Global rigidity of periodic body-bar frameworks

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Joint work with V.E. Kaszanitzky and Cs. Király

Outline

1. Rigidity and global rigidity of periodic frameworks
2. Vertex-redundant rigidity implies global rigidity
3. Characterisation of generic globally rigid periodic body-bar frameworks
4. Open problems

Periodic graphs

- Let $\Gamma \simeq \mathbb{Z}^k$. A Γ-labelled graph is a pair (G, ψ) of a finite directed (multi-) graph G and a map $\psi : E(G) \to \Gamma$.
- We assume that G has no parallel edges with the same label and no loops.

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- (G, ψ) defines a k-periodic graph $\tilde{G} = (\tilde{V}, \tilde{E})$ (the covering of (G, ψ)):
  $V(\tilde{G}) = \{\gamma v_i : v_i \in V(G), \gamma \in \Gamma\}$ and
  $E(\tilde{G}) = \{\{\gamma v_i, \psi(v_i v_j)\gamma v_j\} : (v_i, v_j) \in E(G), \gamma \in \Gamma\}$.
  Γ is the periodicity of $\tilde{G}$, which acts naturally on $V(\tilde{G})$ and $E(\tilde{G})$.

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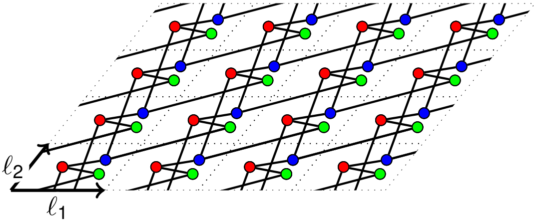
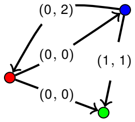
- For a closed walk $C = v_1, e_1, v_2, \ldots, e_k, v_1$ in (G, ψ), let $\psi(C) = \sum_{i=1}^{k} \mathrm{sign}(e_i)\psi(e_i)$, where $\mathrm{sign}(e_i) = 1$ if e_i has forward direction in C, and $\mathrm{sign}(e_i) = -1$ otherwise.
- For $H \subseteq G$, let $\Gamma_H = \langle\{\psi(C) : C \text{ closed walk of } H\}\rangle$. The rank of H is the rank of Γ_H.

Periodic frameworks





L-periodic framework $(\tilde{G}, \tilde{p})$ Γ-labeled framework $((G, \psi), p)$

- Let $\tilde{G} = (\tilde{V}, \tilde{E})$ be a k-periodic graph with periodicity Γ, and let $L : \Gamma \to \mathbb{R}^d$ be a non-singular homomorphism (i.e., $L(\Gamma)$ has rank k) with $k \leq d$.

  A pair $(\tilde{G}, \tilde{p})$ of $\tilde{G}$ and $\tilde{p} : \tilde{V} \to \mathbb{R}^d$ is an L-periodic framework in $\mathbb{R}^d$ if

$$\tilde{p}(v) + L(\gamma) = \tilde{p}(\gamma v) \qquad \text{for all } \gamma \in \Gamma \text{ and all } v \in \tilde{V}. \tag{1}$$

- Γ-labelled framework: (G, ψ, p), where (G, ψ) is Γ-labelled graph and $p : V(G) \to \mathbb{R}^d$.

L-periodic local and global rigidity

- Let $(\tilde{G}, \tilde{p})$ and $(\tilde{G}, \tilde{q})$ be L-periodic frameworks.
- $(\tilde{G}, \tilde{p})$ and $(\tilde{G}, \tilde{q})$ are equivalent if

$$\|\tilde{p}(u) - \tilde{p}(v)\| = \|\tilde{q}(u) - \tilde{q}(v)\| \qquad \text{for all } uv \in \tilde{E}.$$

They are congruent if

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- $(\tilde{G}, \tilde{p})$ is called L-periodically (locally) rigid if in some open neighborhood of $\tilde{p}$ every L-periodic framework $(\tilde{G}, \tilde{q})$ equivalent to $(\tilde{G}, \tilde{p})$ is congruent to $(\tilde{G}, \tilde{p})$.

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- $(\tilde{G}, \tilde{p})$ is called *L-periodically globally rigid* if every L-periodic framework $(\tilde{G}, \tilde{q})$ equivalent to $(\tilde{G}, \tilde{p})$ is congruent to $(\tilde{G}, \tilde{p})$.

