Symbol functions for symmetric frameworks

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Outline

Joint work with Eleftherios Kastis (Fields Institute) and John McCarthy (Washington University in St Louis).

See our recent paper:

Kastis, Kitson, McCarthy. Symbol functions for symmetric frameworks. Journal of Mathematical Analysis and Applications 497 (2021), no. 2, 124895.

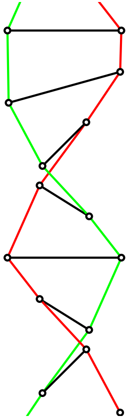


Figure: A bar-joint framework in $\mathbb{R}^3$ with screw-axis symmetry.

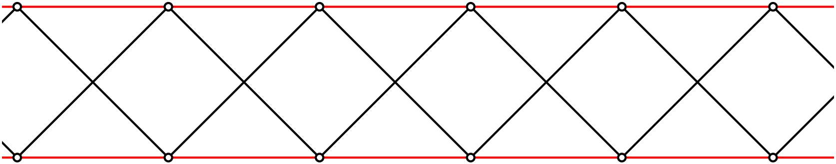


Figure: A direction-length framework in $\mathbb{R}^2$ with both translational and reflectional symmetry.

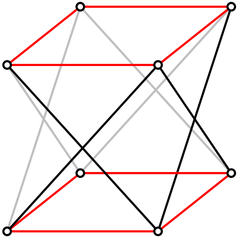


Figure: A symmetric bar-joint framework in $\mathbb{R}^3$ with non-Euclidean norm distance constraints.

$$
\begin{array}{ccc}
\ell^2(V,X) & \xrightarrow{\;C(G,\varphi)\;} & \ell^2(E,Y) \\
\Big\downarrow S_V & & \Big\uparrow S_E^{-1} \\
T_{\tilde{\tau}} \circlearrowright \ \ell^2(\Gamma, X^{V_0}) & \overset{\tilde{C}(G,\varphi)}{\cdots\cdots\cdots\rightarrow} & \ell^2(\Gamma, Y^{E_0}) \\
\Big\downarrow F_{X^{V_0}} & & \Big\uparrow F^{-1}_{Y^{E_0}} \\
L^2(\hat{\Gamma}, X^{V_0}) & \xrightarrow{\;M_\Phi\;} & L^2(\hat{\Gamma}, Y^{E_0})
\end{array}
$$

Figure: Factorisation of the ℓ^2-coboundary operator $C(G,\varphi)$ for a framework (G,φ) with a discrete abelian symmetry group Γ.

Let Γ be a locally compact Hausdorff abelian group.

Denote by $L^2(\Gamma)$ the Hilbert space of square integrable functions, i.e. Borel-measurable functions $f : \Gamma \to \mathbb{C}$ such that,

$$\int_\Gamma |f(\gamma)|^2 \, d\gamma < \infty$$

where we use normalised Haar measure on Γ.

For each $\gamma \in \Gamma$, denote by D_γ the unitary operator

$$D_\gamma : L^2(\Gamma) \to L^2(\Gamma), \quad f(\gamma') \mapsto f(\gamma' - \gamma).$$

Proposition

Let $L \in B(L^2(\Gamma))$. Then L satisfies the commuting property

$$D_\gamma L = L D_\gamma$$

for all $\gamma \in \Gamma$ if and only if L is unitarily equivalent to a multiplication operator $M_\Phi \in B(L^2(\hat{\Gamma}))$ for some $\Phi \in L^\infty(\hat{\Gamma})$.

In particular, $L = F^{-1} M_\Phi F$.

Let X and Y be complex Hilbert spaces.

