On étale wild kernels and Greenberg conjecture

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Today’s theme

We like to study the variation of the étale wild kernels in $p$-adic Lie extensions.

We begin with the basic case of Iwasawa, where he first proved a result on the growth of Sylow $p$-subgroup of class groups in a $\mathbb{Z}_p$-extension.

We then introduce the étale wild kernel and describe the ideas behind our proofs.
Let $F$ be a number field. Consider a sequence of number fields

$$F \subseteq F_1 \subseteq F_2 \subseteq \cdots$$

such that each $F_n$ is a Galois extension of $F$ with $\text{Gal}(F_n/F) \cong \mathbb{Z}/p^n\mathbb{Z}$.

Set $F_\infty = \bigcup_n F_n$. This is a Galois extension of $F$ with $\text{Gal}(F_\infty/F) \cong \mathbb{Z}_p$

(and hence is called a $\mathbb{Z}_p$-extension of $F$).
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Remark: We write $\Gamma$ for the multiplicative group which is isomorphic to
the additive group $\mathbb{Z}_p$ and identify $\text{Gal}(F_\infty/F)$ with $\Gamma$.

Throughout this talk, $p$ is always taken to be an odd prime.
Basic example of $\mathbb{Z}_p$-extension

Set

$$\tilde{F} = \bigcup_{n} F(e^{2\pi i/p^n}).$$

It follows from Galois theory that

$$\text{Gal}(\tilde{F}/F) \cong \mathbb{Z}_p \times \Delta$$

where $\Delta$ is a finite cyclic group of order dividing $p - 1$. 
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The **cyclotomic** $\mathbb{Z}_p$-extension $F^{\text{cyc}}$ of $F$ is defined to be the fixed field of $\Delta$. It follows from the definition that $\text{Gal}(F^{\text{cyc}}/F) \cong \mathbb{Z}_p$. 
Iwasawa asymptotic formula

As before, $p$ denotes an odd prime. Let $F_\infty$ be a $\mathbb{Z}_p$-extension of $F$ with intermediate subfields $F_n$. For a finite abelian group $N$, we write $e(N)$ to be the number such that

$$p^{e(N)} = \left| N[p^\infty] \right|$$
Iwasawa asymptotic formula

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**Theorem (Iwasawa 1959)**

There exist integers $\mu = \mu(F_{\infty}/F)$ and $\lambda = \lambda(F_{\infty}/F)$ (independent of $n$) such that

$$e(\text{Cl}(F_n)) = \mu p^n + \lambda n + O(1) \quad \text{for } n \gg 0.$$
Asymptotic formulas over $p$-adic Lie extensions

Iwasawa’s result was the first of its kind which describes a precise growth of the $p$-class groups in an infinite tower of number fields.

For a $\mathbb{Z}_p^d$-extension, this was established by Cuoco and Monsky.

For certain noncommutative $p$-adic Lie extensions, asymptotic formulas were established by Perbet, Lei and Liang-Lim.
Idea behind Iwasawa’s proof

Basic facts:
(1) A $\mathbb{Z}_p$-extension $F_\infty/F$ is unramified outside $p$.
(2) Every prime of $F$ (above $p$) that ramified in $F_\infty/F$ will be totally ramified in $F_\infty/F_n$ for big enough $n$.

Without loss of generality, we may assume that the ramified primes of $F_\infty/F$ are totally ramified.

Set $H_n$ to be the $p$-Hilbert class field of $F_n$, in other words, $H_n$ is an unramified extension of $F_n$ with $\text{Gal}(H_n/F_n) \cong \text{Cl}(F_n)[p^\infty]$.

