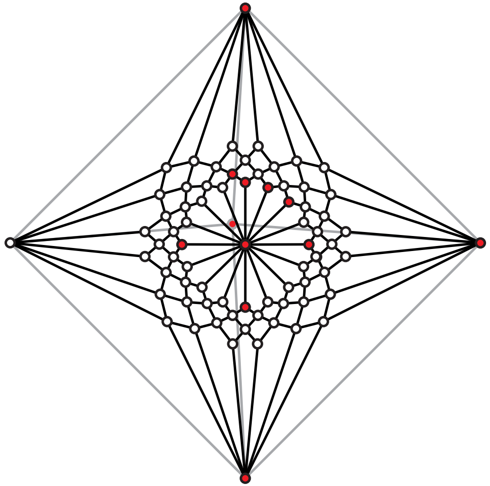
The polyhedron has a *dual polyhedron*. Each face maps to a point, each line maps to a line and each point maps to a plane. This dual polyhedron projects onto a reciprocal diagram.



B. Schulze and W. Whiteley – *Rigidity and Scene Analysis*

Reciprocal Graphs and Rigidity





$b = 152$
$v = 80$
$f = 74$
$2v - b - 3 = 5$
$m - s = 5$

$b^* = 152$
$v^* = 74$
$f^* = 80$
$2v^* - b^* - 3 = -7$
$m^* - s^* = -7$

$$(\boldsymbol{m} - \boldsymbol{s}) + (\boldsymbol{m}^* - \boldsymbol{s}^*) = -\mathbf{2}$$

$$\boldsymbol{m} + \boldsymbol{s} = \boldsymbol{m}^* + \boldsymbol{s}^*$$

$$\boldsymbol{s} = \boldsymbol{m}^* + \mathbf{1}$$

WHY DO STRUCTURAL ENGINEERS CARE?

Gridshells

Gridshells are a collection of rigid bars in 3-space which approximate a surface. The projection of a gridshell is a planar graph.

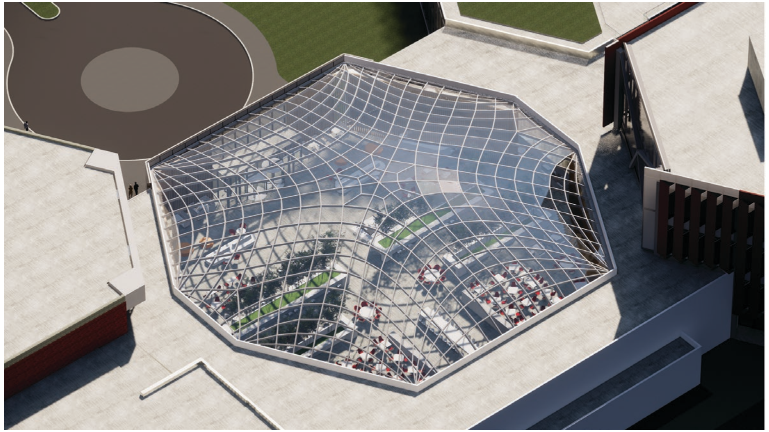
Vertical loads are resisted by edges with axial forces (good) and/or bending moments (not so good).

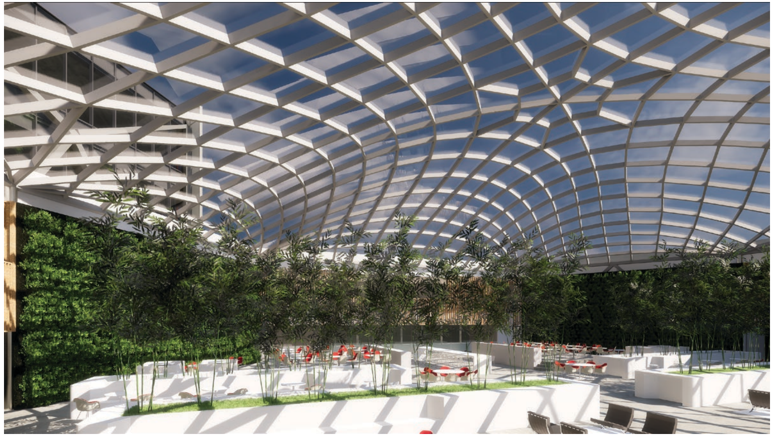
Each linearly independent state of self-stress (planar lift) represents a vertical loading that the gridshell can resist in axial forces without bending.

Gridshells

Gridshells are not always plane-faced, but planarity is desirable.