Denote by $L^2(\Gamma, X)$ the Hilbert space of square integrable X-valued functions. i.e. Bochner-measurable functions $f : \Gamma \to X$ such that,

$$\int_\Gamma \|f(\gamma)\|^2 \, d\gamma < \infty$$

For each $\gamma \in \Gamma$, denote by U_γ and W_γ the unitary operators

$$U_\gamma = D_\gamma \otimes 1_X : L^2(\Gamma, X) \to L^2(\Gamma, X), \quad f(\gamma') \mapsto f(\gamma' - \gamma),$$

$$W_\gamma = D_\gamma \otimes 1_Y : L^2(\Gamma, Y) \to L^2(\Gamma, Y), \quad g(\gamma') \mapsto g(\gamma' - \gamma).$$

Proposition

Let $L \in B(L^2(\Gamma, X), L^2(\Gamma, Y))$. Then L satisfies the intertwining property

$$W_\gamma L = L U_\gamma$$

for all $\gamma \in \Gamma$ if and only if L is unitarily equivalent to a multiplication operator $M_\Phi \in B(L^2(\hat{\Gamma}, X), L^2(\hat{\Gamma}, Y))$ for some $\Phi \in L^\infty(\hat{\Gamma}, B(X, Y))$.

In particular, $L = F_Y^{-1} M_\Phi F_X$.

Let $U(X)$ denote the unitary group of X and let $\pi : \Gamma \to U(X)$ be a unitary representation of Γ on X.

Define $T_\pi \in B(L^2(\Gamma, X))$ by

$$(T_\pi f)(\gamma) = \pi(-\gamma)f(\gamma).$$

For each $\gamma \in \Gamma$, define $U_{\gamma,\pi} \in B(L^2(\Gamma, X))$ by

$$(U_{\gamma,\pi} f)(\gamma') = \pi(\gamma)f(\gamma' - \gamma).$$

Theorem

Let $C \in B(L^2(\Gamma, X), L^2(\Gamma, Y))$ and let $\pi : \Gamma \to U(X)$ be a unitary representation. Then

$$W_\gamma C = C U_{\gamma,\pi}$$

for all $\gamma \in \Gamma$ if and only if $C = L T_\pi$, where L is unitarily equivalent to a multiplication operator $M_\Phi \in B(L^2(\hat{\Gamma}, X), L^2(\hat{\Gamma}, Y))$ for some $\Phi \in L^\infty(\hat{\Gamma}, B(X, Y))$.

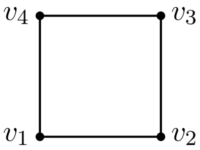
In particular, $L = F_Y^{-1} M_\Phi F_X$.

A framework for X and Y is a pair (G, φ) consisting of a simple undirected graph $G = (V, E)$ and a collection $\varphi = (\varphi_{v,w})_{v,w \in V}$ of linear maps $\varphi_{v,w} : X \to Y$ with the property that $\varphi_{v,w} = 0$ if $vw \notin E$ and $\varphi_{v,w} = -\varphi_{w,v}$ for all $vw \in E$.

A coboundary matrix for (G, φ) is a matrix $C(G, \varphi)$ with rows indexed by E and columns indexed by V. The row entries for a given edge $vw \in E$ are as follows,

$$\begin{array}{c} \\ vw \end{array} \begin{array}{c} \begin{matrix} & & v & & & & w & & \end{matrix} \\ \begin{pmatrix} \cdots & 0 & \varphi_{v,w} & 0 & \cdots & 0 & \varphi_{w,v} & 0 & \cdots \end{pmatrix}. \end{array}$$

Example



$$\begin{array}{c} \\ v_1v_2 \\ v_2v_3 \\ v_3v_4 \\ v_4v_1 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} \varphi_{v_1,v_2} & -\varphi_{v_1,v_2} & 0 & 0 \\ 0 & \varphi_{v_2,v_3} & -\varphi_{v_2,v_3} & 0 \\ 0 & 0 & \varphi_{v_3,v_4} & -\varphi_{v_3,v_4} \\ \varphi_{v_1,v_4} & 0 & 0 & -\varphi_{v_1,v_4} \end{pmatrix} \end{array}$$

Figure: A 4-cycle (left) and coboundary matrix (right).