Example

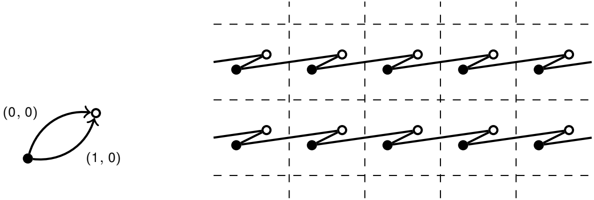

Figure: A rigid but not globally rigid L-periodic framework with rank 2 periodicity in $\mathbb{R}^2$.

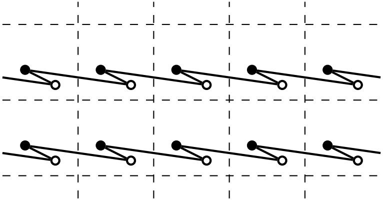

Figure: An equivalent but not congruent L-periodic framework.

Analysing periodic (global) rigidity

- For a Γ-labelled graph (G, ψ) and $L : \Gamma \to \mathbb{R}^d$:

  Rigidity map of (G, ψ):

  $$f_{G,L} : \mathbb{R}^{d|V|} \ni p \mapsto (\dots, \|p(u) - (p(v) + L(\psi(uv)))\|^2, \dots)^T \in \mathbb{R}^{|E|}$$

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- The complete Γ-labelled graph $K(V, \Gamma)$ on V has vertex set V and edge set $\{((u, v); \gamma) : u, v \in V, \gamma \in \Gamma\}$. We denote $f_{K(V,\Gamma),L}$ by $f_{V,L}$.

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- **Prop.:** Let $(\tilde{G}, \tilde{p})$ be an L-periodic framework and let (G, ψ, p) be a quotient Γ-labelled framework of $(\tilde{G}, \tilde{p})$. Then $(\tilde{G}, \tilde{p})$ is L-periodically globally rigid (resp. rigid) if and only if for every $q \in \mathbb{R}^{d|V(G)|}$ (resp. for every q in an open neighborhood of p in $\mathbb{R}^{d|V(G)|}$), $f_{G,L}(p) = f_{G,L}(q)$ implies $f_{V(G),L}(p) = f_{V(G),L}(q)$.

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Periodic local rigidity

Periodic rigidity matrix

- A Γ-labelled framework (G, ψ, p) is generic if the set of coordinates in p is algebraically independent over $\mathbb{Q}$.
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- **Thm. (Ross, 14, Kaszanitzky, S., Tanigawa, 19)**: Let (G, ψ, p) be a generic Γ-labelled framework in $\mathbb{R}^d$ with $|V(G)| \geq d + 1$ and rank k periodicity Γ, and let $L : \Gamma \to \mathbb{R}^d$ be non-singular. Then (G, ψ, p) is L-periodically rigid if and only if

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- **Note:** A rigidity matrix for periodic frameworks with flexible lattice representations was established by Borcea and Streinu in 2010.

Periodic rigidity: combinatorial characterisation in 2D

- **Thm. (Ross, 15)**: Let $(\tilde{G}, \tilde{p})$ be a generic L-periodic framework with $L : \mathbb{Z}^2 \to \mathbb{R}^2$. Then $(\tilde{G}, \tilde{p})$ is L-periodically rigid iff the $\mathbb{Z}^2$-labeled quotient graph (G, ψ) contains a spanning subgraph (H, ψ_H) satisfying:
  (i) $|E(H)| = 2|V(H)| - 2$;
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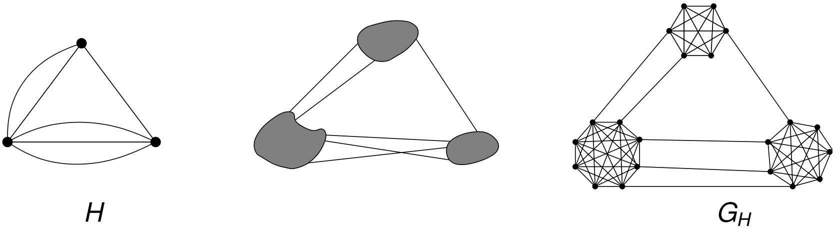
- **Note:** Suppose we just want a rigid periodic framework for some labeling of the edges of the quotient graph. Here we have characterisations for generic rigidity in all dimensions for the fixed lattice (Whiteley, 88) and for the flexible lattice (Borcea and Streinu, 11).