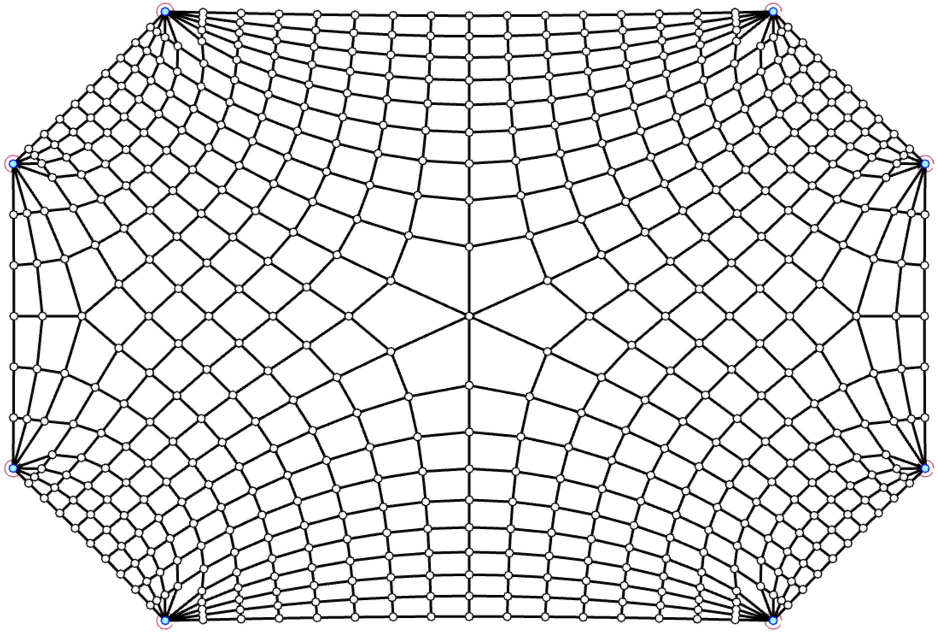
Convex liftings are not always achievable or desirable.



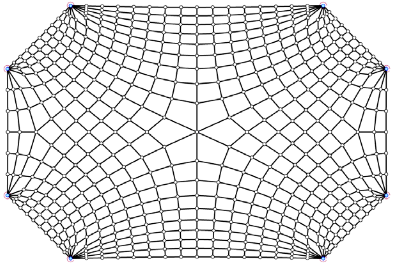


Quad-Dominant Gridshells

Quad dominant gridshells have a graph which consists mainly of quads.
These quads are often regular in shape.
The gridshells often contain a few non-quad faces such as triangles, pentagons and hexagons.



Desirable Attributes of a Quad-Dominant Graph



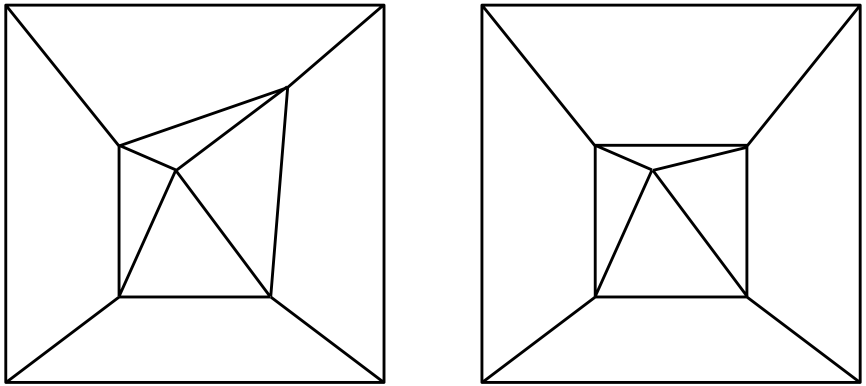
- Maximize the number of states of self-stress (planar lifts).
  - Maximizes the number of load combinations that the gridshell can resist with axial forces.
- States of self-stress should be “well distributed” – the forces in each edge depend on at least one state of self-stress and preferably more than one state of self-stress. Some small edge members adjacent to the perimeter may have zero force.
- Fine grain to have reasonable panel sizes.
- Symmetry and antisymmetry.
  - A symmetrical state of self-stress (planar lift) is desirable – it helps the gridshell resist uniform loads such as self-weight and uniform snow.
  - An antisymmetric state of self-stress (planar lift) is desirable – it helps the gridshell resist antisymmetric loads such as wind or snow drifts. Antisymmetric lifts are often not possible in practice.