Then by the above ramification assumption, we have $H_n \cap F_\infty = F_n$ and hence $H_n \subseteq H_n F_{n+1} \subseteq H_{n+1}$.
Idea behind Iwasawa’s proof (cont’d)

The inclusion $H_n \subseteq H_n F_{n+1} \subseteq H_{n+1}$ induces a group homomorphism

$$
\text{Gal}(H_{n+1}/F_{n+1}) \rightarrow \text{Gal}(H_n F_{n+1}/F_{n+1}) \cong \text{Gal}(H_n/F_n)
$$

which fits into the following diagram

$$
\begin{array}{ccc}
\text{Gal}(H_{n+1}/F_{n+1}) & \cong & \text{Cl}(F_{n+1})[p^\infty] \\
\downarrow & & \downarrow \text{Norm} \\
\text{Gal}(H_n/F_n) & \cong & \text{Cl}(F_n)[p^\infty]
\end{array}
$$

Taking inverse limit, we obtain an isomorphism

$$
\text{Gal}(H_\infty/F_\infty) \cong \lim_{\leftarrow} \text{Gal}(H_n/F_n) \cong \lim_{\leftarrow} \text{Cl}(F_n)[p^\infty]
$$

of $\mathbb{Z}_p[[\Gamma]]$-modules.
Two further observations

**Theorem (Iwasawa)**

\[
\text{Gal}(H_\infty / F_\infty) \text{ is a torsion } \mathbb{Z}_p[\Gamma]\text{-module.}
\]

From the structure of \( \mathbb{Z}_p[\Gamma]\)-module, we have the following

**Theorem**

Write \( \Gamma_n = \Gamma p^n \). Let \( M \) be a torsion \( \mathbb{Z}_p[\Gamma]\)-module such that \( M_{\Gamma_n} \) is finite for all \( n \). Then there exist \( \mu, \lambda \) such that

\[
e(M_{\Gamma_n}) = \mu p^n + \lambda n + O(1).
\]
Completing the proof of Iwasawa’s formula

By appealing to Galois theory and class field theory, one can show that the map

$$\text{Gal}(H_{\infty}/F_{\infty})_{\Gamma_n} \rightarrow \text{Gal}(H_n/F_n)$$

has finite kernel and cokernel which is bounded independent of $n$, and whence

$$\left| e\left( \text{Gal}(H_{\infty}/F_{\infty})_{\Gamma_n} \right) - e\left( \text{Gal}(H_n/F_n) \right) \right| = O(1).$$

By the two observations in the previous slide, there exist $\mu, \lambda$ such that

$$e\left( \text{Gal}(H_{\infty}/F_{\infty})_{\Gamma_n} \right) = \mu p^n + \lambda n + O(1).$$

In conclusion, we have

$$e(\text{Cl}(F_n)) = e\left( \text{Gal}(H_n/F_n) \right) = \mu p^n + \lambda n + O(1)$$
Cohomological interpretation: preparation

We recall some terminology.

\[ S_\infty \] (resp., \( S_p \)) for the set of infinite primes (resp., primes above \( p \)) in \( S \).

Denote by \( F_{S\infty} \) the maximal algebraic extension of \( F \) unramified outside \( S \).

Write \( G_{S}(F) \) for the Galois group \( \text{Gal}(F_{S\infty}/F) \).

Let \( \mu_p^n \) be the cyclic group generated by a primitive \( p^n \)-root of unity.

This has a natural \( G_{S}(F) \)-module structure. Furthermore, for \( j \geq 2 \), via the diagonal action, we see that \( j \)-fold tensor product \( \mu \otimes_p j \) can be endowed with a \( G_{S}(F) \)-module structure.
We recall some terminology.

Write $S_{\infty}$ (resp., $S_p$) for the set of infinite primes (resp., primes above $p$) in $S$. Denote by $F_S$ the maximal algebraic extension of $F$ unramified outside $S$. Write $G_S(F)$ for the Galois group $\text{Gal}(F_S/F)$.

