Zero-sum cycles in flexible polyhedra

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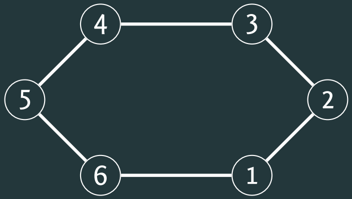
A cycle on a line

Let us consider a cycle



and realize it on a line

A cycle on a line

If we "follow" the cycle and we sum algebraically the lengths of the realizations of the edges, namely:

$$|12| - |23| + |34| - |45| + |56| - |61|$$

we clearly obtain zero by telescoping.

Bricard's octahedra

Bricard proved that there are three types of flexible octahedra:

- line-symmetric ones;
- plane-symmetric ones;
- octahedra whose edge lengths ℓ_{ij} of the edges satisfy three linear equations of the form:

$$\begin{aligned} \eta_{35}\,\ell_{35} + \eta_{45}\,\ell_{45} + \eta_{46}\,\ell_{46} + \eta_{36}\,\ell_{36} &= 0\,, \\ \eta_{14}\,\ell_{14} + \eta_{24}\,\ell_{24} + \eta_{23}\,\ell_{23} + \eta_{13}\,\ell_{13} &= 0\,, \\ \eta_{15}\,\ell_{15} + \eta_{25}\,\ell_{25} + \eta_{26}\,\ell_{26} + \eta_{16}\,\ell_{16} &= 0\,, \end{aligned} \tag{1}$$

  where $\eta_{ij} \in \{1, -1\}$ and the angles between edges satisfy a certain property.

See Jan Legerský's website for nice animations:

https://jan.legersky.cz/project/bricard_octahedra/

Suspensions

Notice that the symmetry of the first two cases forces their edge lengths to satisfy similar linear equations as the ones in the third case.

This is an example of a more general phenomenon: Mikhalëv proved in 2001 that if we have a flexible suspension, then the sum of the edge lengths of the equator, multiplied by 1 or -1, is zero.

Our result

We aim at generalizing the result about suspensions.

We fix once and for all a graph $G = (V, E)$ that is the 1-skeleton of a polyhedron with triangular faces.

Definition

A realization is a map

$$\rho \colon V \longrightarrow \mathbb{R}^3$$

A realization induces edge lengths: if $\{i, j\} \in E$, then

$$\lambda_{ij} := \|\rho(i) - \rho(j)\| \,.$$

We ask that $\lambda_{ij} \neq 0$ for all $\{i, j\} \in E$.

Definition

A realization ρ has a flex f if there exists a map

$$f\colon [0,1) \longrightarrow (\mathbb{R}^3)^V$$

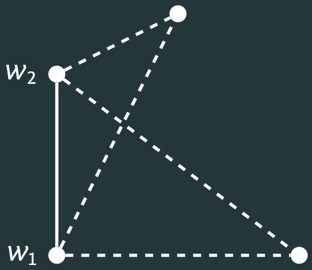
where

- $f(0) = \rho$;
- $f(t)$ determines the same edge lengths as $f(0)$ for any $t \in [0,1)$;
- $f(t_1)$ and $f(t_2)$ are not isometric for any $t_1 \neq t_2$.

Our result

Theorem

Suppose that ρ is a realization that admits a flex. Suppose that $\{w_1, w_2\} \in E$ is an edge such that its dihedral angle changes during the flex. Then there exists an induced cycle in the graph containing $\{w_1, w_2\}$ such that the sum of its edge lengths, weighted by 1 or -1, is zero.



The technique

We use two guiding principles in proving the result:

- compactify $\mathbb{R}^3$ and look at what happens at infinity;
- use complex geometry to obtain real results.

The compactification

We use a compactification of $\mathbb{R}^3$ that behaves well with respect to the Euclidean norm:

$$\begin{array}{rcl} \mathbb{R}^3 & \hookrightarrow & \mathbb{P}^4 \\ (x,y,z) & \mapsto & (x : y : z : \underbrace{x^2+y^2+z^2}_{=:r} : \underbrace{1}_{h}) \end{array}$$

Definition

The Zariski closure of the image of $\mathbb{R}^3$ is the quartic

$$M := \{x^2+y^2+z^2-rh=0\} \subset \mathbb{P}^4$$

The link to the Euclidean norm

Since M is a quadric, we get an associated bilinear form

$$\langle \cdot, \cdot \rangle_M : \mathbb{R}^5 \times \mathbb{R}^5 \longrightarrow \mathbb{R}$$

Lemma

If $p, q \in \mathbb{R}^3$ and $\hat{p}, \hat{q} \in \mathbb{R}^5$ are corresponding vectors of their images in $\mathbb{P}^4$, where the h-coordinate is 1. Then

$$\|p - q\|^2 = -2 \langle \hat{p}, \hat{q} \rangle_M$$

Extending the distance

The lemma allows us to extend the Euclidean (squared) distance to a rational function:

$$\begin{aligned} d\colon \quad M \times M &\dashrightarrow \mathbb{P}^1 \\ (q_1, q_2) &\mapsto \big(\langle q_1, q_2\rangle_M : h_1 h_2\big) \end{aligned}$$

where $q_i = (x_i : y_i : z_i : r_i : h_i)$.