A Γ-symmetric graph is a pair (G, θ) where $G = (V, E)$ is a simple undirected graph with automorphism group $\mathrm{Aut}(G)$ and $\theta : \Gamma \to \mathrm{Aut}(G)$ is a group homomorphism.

A Γ-symmetric framework is a tuple $\mathcal{G} = (G, \varphi, \theta, \tau)$ where $\tau : \Gamma \to \mathrm{Isom}(X)$ is a group homomorphism, (G, θ) is a Γ-symmetric graph and (G, φ) is a framework with the property that,

$$\varphi_{\gamma v, \gamma w} = \varphi_{v,w} \circ \tau(-\gamma)$$

for all $\gamma \in \Gamma$ and all $v, w \in V$.

Theorem

Let $\mathcal{G} = (G, \varphi, \theta, \tau)$ be a Γ-symmetric framework for X and Y where G has a finite or a countably infinite vertex set, Γ is a discrete abelian group, θ acts freely on the vertices and edges of G and V_0 and E_0 are finite sets.

Then $C(G, \varphi) \in B(\ell^2(V, X), \ell^2(E, Y))$ and,

$$C(G, \varphi) = S_E^{-1} \circ F_{Y^{E_0}}^{-1} \circ M_\Phi \circ F_{X^{V_0}} \circ T_{\tilde{\tau}} \circ S_V,$$

for some $\Phi \in L^\infty(\hat{\Gamma}, B(X^{V_0}, Y^{E_0}))$.

Let $\chi \in \hat{\Gamma}$. A χ-orbit matrix for $\mathcal{G} = (G, \varphi, \theta, \tau)$ is a matrix $O_{\mathcal{G}}(\chi)$ with rows indexed by the directed edges of the gain graph and with columns indexed by the vertex orbits.

The row entries for a non-loop directed edge $([v], [w]) \in E_0$ with gain $\gamma \in \Gamma$ are as follows,

$$\begin{pmatrix} \cdots & 0 & \overset{[v]}{\varphi_{\tilde{v},\gamma\tilde{w}}} & 0 & \cdots & 0 & \overset{[w]}{\chi(\gamma)\varphi_{\tilde{w},-\gamma\tilde{v}}} & 0 & \cdots \end{pmatrix}.$$

The row entries for a loop edge $([v], [v]) \in E_0$ with gain γ are,

$$\begin{pmatrix} \cdots & 0 & \overset{[v]}{\varphi_{\tilde{v},\gamma\tilde{v}} + \chi(\gamma)\varphi_{\tilde{v},-\gamma\tilde{v}}} & 0 & \cdots \end{pmatrix}.$$

Figure: A $\mathbb{Z}_2$-symmetric graph (left) and gain graph (right).

$$O_{\mathcal{G}}(\chi_0) = \begin{array}{c} \\ [e_1] \\ [e_2] \end{array} \begin{array}{c} \begin{array}{cc} [v_1] & [v_2] \end{array} \\ \begin{pmatrix} \varphi_{\tilde{v}_1,\tilde{v}_2} & -\varphi_{\tilde{v}_1,\tilde{v}_2} \\ \varphi_{\tilde{v}_1,\tilde{v}_4} & \varphi_{\tilde{v}_2,\tilde{v}_3} \end{pmatrix} \end{array}$$

$$O_{\mathcal{G}}(\chi_1) = \begin{array}{c} \\ [e_1] \\ [e_2] \end{array} \begin{array}{c} \begin{array}{cc} [v_1] & [v_2] \end{array} \\ \begin{pmatrix} \varphi_{\tilde{v}_1,\tilde{v}_2} & -\varphi_{\tilde{v}_1,\tilde{v}_2} \\ \varphi_{\tilde{v}_1,\tilde{v}_4} & -\varphi_{\tilde{v}_2,\tilde{v}_3} \end{pmatrix} \end{array}$$