Periodic body-bar frameworks in $\mathbb{R}^d$, $d \geq 3$

- A (finite) body-bar framework is a bar-joint framework (G_H, p) with $p : V(G_H) \to \mathbb{R}^d$, where H is a multi-graph with no loops and G_H is the simple body-bar graph induced by H.



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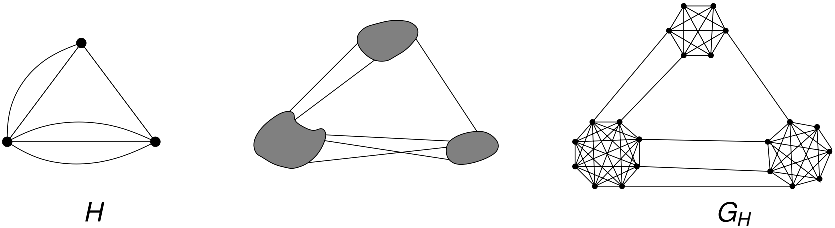

- L-periodic body-bar framework is a bar-joint framework $(G_{\tilde{H}}, \tilde{p})$:
  - (H, ψ) Γ-labelled graph which may have loops with non-trivial labels and parallel edges with equal labels.
  - (H, ψ) defines a *k-periodic multi-graph* $\tilde{H}$ with no loops.
  - $\tilde{H}$ induces k-periodic body-bar graph $G_{\tilde{H}}$ as above, but for any $e \in E(\tilde{H})$ joining v with γv for some $\gamma \neq \mathrm{id}$, we add two vertices v_{e^-} and v_{e^+} (instead of just one vertex v_e) to the 'body of v', and define e' to be the edge $v_{e^-}\gamma v_{e^+}$ (instead of $v_e \gamma v_e$).

Periodic rigidity of body-bar frameworks in $\mathbb{R}^d$, $d \geq 3$

- **Thm. (Tanigawa, 15)**: Let $(G_{\tilde{H}}, \tilde{p})$ be a generic L-periodic body-bar realisation of the multi-graph $\tilde{H}$ in $\mathbb{R}^d$ with rank k periodicity Γ, and $L : \Gamma \to \mathbb{R}^d$ be non-singular.

  $(G_{\tilde{H}}, \tilde{p})$ is L-periodically rigid in $\mathbb{R}^d$ if and only if the quotient Γ-labelled graph H of $\tilde{H}$ contains a spanning subgraph (V, E) satisfying

  - $|E| = \binom{d+1}{2}|V| - d - \binom{d-k}{2}$;
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- Extensions to body-hinge or molecular frameworks are open.

- Suppose we just want a rigid periodic framework for some labeling of the edges of the quotient graph. Characterisations for generic rigidity of body-bar frameworks (with a flexible lattice) in $\mathbb{R}^d$ established by Borcea, Streinu and Tanigawa in 2012.

Periodic global rigidity

Global periodic rigidity: comb. characterisation in 2D

- **Thm. (Kaszanitzky, S., Tanigawa, 19)**: Let $(\tilde{G}, \tilde{p})$ be a generic L-periodic framework with rank $k = 1$ periodicity Γ and $|V(G)| \geq 3$. Then $(\tilde{G}, \tilde{p})$ is L-periodically globally rigid if and only if the Γ-labeled quotient graph (G, ψ) is bar-redundantly periodically rigid in $\mathbb{R}^2$, 2-connected, and has no $(0, 2)$-block.

- **Thm. (Kaszanitzky, S., Tanigawa, 19)**: Let $(\tilde{G}, \tilde{p})$ be a generic L-periodic framework with rank $k = 2$ periodicity Γ and $|V(G)| \geq 3$. Then $(\tilde{G}, \tilde{p})$ is L-periodically globally rigid if and only if the Γ-labeled quotient graph (G, ψ) is connected and each 2-connected component (G', ψ') of (G, ψ) is bar-redundantly periodically rigid in $\mathbb{R}^2$, has no $(0, 2)$-block, and has rank two.

