Does it matter if the Geometric Constraint is projectively invariant?

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Does it matter if the Constraints are projectively invariant?

Ways it can matter:

- Vocabulary and Notation: $\mathcal{P}^d$: Basic Structure and Vocabulary: projective notation Plücker, Grassman, Cayley;
- for statics as 2-extensors,
- centers of motion for structures,
- Techniques to test: e.g. Coning, Projection, Projective constructions;
- Change of Metrics: plane, sphere, hyperbolic
- Vision to Recognize What happens with points at infinity? - sliders; polarity and sheet structures, buildings.
- Analogies with other projective constraint systems: parallel drawing and scene analysis;

Does it matter if the Constraints are projectively invariant?

Vocabulary and notion:
Use projective coordinates - homogeneous and projective operations,
Use extensors as joins of points, for a force and for static equilibrium equation
Names such as Plücker, Grassman, Cayley for algebra.
All worked with projective algebra, as did Klein (student of Plücker)

References

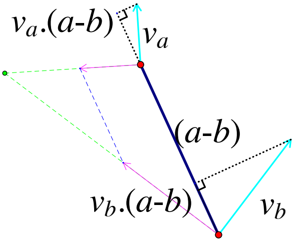
1. Classic: Reference: Klein Elementary Mathematics from an advanced standpoint
2. Coming references: B. Schulze, A. Nixon and W. Whiteley: Paper 1 Rigidity Through a Projective Lens, to appear March 2021
3. Paper 2 Projective Theory of Scene Analysis, Parallel Drawings and Reciprocal Diagrams - to appear: August 2021?
4. 50 years of papers coauthored by W. Whiteley, ...

Statics as projective

Móbius barycentric coordinates with weighted points and balance of forces.
Rankine, and engineer writing a book on statics immediately realized after hearing one talk that statics was projectively invariant.
$f \vee p$ as 2-extensor for point p and force f (free vector - point at infinity) which has $\binom{d+1}{2}$ coordinates in $\mathcal{P}^d$.
3 equilibrium equations in the plane, 6 equilibrium equations in 3-space, ...
Clean and effective.

infinitesimal motions as projective

Centers of motion as projective representation of infinitesimal motions
- point centers for bodies including bars, in plane
- *momenta* as projective centers for vertices weighted piece of line, plane,
... through point perpendicular to 'velocity'



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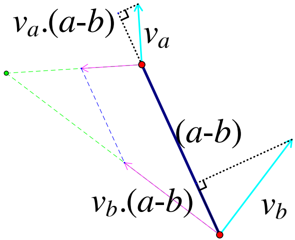

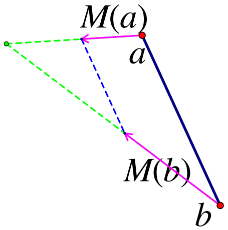

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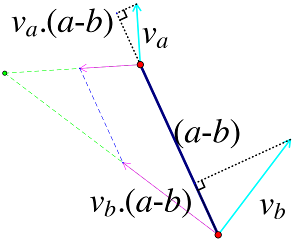
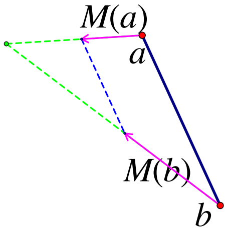
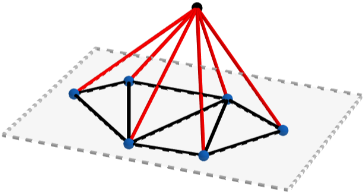




- screw centers in 3 space
- weighted plane segments through points as momenta $M(a)$
- notice parallel drawing above

Coning and Projecting

Take a new joint in the next dimension, join it to all joints in a framework with bars



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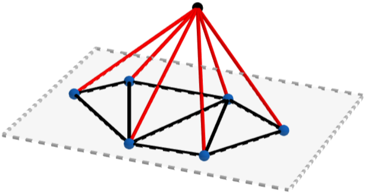

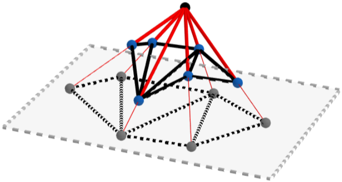

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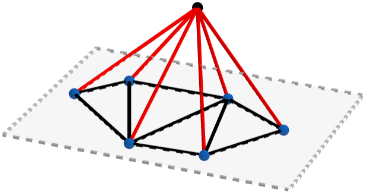

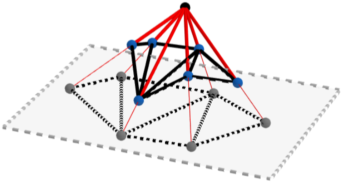

Geometrically, preserves all statics and infinitesimal mechanics

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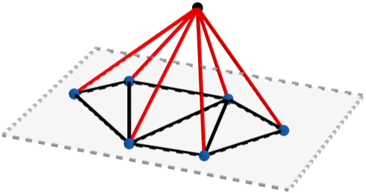

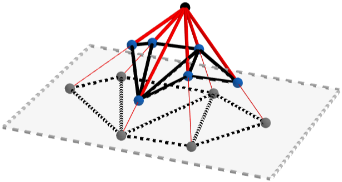

Geometrically, preserves all statics and infinitesimal mechanics
Push-pull vertices along cone-rays - preserves statics.