SIMPLE EXAMPLE

Simple Example

$b = 16$
$v = 9$
$f = 9$
$2v - b - 3 = -1$
$m - s = -1$

$m = 0$
$s = 1$



$b = 16$
$v = 9$
$f = 9$
$2v - b - 3 = -1$
$m - s = -1$

$m = 1$
$s = 2$

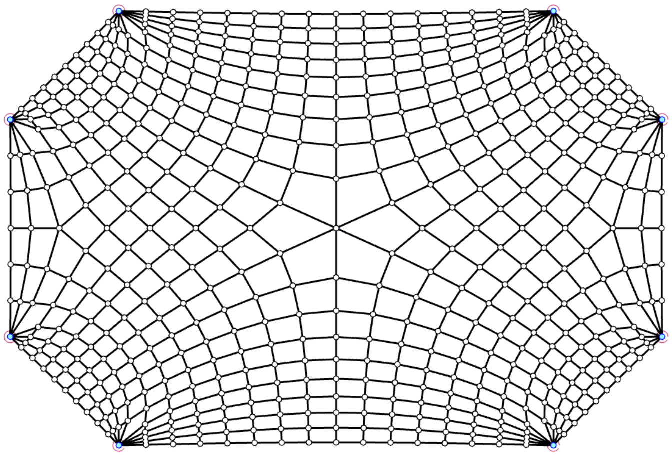
Only one reciprocal diagram.
Load space of size 4 cannot be resisted by axial forces only.

Can construct two reciprocal diagrams.
Load space of size 3 cannot be resisted by axial forces only.

PROBLEM STATEMENTS

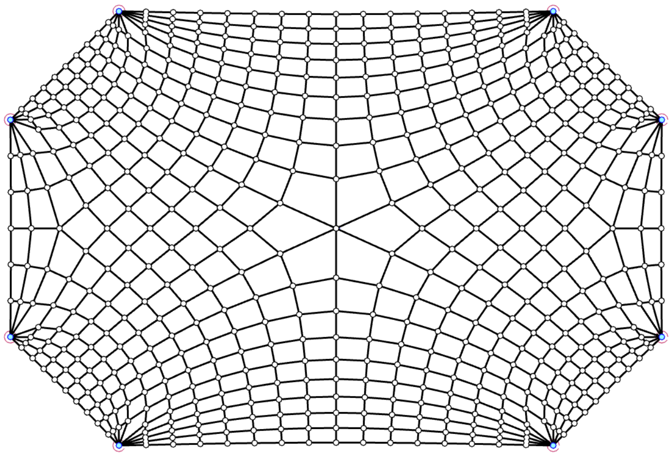
Problem 1

For a given planar graph (often quad-dominant), what vertex positions in the plane achieve the maximum number of planar lifts under the conditions of non-degeneracy (no faces collapsed to lines, no lines collapsed to points…)?



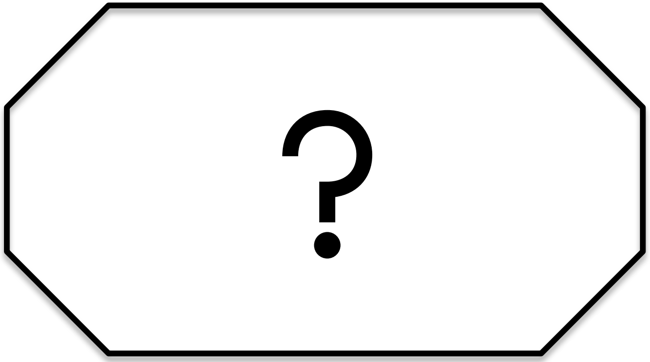
Related Questions on non-degenerate Quad-Dominant Graphs

- For a given graph topology…
  - Is it known how to determine the maximum number of achievable states of self-stress (planar lifts)?
  - Is it known how to find the vertex positions in the plane to achieve the maximum number of states of self-stress (planar lifts)?



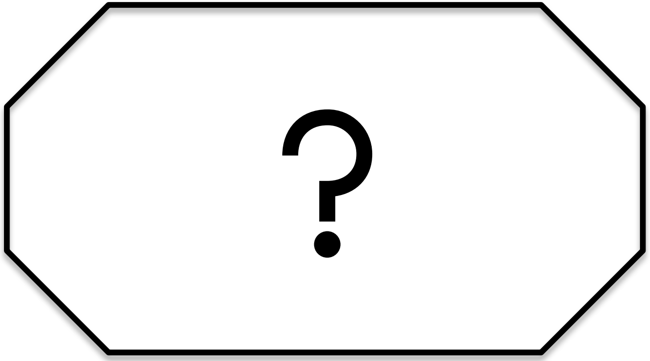
Problem 2

How would one design a quad-dominant planar graph that maximizes the number of lifts for a given convex, polygonal boundary that is also held plane in the lift?



Related Questions on non-degenerate Quad-Dominant Graphs

- With free choice of graph topology…
  - Is it known how to create topologies that can maximize the number of states of self-stress?
  - Are there subgraphs (primitives) that can be combined to make topologies with a large number of states of self-stress?



ANY QUESTIONS?
ANY ANSWERS?