Class groups of n-monogenic cubic fields

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Joint work with Manjul Bhargava and Jonathan Hanke

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In this talk, we will be imposing certain *global* conditions on the family of cubic fields.

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For fixed n, a cubic field K has an absolutely bounded number of n-monogenizers.

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But $\mathcal{F}(\delta)$ expected to be "thin" for $\delta<1/4$.

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That is, monogenicity has a doubling effect (on average) on the nontrivial part of the 2-torsion in the class groups of cubic fields!

Statements of the results (Part 2)

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Fix $n = m^2 k$ and let $\mathcal{F} \subset \mathcal{F}_n$ be a subfamily defined by local conditions at finitely many primes p.

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Theorem (Bhargava–Hanke–S.)

When $(K, \alpha) \in \mathcal{F}$ are ordered by height:

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The general n-monogenized theorem

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The increase for nonsquare n is within sufficiently ramified fields.

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1	1.5003	1.5
2	1.3203	1.333
3	1.3008	1.313
4	1.5117	1.5
5	1.2892	1.292
6	1.2638	1.271
7	1.2688	1.281
8	1.3378	1.333
9	1.4773	1.5
10	1.2625	1.264

Parametrizing the arithmetic objects

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Theorem (Bhargava)

Let $f(x) \in U(\mathbb{Z})$ correspond to $\mathcal{O}_K$, for some cubic field K. There is a bijection between $SL_3(\mathbb{Z})$-orbits on $Res^{-1}(f)$ and index-2 subgroups of $Cl(K)$.

Counting the relevant lattice points

For the first result (where n varies), we need to count the relevant $\mathbb{Z}$-orbits on $U(\mathbb{Z})$, and the relevant $\mathbb{Z} \times \mathrm{SL}_3(\mathbb{Z})$-orbits on $V(\mathbb{Z})$, having bounded height and index.

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These methods yields answers in terms of products of local volumes of sets within $U(\mathbb{Q}_v)$, $U_n(\mathbb{Q}_v)$, $V(\mathbb{Q}_v)$, and $V_A(\mathbb{Q}_v)$.

For each p, pick a subset $S_p \subset U(\mathbb{Z}_p)$ of maximal forms. Our family $\mathcal{F}$ of cubic fields is in bijection with

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The necessary result follows from the computation of $\mathrm{Mass}_\infty^{\pm}$, which is $\frac{1}{4}$ for $+$ and $\frac{1}{2}$ for $-$.

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Evaluating this sum using the theory of quadratic forms yields the result.

Generalizations to higher degrees: monogenic case

For an integer $d \geq 0$, let $\mathcal{F}(d)$ denote the family of monogenized degree-d number fields. For integers r_1, r_2 with $r_1 + 2r_2 = d$, let $\mathcal{F}(d, r_1, r_2) \subset \mathcal{F}(d)$ be the set of fields with signature (r_1, r_2).

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Theorem (Siad)

Let $d \geq 3$ be an odd integer. The average of $|Cl_2(K)|$ over $(K, \alpha) \in \mathcal{F}(d, r_1, r_2)$ is bounded above by $1 + 2^{2-r_1-r_2}$, with equality conditional on a widely expected tail estimate.

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Siad also proves the result for all signatures. These results use a generalization of Bhargava's parametrization due to Wood.

Higher degrees: more general families

For positive integers d and n, let $\mathcal{F}(d, n)$ denote the family of degree-d number fields corresponding to the family of degree-d polynomials whose first coefficient is n.

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Let $d \geq 3$ be an odd integer. The average of $|Cl_2(K)|$ over $(K, \alpha) \in \mathcal{F}(d, n; r_1, r_2)$ is bounded above by

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Both Siad's and Swaminathan's work have a host of applications towards finding fields with odd class number, various unit signatures, and much more!

Thank you!