Let $\mu_{p^n}$ be the cyclic group generated by a primitive $p^n$-root of unity. This has a natural $G_S(F)$-module structure. Furthermore, for $j \geq 2$, via the diagonal action, we see that $j$-fold tensor product $\mu_{p^n} \otimes^j$ can be endowed with a $G_S(F)$-module structure.
Galois cohomology

For $j \geq 1$, we write $\mathbb{Z}_p(j) = \lim_{\leftarrow} \mu_{p^n} \otimes j$. 

$\Rightarrow \hspace{1cm} \Rightarrow \hspace{1cm} \Rightarrow$
For $j \geq 1$, we write $\mathbb{Z}_p(j) = \lim_{\leftarrow n} \mu_{p^n}^\otimes j$.

Then we have

$$\lim_{\leftarrow n} H^k_{\text{ét}}(\text{spec}(\mathcal{O}_F, S), \mu_{p^n}^\otimes j) \cong \lim_{\leftarrow n} H^k(G_S(F), \mu_{p^n}^\otimes j) \cong H^k_{\text{cts}}(G_S(F), \mathbb{Z}_p(j)),$$

where $H^k_{\text{cts}}(\ , \ )$ is the continuous cohomology group of Tate.
Étale wild kernel

For $i \geq 1$, the étale wild kernels $WK_{2i}^{\text{ét}}(F)$ is defined to be the kernel of the localization map

$$H^2(G_S(F), \mathbb{Z}_p(i + 1)) \longrightarrow \bigoplus_{v \in S} H^2(F_v, \mathbb{Z}_p(i + 1)).$$

Fact: $WK_{2i}^{\text{ét}}(F)$ is finite. As a matter of fact, $H^2(G_S(F), \mathbb{Z}_p(i + 1))$ is finite by a theorem of Borel and Soulé.
Lichtenbaum’s conjecture

**Theorem (Kolster-Nyugen Quand Do-Fleckinger, Wiles)**

Let $F$ be a totally real number field and $\zeta_F(z)$ the Dedekind zeta function of $F$. Let $p$ be an odd prime. Then for odd $i \geq 1$, we have an equality

$$|\mathcal{WK}_2^1(F)| = \left| \frac{w_{i+1}(F)\zeta_F(-i)}{\prod_{v|p} w_i(F_v)} \right|_p^{-1}.$$

Here $w_{i+1}(F) = |H^0(F, \mu_p^\otimes i+1)|$ and $w_i(F_v) = |H^0(F_v, \mu_p^\otimes i)|$.
To understand the growth of étale wild kernel in a $p$-adic Lie extension $F_\infty/F$, one is led to analysing the codescent map

$$(\limleftarrow_n WK^{\text{ét}}_{2i}(F_n))_{G_n} \longrightarrow WK^{\text{ét}}_{2i}(F_n),$$

where the $F_n$'s are certain appropriate subextensions of $F_\infty/F$ with $G_n = \text{Gal}(F_\infty/F_n)$.

For a finite $p$-extension, the kernel and cokernel of this codescent map has been determined. However, despite their explicit nature, it is not easy to obtain good enough estimates from them on the kernels and cokernels for our problem in hand.
Reinterpretation of étale wild kernel

Write $R_i(F) = WK^\text{ét}_{2i}(F)^\vee := \text{Hom}(WK^\text{ét}_{2i}(F), Q_p/\mathbb{Z}_p)$. By Poitou-Tate duality, $R_i(F)$ can be identified as the kernel of

$$H^1(G_S(F), Q_p/\mathbb{Z}_p(-i)) \longrightarrow \bigoplus_{v \in S} H^1(F_v, Q_p/\mathbb{Z}_p(-i))$$