Theorem

Let $\mathcal{G} = (G, \varphi, \theta, \tau)$ be a Γ-symmetric framework with symbol function $\Phi \in L^\infty(\hat{\Gamma}, B(X^{V_0}, Y^{E_0}))$. Then,

$$\Phi(\chi) = O_{\mathcal{G}}(\chi), \quad \textit{a.e.} \ \chi \in \hat{\Gamma}.$$

Corollary

Let $\mathcal{G} = (G, \varphi, \theta, \tau)$ be a Γ-symmetric framework. If G is a finite graph then the coboundary matrix $C(G, \varphi)$ is equivalent to the direct sum,

$$\bigoplus_{\chi \in \hat{\Gamma}} O_{\mathcal{G}}(\chi) : \bigoplus_{\chi \in \hat{\Gamma}} X^{V_0} \to \bigoplus_{\chi \in \hat{\Gamma}} Y^{E_0}.$$

Corollary

Let $\mathcal{G} = (G, \varphi, \theta, \tau)$ be a Γ-symmetric framework with symbol function Φ. Let $\Gamma_0 \subset \Gamma$ be the finite set of non-zero gains on the edges of this gain graph.

(i) *Φ is the operator-valued trigonometric polynomial with,*

$$\Phi(\chi) = \hat{\Phi}(0) + \sum_{\gamma \in \Gamma_0} \hat{\Phi}(\gamma)\chi(\gamma), \quad \forall\, \chi \in \hat{\Gamma}.$$

(ii) *For each $\gamma \in \Gamma_0$, each $[v] \in V_0$ and each $[e] \in E_0$,*

$$\hat{\Phi}(\gamma)_{[e],[v]} = C(G, \varphi)_{\tilde{e}, \gamma\tilde{v}} \circ d\tau(\gamma),$$

where $\hat{\Phi}(\gamma)_{[e],[v]}$ is the $([e],[v])$-entry of $\hat{\Phi}(\gamma)$ and $C(G, \varphi)_{\tilde{e}, \gamma\tilde{v}}$ is the $(\tilde{e}, \gamma\tilde{v})$-entry of $C(G, \varphi)$.

Let $\mathcal{G} = (G, \varphi, \theta, \tau)$ be a Γ-symmetric framework for X and Y with symbol function Φ.

Fix $\chi \in \hat{\Gamma}$ and $a \in X^{V_0}$ and define $z(\chi, a) = (z_v)_{v \in V} \in \ell^\infty(V, X)$ to be the bounded vector with components,

$$z_v = \chi(\gamma) d\tau(\gamma) a_{[v]}, \quad \text{for } v = \gamma \tilde{v}.$$

We refer to $z(\chi, a)$ as a χ-symmetric vector in $\ell^\infty(V, X)$.

Theorem

If $a \in \ker \Phi(\chi)$ *then* $z(\chi, a) \in \ker C(G, \varphi)$*.*

The Rigid Unit Mode (RUM) spectrum of $\mathcal{G}$ is defined as follows,

$$\Omega(\mathcal{G}) = \{\chi \in \hat{\Gamma} : \ker \Phi(\chi) \neq \{0\}\}.$$

Double helix bar-joint framework

$$\Phi(\omega) \quad = \quad \begin{array}{c} \\ ([v_{0,0}],[v_{1,0}]) \\ ([v_{0,0}],[v_{0,0}]) \\ ([v_{1,0}],[v_{1,0}]) \end{array} \begin{array}{c} \begin{array}{cccccc} ([v_{0,0}],1) & ([v_{0,0}],2) & ([v_{0,0}],3) & ([v_{1,0}],1) & ([v_{1,0}],2) & ([v_{1,0}],3) \end{array} \\ \begin{pmatrix} 2 & 0 & 0 & -2 & 0 & 0 \\ 1-\frac{\sqrt{2}}{2}(1+\omega) & \omega-\frac{\sqrt{2}}{2}(1+\omega) & \omega-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{2}}{2}(1+\omega)-1 & \frac{\sqrt{2}}{2}(1+\omega)-\omega & \omega-1 \end{pmatrix} \end{array}$$