$k = 2$ connectivity conditions: no $(0,2)$-block

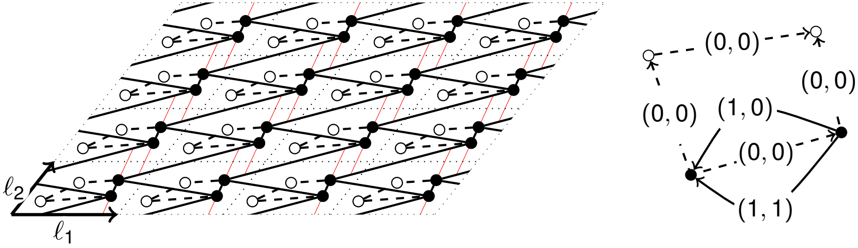
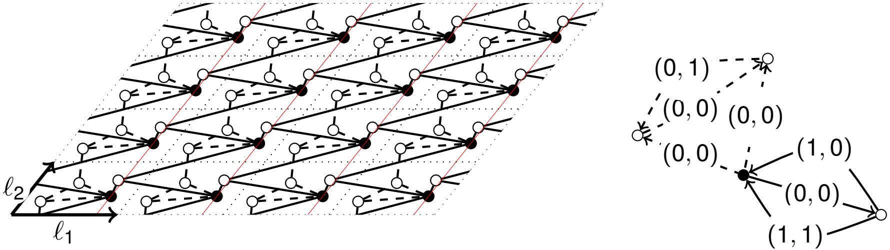


Figure: $(0,2)$-block shown with dashed edges.

$k = 2$ connectivity conditions: no rank 1 component



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Global periodic rigidity in $\mathbb{R}^d$, $d \geq 3$

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- Combinatorial characterisations for generic periodic global rigidity in dimensions $d \geq 3$ are open.

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- Combinatorial characterisations for generic periodic global rigidity in dimensions $d \geq 3$ are open.

- The special case of body-bar frameworks is solved:

  Thm (Kaszanitzky, Király, S., 21): Let $(G_{\tilde{H}}, \tilde{p})$ be a generic L-periodic body-bar realisation of the multi-graph $\tilde{H}$ in $\mathbb{R}^d$ with rank k periodicity Γ, and let $L : \Gamma \to \mathbb{R}^d$ be non-singular.

  $(G_{\tilde{H}}, \tilde{p})$ is L-periodically globally rigid in $\mathbb{R}^d$ if and only if $(G_{\tilde{H}}, \tilde{p})$ is L-periodically bar-redundantly rigid in $\mathbb{R}^d$, and the quotient Γ-labelled graph of $G_{\tilde{H}}$ is of rank d in the case when $k = d$.

Vertex-redundant rigidity implies global rigidity

The main result

- A Γ-labelled framework (G, ψ, p) is *L*-periodically vertex-redundantly rigid if for every vertex v of G, the Γ-labelled framework $(G - v, \psi|_{G-v}, p|_{V(G)-v})$ is L-periodically rigid.

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- **Thm (Kaszanitzky, Király, S., 21):** Let (G, ψ, p) be a generic Γ-labelled framework in $\mathbb{R}^d$ with rank k periodicity Γ, and let $L : \Gamma \to \mathbb{R}^d$ be non-singular.

  If (G, ψ, p) is L-periodically vertex-redundantly rigid, and if (G, ψ) is also of rank d in the case when $k = d$, then (G, ψ, p) is L-periodically globally rigid in $\mathbb{R}^d$.

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- The proof is similar to Tanigawa's proof for finite frameworks, but there are some added difficulties:
  - global rigidity of small rigid graphs
  - periodic global rigidity a generic property?

Global rigidity of small graphs: the key lemma

- Need to show that for small periodic graphs, rigidity implies global rigidity.
- To this end, we extend a result of Bezdek and Connelly (2002).

- **Lemma:** Let (G, ψ) be the $\mathbb{Z}^k$-labelled graph with vertices $v_1, \ldots, v_n$ and no edges, and let $L : \mathbb{Z}^k \to \mathbb{R}^d$ be non-singular. Let (G, ψ, p) and (G, ψ, q) be two $\mathbb{Z}^k$-labelled frameworks whose coverings are the L-periodic frameworks $(\tilde{G}, \tilde{p})$ and $(\tilde{G}, \tilde{q})$ in $\mathbb{R}^d$.