If we write $R_i(F) = (\varprojlim_{n} WK^\text{ét}_{2i}(F_n))^\vee$, then $R_i(F)$ can be identified as the kernel of

$$H^1(G_S(F_\infty), Q_p/\mathbb{Z}_p(-i)) \longrightarrow \bigoplus_{w \in S(F_\infty)} H^1(F_\infty, w, Q_p/\mathbb{Z}_p(-i))$$
A commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & R_i(F_n) \\
& & \downarrow r_n \\
& & H^1(G_S(F_n), \mathbb{Q}_p/\mathbb{Z}_p(-i)) \\
& & \downarrow h_n \\
& & \bigoplus_{v \in S(F_n)} H^1(F_v, \mathbb{Q}_p/\mathbb{Z}_p(-i)) \\
& & \downarrow g_n \\
0 & \rightarrow & R_i(F_\infty)^G \rightarrow H^1(G_S(F_\infty), \mathbb{Q}_p/\mathbb{Z}_p(-i))^G \\
& & \bigoplus_{v \in S(F_\infty)} H^1(F_\infty, v, \mathbb{Q}_p/\mathbb{Z}_p(-i))^G \\
& & G_n \\
\end{array}
\]

By the snake lemma, we have an exact sequence

\[
0 \rightarrow \ker r_n \rightarrow \ker h_n \rightarrow C_n \rightarrow \coker r_n \rightarrow \coker h_n,
\]

where \( C_n \) is a subgroup of \( \ker g_n \). Therefore, in estimating \( \ker r_n \) and \( \coker r_n \), one is reduced to studying \( \ker h_n \), \( \coker h_n \) and \( \ker g_n \).
From the Hochschild-Serre spectral sequence, we see that

\[ \ker h_n = H^1(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)(F_\infty)), \]

\[ \text{coker } h_n \subseteq H^2(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)(F_\infty)), \]

where we write \( \mathbb{Q}_p/\mathbb{Z}_p(-i)(F_\infty) = (\mathbb{Q}_p/\mathbb{Z}_p(-i))^{G_S(F_\infty)}. \)

This therefore leads us to estimating the groups \( H^1(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)(F_\infty)) \) and \( H^2(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)(F_\infty)). \) For the estimate of \( \ker g_n \), one needs to estimate

\[ H^1(\text{Gal}(F_\infty,w/F_{v_n}), \mathbb{Q}_p/\mathbb{Z}_p(-i)(F_\infty,w)) \]

and the number of primes of \( F_n \) above \( S \).
A lemma of Tate

For our estimate, we require a lemma of Tate which we now recall. In this slide, $K$ denotes a field of characteristic 0.

**Tate Lemma**

Suppose that $H^0(K, \mathbb{Q}_p/\mathbb{Z}_p(i))$ is finite and that the Galois group $\text{Gal}(K(\mu_{p^\infty})/K)$ is infinite. Then one has

$$H^1(\text{Gal}(K(\mu_{p^\infty})/K), \mathbb{Q}_p/\mathbb{Z}_p(i)) = 0.$$
Let $F$ be a number field which contains a primitive $p$th root of unity. Set $F_n = F(\mu_{p^n}, \sqrt[p^n]{\alpha})$ and $F_\infty = \cup_n F_n$. Set $H_n = \text{Gal}(F_\infty/F(\mu_{p\infty}, \sqrt[p\infty]{\alpha}))$ and $\Gamma_n = \text{Gal}(F(\mu_{p\infty}, \sqrt[p\infty]{\alpha})/F_n)$.\[\]
Let $F$ be a number field which contains a primitive $p$th root of unity. Set $F_n = F(\mu_p^n, \sqrt[p^n]{\alpha})$ and $F_\infty = \bigcup_n F_n$. Set $H_n = \text{Gal}(F_\infty/F(\mu_p, \sqrt[p]{\alpha}))$ and $\Gamma_n = \text{Gal}(F(\mu_p, \sqrt[p]{\alpha})/F_n)$.