$\Omega(\mathcal{G}_{dh}) = \mathbb{T}$

The function $z(\chi_\omega, a) : V \to \mathbb{C}^3$ with

$$v_{j,k} \mapsto \omega^k \begin{pmatrix} \cos(\frac{k\pi}{4}) + \sin(\frac{k\pi}{4}) \\ \sin(\frac{k\pi}{4}) - \cos(\frac{k\pi}{4}) \\ (-1)^j \end{pmatrix}, \quad j \in \{0,1\},\ k \in \mathbb{Z}.$$

is a χ_ω-symmetric infinitesimal flex, where $\omega \in \mathbb{T}$ and $a = (1,-1,1,1,-1,-1)^T$.

Diamond lattice direction-length framework

$$\Phi(\omega,\iota) = \begin{matrix} [e_{1,(0,0)}] \\ [e_{2,(0,0)}] \end{matrix} \overset{\begin{matrix} ([v_{0,0}],1) & ([v_{0,0}],2) \end{matrix}}{\begin{pmatrix} 0 & 1-\omega \\ -1+\omega\iota & -2(1+\omega\iota) \end{pmatrix}}$$

where $\omega \in \mathbb{T}$ and $\iota \in \hat{\mathbb{Z}}_2$.

$\Omega(\mathcal{G}_{dl}) = \{(1,1),(1,-1),(-1,-1)\}$

The vector $z(\chi_{-1,-1}, a)$ with components,

$$z_{v_{m,j}} = (-1)^m(-1)^j d\tau(m,j)a = \begin{pmatrix} (-1)^{m+j} \\ 0 \end{pmatrix}, \quad m \in \mathbb{Z},\ j \in \mathbb{Z}_2.$$

is a $\chi_{-1,-1}$-symmetric infinitesimal flex, where $a := \binom{1}{0}$.

Box-kite non-Euclidean bar-joint framework

$$\|(x,y,z)\|_{2,q} = ((x^2+y^2)^{\frac{q}{2}} + |z|^q)^{\frac{1}{q}}.$$

$$\Phi(\eta,\iota) = \begin{matrix} [e_{1,(0,0)}] \\ [e_{2,(0,0)}] \end{matrix} \overset{\begin{matrix} ([v_{0,0}],1) & ([v_{0,0}],2) & ([v_{0,0}],3) \end{matrix}}{\begin{pmatrix} -1 & -\eta & 0 \\ -2^{q-1}\alpha & -2^{q-1}\alpha\eta\iota & -\alpha(1+\eta\iota) \end{pmatrix}},$$

where $\alpha = (2^q+1)^{\frac{1}{q}-1}$.
$\Omega(\mathcal{G}_{bk}) = \hat{\mathbb{Z}}_4 \times \hat{\mathbb{Z}}_2$

$z(\chi_{-1,-1},a)$ is a $\chi_{-1,-1}$-symmetric infinitesimal flex of $\mathcal{G}_{bk}$ where $a = \begin{pmatrix} 1 \\ 1 \\ -2^{q-1} \end{pmatrix}$ and, for $j \in \mathbb{Z}_2$,

$$z_{v_{0,j}} = \begin{pmatrix} 1 \\ 1 \\ (-1)^{j+1}2^{q-1} \end{pmatrix}, \quad z_{v_{1,j}} = \begin{pmatrix} 1 \\ -1 \\ (-1)^{j}2^{q-1} \end{pmatrix}, \quad z_{v_{2,j}} = \begin{pmatrix} -1 \\ -1 \\ (-1)^{j+1}2^{q-1} \end{pmatrix}, \quad z_{v_{3,j}} = \begin{pmatrix} -1 \\ 1 \\ (-1)^{j}2^{q-1} \end{pmatrix}.$$

Thank you