  Denote $p_{\gamma,i} = \tilde{p}(\gamma v_i) = p(v_i) + L(\gamma)$ and $q_{\gamma,i} = \tilde{q}(\gamma v_i) = q(v_i) + L(\gamma)$ for $i = 1, \ldots, n$ and $\gamma \in \mathbb{Z}^k$. Let $\bar{p}_{\gamma,i} : [0,1] \to \mathbb{R}^{2d}$ be the following continuous maps for $i = 1, \ldots, n$:

$$\bar{p}_{\gamma,i}(t) = \left(\frac{p_{\gamma,i} + q_{\gamma,i}}{2} + (\cos(\pi t)) \frac{p_{\gamma,i} - q_{\gamma,i}}{2}, (\sin(\pi t)) \frac{p_{\gamma,i} - q_{\gamma,i}}{2} \right).$$

  Then $\bar{p}_{\gamma,i}(0) = (p_{\gamma,i}, 0^d)$ and $\bar{p}_{\gamma,i}(1) = (q_{\gamma,i}, 0^d)$.

  Also: $|\bar{p}_{\gamma,i}(t) - \bar{p}_{\gamma',j}(t)|$ is monotone and $\bar{p}_{\gamma,i}(t) = \bar{p}_{0^k,i}(t) + (L(\gamma), 0^d)$ for every $i, j \in \{1, \ldots, n\}$ and $\gamma, \gamma' \in \mathbb{Z}^k$.

Global rigidity of small graphs

- **Theorem.**: Let $L : \Gamma \to \mathbb{R}^d$ be non-singular and let (G, ψ, p) be not L-periodically globally rigid in $\mathbb{R}^d$. Then $(G, \psi, (p, 0^d))$ is $(L, 0^d)$-periodically flexible in $\mathbb{R}^{2d}$.

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  Proof:

  - There exists (G, ψ, q) which is equivalent but not congruent to (G, ψ, p).
  - By Lemma, there exists a continuous deformation between $(\tilde{G}, (\tilde{p}, 0^d))$ and $(\tilde{G}, (\tilde{q}, 0^d))$ in $\mathbb{R}^{2d}$ that maintains lattice $(L, 0^d)$ and, by the monotonicity of the distances, also maintains the edge lengths.

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- **Corollary**: Let (G, ψ, p) be a Γ-labelled framework in $\mathbb{R}^d$ with rank k periodicity and $L : \Gamma \to \mathbb{R}^d$. Suppose (G, ψ, p) is L-periodically rigid and $|V(G)| \leq d - k + 1$. Then (G, ψ, p) is also L-periodically globally rigid.

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  Proof:
  - For $D \geq d$, the points $(\tilde{q}(v), 0^{D-d})$ of the $(L, 0^{D-d})$-periodic framework $(\tilde{G}, (\tilde{q}, 0^{D-d}))$ in $\mathbb{R}^D$ affinely span a space of dim. $\leq |V(G)| + k - 1 \leq d$.
  - If (G, ψ, p) is not L-periodically globally rigid in $\mathbb{R}^d$, then the points of the frameworks along the motion guaranteed by the Theorem span an at most d-dimensional subspace, a contradiction.

Key geometric lemma

- Assuming that every edge incident to v is directed from v, let (G_v, ψ_v) be the Γ-labeled graph obtained from (G, ψ) by removing v and inserting a new edge from u to w with label $\psi(vu)^{-1}\psi(vw)$ for every pair of edges vu, vw incident to v (unless uw with that label is already present).

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- **Lemma:** Let (G, ψ, p) be a generic Γ-labelled framework in $\mathbb{R}^d$ with rank k periodicity Γ and with $|V(G)| \geq d - k + 2$ and let $L : \Gamma \to \mathbb{R}^d$ be non-singular.

  Suppose that the covering $(\tilde{G}, \tilde{p})$ has a vertex v with at least $d + 1$ neighbours $\gamma_0 v_0, \gamma_1 v_1, \dots, \gamma_d v_d$, where $v, v_i \in V(G), \gamma_i \in \Gamma$, so that the points $\tilde{p}(\gamma_0 v_0), \tilde{p}(\gamma_1 v_1), \dots, \tilde{p}(\gamma_d v_d)$ affinely span $\mathbb{R}^d$. If

  - $(G - v, \psi|_{G-v}, p')$ is L-periodically rigid in $\mathbb{R}^d$, with $p' = p|_{V(G)-v}$
  - (G_v, ψ_v, p') is L-periodically globally rigid in $\mathbb{R}^d$.

  Then (G, ψ, p) is L-periodically globally rigid in $\mathbb{R}^d$.