From the Hochschild-Serre spectral sequence

$$H^r(\Gamma_n, H^s(H_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))) \Rightarrow H^{r+s}(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)),$$

we have a short exact sequence

$$0 \rightarrow H^1(\Gamma_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)) \rightarrow H^1(G_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)) \rightarrow H^1(H_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))^{\Gamma_n} \rightarrow 0$$
Tate’s lemma tells us that $H^1(\Gamma_n, \mathbb{Q}_p/\mathbb{Z}_p(-i)) = 0$.

On the other hand, since $H_n$ acts trivially on $\mathbb{Q}_p/\mathbb{Z}_p(-i)$ and $H_n \cong \mathbb{Z}_p(1)$ as $\Gamma_n$-modules by Kummer theory, we have

$$H^1(H_n, \mathbb{Q}_p/\mathbb{Z}_p(-i))^\Gamma_n \cong \text{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p(1), \mathbb{Q}_p/\mathbb{Z}_p(-i))^\Gamma_n = \mathbb{Q}_p/\mathbb{Z}_p(-1-i)^\Gamma_n$$

which can be seen to be finite with $\text{ord}_p$-growth $O(n)$.

One also perform same kind of argument for the local primes. In conclusion, we obtain

$$\left| e(R_i(F_{\infty})^{G_n}) - e(R_i(F_n)) \right| = O(n).$$
Estimates

We can obtain estimates for these cohomology groups when the $p$-adic Lie extension is one of the following

1. $\mathbb{Z}_p^d$-extension,
2. a multi-false Tate extension,
3. a $GL_2$-extension cut out by an elliptic curve without complex multiplication,
4. a compositum of a $GL_2$-extension with multi-false-Tate extension.

The estimates give us

$$\left| \left| e \left( R_i \left( F_\infty \right)^G \right) - e \left( R_i \left( F_n \right)^G \right) \right|\right| = O \left( \frac{p^n (d-2)^n}{n} \right).$$
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The estimates give us

$$\left| e(R_i(F_{\infty})^{G_n}) - e(R_i(F_n)) \right| = O(np^{(d-2)n}).$$
Theorem (Lim)

\[ \lim_{n \to \infty} WK_{i}^{\text{ét}}(F_n) \text{ is a torsion } \mathbb{Z}_p[G]\text{-module}. \]
Algebraic Ingredient

**Theorem (Lim)**

\[
\lim_{n \to \infty} WK_i^{\text{ét}}(F_n) \text{ is a torsion } \mathbb{Z}_p[G]-\text{module.}
\]

**Lemma (Lei, Liang-Lim)**

Let \( G \) be a compact pro-\( p \) \( p \)-adic Lie group which contains a closed normal subgroup \( H \cong \mathbb{Z}_p^{d-1} \) such that \( G/H \cong \mathbb{Z}_p \). Let \( M \) be a \( \mathbb{Z}_p[G] \)-module which is finitely generated over \( \mathbb{Z}_p[H] \). Suppose further that \( M_{G_n} \) is finite for every \( n \). Then we have

\[
e(M_{G_n}) = \text{rank}_{\mathbb{Z}_p[H]}(M) np^{(d-1)n} + O(p^{(d-1)n}).
\]
Growth in $\mathbb{Z}_p^{(d-1)} \rtimes \mathbb{Z}_p$-extensions

Now set $F_\infty = F(\mu_p, \sqrt[p]{\alpha_1}, \ldots, \sqrt[p]{\alpha_{d-1}})$, where $\alpha_1, \ldots, \alpha_{d-1} \in F^\times$ whose image in $F^\times/(F^\times)^p$ are linearly independent over $\mathbb{F}_p$. 
Growth in $\mathbb{Z}_p^{(d-1)} \times \mathbb{Z}_p$-extensions

Now set $F_{\infty} = F(\mu_{p^{\infty}}, p^{\infty}\sqrt[\alpha]{1}, \ldots, p^{\infty}\sqrt[\alpha_{d-1}]{1})$, where $\alpha_1, \ldots, \alpha_{d-1} \in F^\times$ whose image in $F^\times/(F^\times)^p$ are linearly independent over $F_p$.