Lemma

d is not defined at (q_1, q_2) if and only if

$$q_1 \in \{h = 0\} \textit{ and } q_2 \in \mathbb{T}_{q_1} M \qquad \textit{or}$$
$$q_2 \in \{h = 0\} \textit{ and } q_1 \in \mathbb{T}_{q_2} M$$

Let us look at infinity!

We now focus on the complement of $\mathbb{R}^3$ in M, namely

$$M_\infty := M \cap \{h = 0\}$$

This is a cone over a plane curve

$$M_\infty := \{x^2 + y^2 + z^2 = 0\}$$

with a single real point, namely $(0 : 0 : 0 : 1 : 0)$.

Better extend to the complex numbers.

At infinity, things get easy

Key technical result:

Lemma

Let $p \in M_\infty$, we set

$$\mathrm{Fin}_p := \mathbb{T}_p M \cap M \cap \{h \neq 0\}\,.$$

There exists a function $\pi : \mathrm{Fin}_p \longrightarrow \mathbb{R}$ such that for any $q_1, q_2 \in \mathrm{Fin}_p$

$$d(q_1, q_2) = \Big(\big(\pi(q_1) - \pi(q_2)\big)^2 : 1 \Big)$$

Namely, on Fin_p, the function d behaves like a 1-dimensional (squared) distance.

A new goal

Using the key technical lemma, our goal now becomes

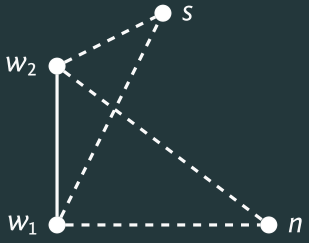
Find a realization such that there is a whole cycle containing our given edge that lies on a set of the form Fin_p.

Complex configuration spaces

If ρ is the realization that admits the flex, we define

$$W := \{\rho' \colon V \longrightarrow \mathbb{C}^3 \mid \\ \rho' \text{ determines the same edge lengths of } \rho\} \subset (\mathbb{C}^3)^V$$

W is rather redundant: any time a realization lies on W, all the isometric realizations also lie on W. So we stop the isometries:



$$Z := \{\rho' \in W \mid \\ \rho'(w_1) = \rho(w_1), \\ \rho'(w_2) = \rho(w_2), \\ \rho'(n) = \rho(n)\}$$

Complex configuration spaces

Using our compactification

$$Z \hookrightarrow M^V$$

Let Y be the Zariski closure of the image of Z in M^V.

Y has positive dimension since ρ has a flex!

Not only that! Since the angle at $\{w_1, w_2\}$ is not frozen during the flex, if we consider the projection

$$Y \longrightarrow M$$

on the coordinate of vertex s, the the image of this map is positive-dimensional!

The wanted realization

Hence there exists a (complex!) realization ρ_∞ such that

$$\rho_\infty(s) \in M_\infty$$

Coloring the vertices

Using this special realization ρ_∞ we can define a coloring of the vertices of the 1-skeleton

$$v \in V \text{ is colored } \begin{cases} \text{red} & \text{if } \rho_\infty(v) \in M \cap \{h \neq 0\} \\ \text{blue} & \text{if } \rho_\infty(v) \in M_\infty \text{ and } (*) \\ \text{gold} & \text{otherwise} \end{cases}$$

where $(*)$ says that if we project $\rho_\infty(v)$ from the vertex of the cone M_∞ to the plane quadric

$$\{x^2 + y^2 + z^2 = 0\}$$

then we obtain the same point we would have obtained from $\rho_\infty(s)$.

Properties of the coloring

The coloring determines two cycles:

- a cycle of red vertices containing $\{w_1, w_2\}$
- a cycle of blue vertices containing s

The properties of the function d imply that the two cycles satisfy the properties:

(1) For each vertex v in the blue cycle

$$\rho_\infty(v) = \rho_\infty(s)$$

(2) For each vertex v in the red cycle

$$\rho_\infty(v) \in \mathrm{Fin}_{\rho_\infty(s)}$$

Summing up

Starting from a realization ρ admitting a flex, we found a "realization" ρ_∞ of the polyhedron such that a whole cycle containing our lies on a set of the form Fin_p for $p = \rho_\infty$.

Due to the fact that ρ_∞ was defined using polynomial equations, the lengths of the edges contained in the cycle are the same as in the realization ρ.

Now, due to the key technical lemma, the function d behaves on $\mathrm{Fin}_{\rho_\infty}$ as a (squared) distance on a line, so there is a choice of weights 1 and -1 for the edges so that the weighted sum of edge lengths is zero.

Open questions

- Can we use results like this to classify motions of polyhedra that are more complex than octahedra?
- Can the compactification used here be useful in other problems involving flexibility?
- Are there other compactifications of $\mathbb{R}^2$ or $\mathbb{R}^3$ that could encode interesting properties concerning rigidity or flexibility?