Proof sketch of the main result

- **Thm (Kaszanitzky, Király, S., 21):** Let (G, ψ, p) be a generic Γ-labelled framework in $\mathbb{R}^d$ with rank k periodicity Γ, and let $L : \Gamma \to \mathbb{R}^d$ be non-singular.

  If (G, ψ, p) is L-periodically vertex-redundantly rigid, and if (G, ψ) is also of rank d in the case when $k = d$, then (G, ψ, p) is L-periodically globally rigid in $\mathbb{R}^d$.

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  • Suppose $|V(G)| = d - k + 2$. Then $|V(G - v)| = d - k + 1$ and since $(G - v, \psi|_{G-v}, p|_{V(G)-v})$ is L-periodically rigid, it is also L-periodically globally rigid (by Cor.).

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  • Claim: for any occurrence of any $v \in V(G)$ in the covering $\tilde{G}$, the affine span of $\{\tilde{p}(w) | \, vw \in E(\tilde{G})\}$ is all of $\mathbb{R}^d$. The result then follows from the key geometric lemma.

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  • Suppose $d > k$ (and hence $|V(G)| = d - k + 2 > 2$). Suppose the claim is false.
  Removal of a neighbour of v (and the whole vertex orbit) results in an L-periodic framework with at least two vertex orbits. By genericity, all the points connected to $\tilde{p}(v)$ in this framework affinely span a space of dimension at most $d - 2$. $\tilde{p}(v)$ can be rotated about this $(d-2)$-dim. axis. Contradiction to vertex-redundant rigidity of $(\tilde{G}, \tilde{p})$

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  • Suppose $|V(G)| > d - k + 2$.

  • Claim: (G_v, ψ_v, p') is L-periodically vertex-redundantly rigid for any $v \in V(G)$.
  Easy to see that (G, ψ) is of rank d then so is (G_v, ψ_v). By induction hypothesis, (G_v, ψ_v, p') is L-periodically globally rigid. Further, as above, the affine span of $\{\tilde{p}(w) |\, vw \in E(\tilde{G})\}$ is all of $\mathbb{R}^d$. So the result then follows from the key geometric lemma.

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  • Suppose the claim is false. Then there is a vertex u whose removal results in an L-periodically flexible framework.

  • The neighbours of one occurrence of v in $\tilde{G}$ induce a complete graph in $\tilde{G}_v$. So adding v and its incident edges to $(G_v - u, \psi_v|_{G_v - u}, p|_{V(G)-\{u,v\}})$ still yields an L-periodically flexible framework.

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  • This is a contradiction, as $(G - u, \psi|_{G-u}, p|_{V(G)-u})$ is an L-periodically rigid spanning subframework of the framework obtained from $(G_v - u, \psi_v|_{G_v - u}, p|_{V(G)-\{u,v\}})$ by adding v and its incident edges. $\square$

Characterisation of generic globally rigid periodic body-bar frameworks

Global rigidity of periodic body-bar frameworks

- **Thm (Kaszanitzky, Király, S., 21):** Let $(G_{\tilde{H}}, \tilde{p})$ be a generic L-periodic body-bar realisation of the multi-graph $\tilde{H}$ in $\mathbb{R}^d$ with rank k periodicity Γ, and let $L : \Gamma \rightarrow \mathbb{R}^d$ be non-singular.

  $(G_{\tilde{H}}, \tilde{p})$ is L-periodically globally rigid in $\mathbb{R}^d$ if and only if $(G_{\tilde{H}}, \tilde{p})$ is L-periodically bar-redundantly rigid in $\mathbb{R}^d$, and the quotient Γ-labelled graph of $G_{\tilde{H}}$ is of rank d in the case when $k = d$.

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  • If $k = d$, then we must have $\mathrm{rank}\ G_{\tilde{H}} = d$ for L-periodic global rigidity.

  • L-periodic bar-redundant rigidity implies L-periodic vertex-redundant rigidity, since the edges connecting the bodies are all disjoint. The result now follows from the main theorem.

Combinatorial periodic rigidity: summary

Rigidity:

	fixed lattice	flexible lattice
bar-joint $d = 2$	Ross 15	Malestein-Theran 13
bar-joint $d \geq 3$	open	open
body-bar	Tanigawa 15	open
body-hinge	open	open
molecular	open	open

Global rigidity:

	fixed lattice	flexible lattice
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bar-joint $d \geq 3$	open	open
body-bar	Kaszanitzky-Király-S. 21	open
body-hinge	open	open
molecular	open	open

Thank you!

Questions?