Projective Conditions

Projective polynomial for when generically isostatic graph is singular with an infinitesimal motion and stress

Pure Condition

$$[a_1 b_1 a_2][a_1 a_3 b_3][b_2 a_3 a_2][b_2 b_1 b_3] - [b_2 b_1 a_2][b_2 a_3 b_3][a_1 a_3 a_2][a_1 b_1 b_3]$$

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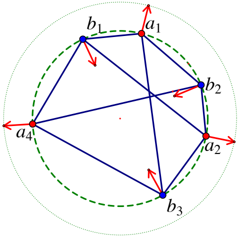
Complete Bipartite Frameworks $K_{3,3}$ on a circle



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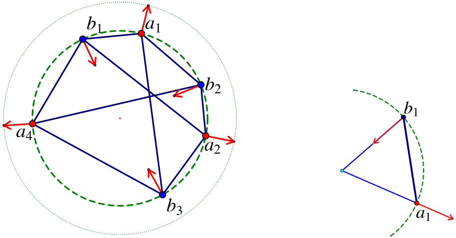

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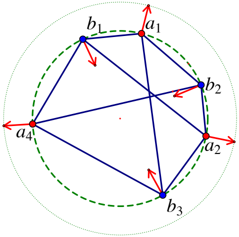

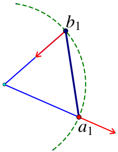

Not yet projective!

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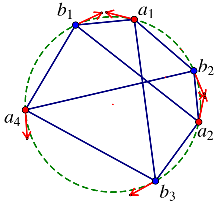

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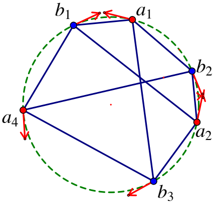

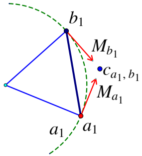

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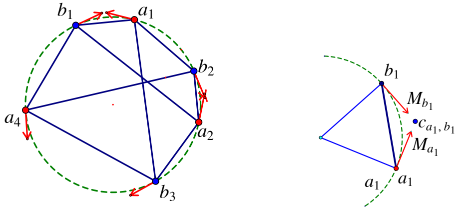

Projectively - circle goes to all conics
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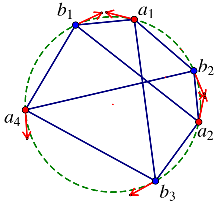
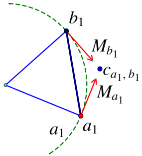
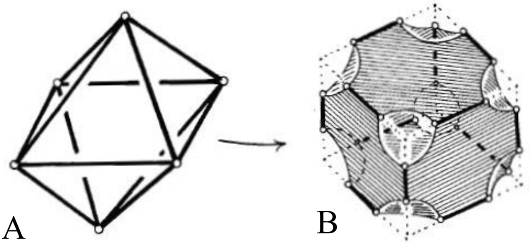




Projectively - circle goes to all conics
- point momenta are tangent to conic
- plus taking limits to get two lines
- extends to larger complete bipartite

Polarity

bar-joint polarizes to plane sheet-hinge
same $3d$ counts, projective conditions.



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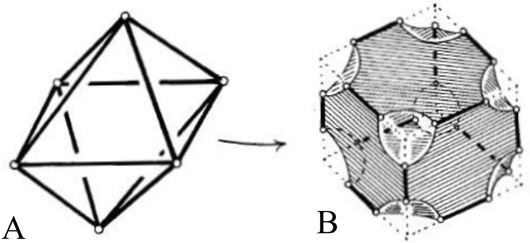
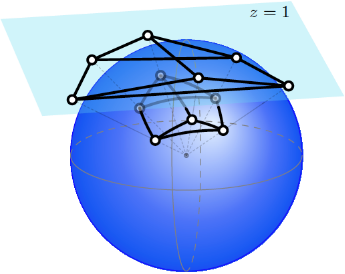


Sheet is rigid within its plane - has a dual momentum - which is a weighted point center in the plane.
Related to Alexandrov's extension of Cauchy
Polarity for rigidity does not easily adapt to $d \neq 3$
Sheets are $3d$ analogs of collinear points with a tree of edges in the plane

Change of Metric

Spherical as slice of cone



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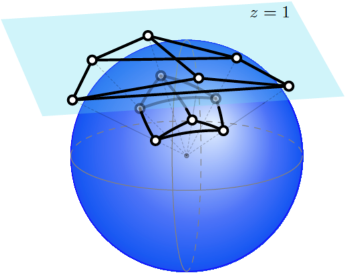