**Theorem (Lim)**

Let $i \geq 1$ be given. Let $F_{\infty}$ be the multi-false-Tate extension of $F$ as above. Suppose that the Iwasawa $\mu$-conjecture is valid for $F_{\text{cyc}}/F$. Then we have

$$e(\text{WK}_i^{\text{et}}(F_n)) = \text{rank}_{\mathbb{Z}_p[\mathcal{H}]}(Y_i(F_{\infty})) np^{(d-1)n} + O(p^{(d-1)n}).$$
For a given $p$-adic Lie extension $\mathcal{L}$ of $F$, denote by $K(\mathcal{L})$ the maximal unramified abelian pro-$p$ extension of $\mathcal{L}$ in which every prime above $p$ splits completely.

**Conjecture (Greenberg, 2001)**

Let $F$ be a number field and $\tilde{F}$ the compositum of all $\mathbb{Z}_p$-extensions of $F$. Then $\text{Gal}(K(\tilde{F})/\tilde{F})$ is a pseudo-null $\mathbb{Z}_p[\text{Gal}(\tilde{F}/F)]$-module.
The noncommutative analogue of Greenberg conjecture is not always true. Counterexamples have been constructed by Hachimori-Sharifi. But they have expressed belief and some evidences that the case of $F_\infty$ being a multi false-Tate extension should be plausibly true.

When $F_\infty = F(E[p^\infty])$, where $E$ is an elliptic curve, the noncommutative analogue of Greenberg conjecture is equivalent to a conjecture of Coates-Sujatha on the fine Selmer group of an elliptic curve.
Theorem (Lim)

Let $F_\infty$ be a multi false Tate extension of $F$. Then the following statements are equivalent.

(a) The noncommutative analog of Greenberg's conjecture is valid for $F_\infty$. In other words, the module $\text{Gal}(K(F_\infty)/F_\infty)$ is pseudo-null over $\mathbb{Z}_p[\mathcal{G}]$.

(b) $e(\mathcal{W}K_\text{ét}^i(F_n)) = O(p^{(d-1)n})$ for some $i \geq 1$.

(c) $e(\mathcal{W}K_\text{ét}^i(F_n)) = O(p^{(d-1)n})$ for all $i \geq 1$. 
Examples

Let $F = \mathbb{Q}(\mu_p)$, where $p$ is an irregular prime $< 1000$. Calculations of Sharifi shows that $\text{Gal}(K(\tilde{F})/\tilde{F})$ is pseudo-null over $\mathbb{Z}_p[\text{Gal}(\tilde{F}/F)]$. From this, we have

$$e(\mathcal{W}_K^{\text{ét}}(F_n)) = O(p^{(p-3)n/2})$$

for all $i \geq 1$. 
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$$e(\mathcal{W}_K^{\text{ét}}(F_n)) = O(p^{(p-3)n/2})$$

for all $i \geq 1$.

Consider the case $L_\infty = \mathbb{Q}(\mu_{p^\infty}, p^{-p^\infty})$, where $p$ is an irregular prime $< 1000$. Then Sharifi’s calculations can give us

$$\text{ord}_p(\mathcal{W}_K^{\text{ét}}(L_n)) = O(n).$$
Some final remark: An analogue of Tate lemma

We end mentioning the following lemma which can be viewed as a generalization of Tate’s lemma.

**Lemma**

Let $\Gamma \cong \mathbb{Z}_p$ and $N$ a $\Gamma$-module which is cofinitely generated and cofree over $\mathbb{Z}_p$. Suppose that $H^0(\Gamma, N)$ is finite. Then one has

$$H^1(\Gamma, N) = 0.$$
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In a joint work with Debanjana Kundu, we make use of the above lemma to consider the case of fine Selmer groups of elliptic curves in $p$-adic Lie extensions.
THE END
THANK YOU!