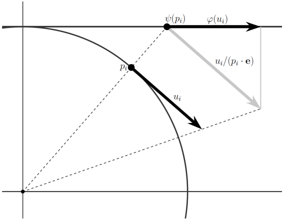

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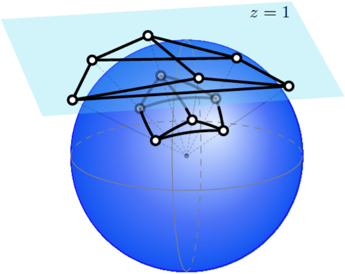
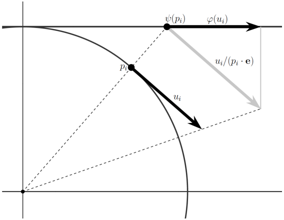
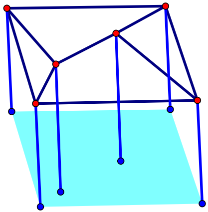




Also extends to Hyperbolic

Expanding our vision

One story building as cone



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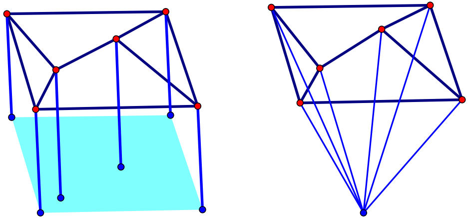
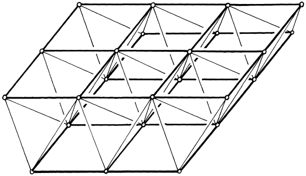


Expanding vision

Truss as bipartite - can you see this as a ruled quadric with a bipartite framework?



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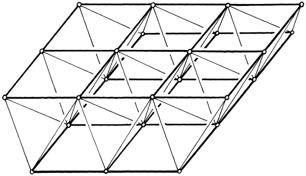

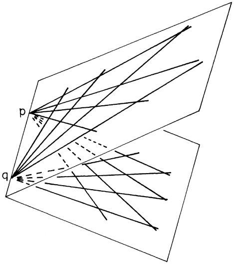

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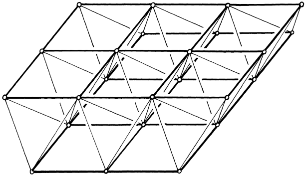
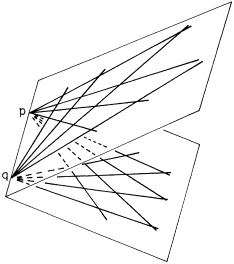
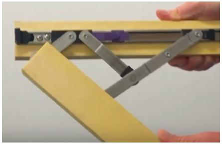




Collapse - did the projective geometry matter?

Expanding vision

Sliders in mechanical engineering - as points at infinity



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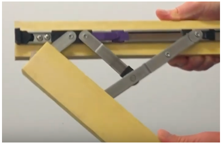

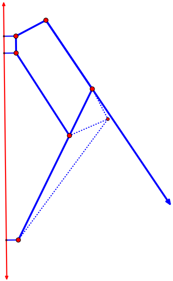

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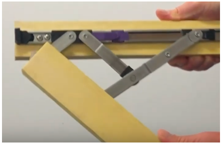

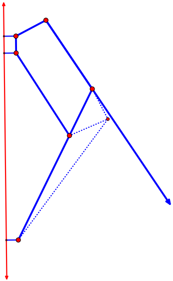

Both geometry and combinatorics
Includes conditions for collinear vertices

Other projective constraints

Liftings: Given picture with points, lines and faces in plane - what plane faced polyhedra project to this? How many?

Picture Theorem [Whiteley 1989]

Incidence structure $S = (V, F; I)$ with a generic picture has $d + k$ space of of liftings if and only if $i = v + df - (d + k)$ and $i' \leq v' + df' - (d + 1)$ for any substructure I' with at least two faces.

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Parallel drawing: theory is projective dual of liftings, in all dimensions,
For graphs $G = (V, E)$: $(d - 1)$ copies of edges and subgraphs G^*:
$|E^*| = d|V| - (d + 1)$ and $|E^*| \leq d|V| - (d + 1)$.
Generalizes Laman's Theorem $(d = 2)$
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Connects directly to rigidity in the plane, in part though reciprocal diagrams, and momenta

Discussion

Questions?

Other Connections

New research problems?

Discussion

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Symmetry and Projection

In the plane, the projective transformation of a mirror symmetry is a half-turn!

Send mirror to infinity and normal to mirror becomes center of half-turn

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In the sphere, inversion of mirror symmetry is half-turn

Symmetry and Projection

In the plane, the projective transformation of a mirror symmetry is a half-turn!
Send mirror to infinity and normal to mirror becomes center of half-turn

In the sphere, inversion of mirror symmetry is half-turn

Pairing of symmetries found in elliptic space - sphere with antipodal points identified

Global Rigidity with sliders

If we add sliders - what happens to global rigidity?

In plane? in higher dimensions?

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In plane? in higher dimensions?

What happens with global rigidity of sheet structures?