Class groups of $n$-monogenic cubic fields

Arul Shankar

Joint work with Manjul Bhargava and Jonathan Hanke

December 4
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(a) The average of $|\text{Cl}_3(K)|$ over real quadratic fields $K$ is $4/3$. 
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*The values in the above two results are unchanged if we impose any finite set of local conditions on the family of quadratic/cubic fields.*
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In this talk, we will be imposing certain *global* conditions on the family of cubic fields.
A number field $K$, with ring of integers $\mathcal{O}_K$ is said to be monogenic if $\mathcal{O}_K = \mathbb{Z}[\alpha]$ for some $\alpha \in \mathcal{O}_K$. 
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More generally, for $\alpha \in \mathcal{O}_K$, a pair $(K, \alpha)$ is an $n$-monogenized field if

$$\text{index } \mathcal{O}_K : \mathbb{Z}[\alpha] = n;$$

the element $\alpha$ is primitive in $\mathcal{O}_K / \mathbb{Z}$.

The $n$-monogenized fields $(K, \alpha)$ and $(K', \alpha')$ are isomorphic if there is an isomorphism $f : K \to K'$ such that $f(\alpha) = \alpha' + m$ for some $m \in \mathbb{Z}$.

Every cubic field $K$ has an "$n$-monogenizer" $\alpha$ for some $n \ll |\Delta(K)|^{1/4}$.

For fixed $n$, a cubic field $K$ has an absolutely bounded number of $n$-monogenizers.
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The $n$-monogenized fields $(K, \alpha)$ and $(K', \alpha')$ are \textit{isomorphic} if there is an isomorphism $f : K \to K'$ such that $f(\alpha) = \alpha' + m$ for some $m \in \mathbb{Z}$.

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Let \((K, \alpha)\) be an \(n\)-monogenized cubic field, and denote the characteristic polynomial of \(\alpha\) by \(f(x) = x^3 + ax^2 + bx + c\).
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H(K, \alpha) = n^{-2} \max\{4|b|^3, 27c^2\}.
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But \(\mathcal{F}(\delta)\) expected to be “thin” for \(\delta < 1/4\).
Theorem (Bhargava–Hanke–S.)

Fix $\delta > 0$. When cubic fields in $\mathcal{F}(\delta)$ are ordered by height:

(a) The average of $|\text{Cl}_2(K)|$ over real $(K,\alpha)$ is $\frac{5}{4}$;

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Imposing finitely many local conditions leave the result unchanged.
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Fix \( n = m^2 k \), with \( k \) squarefree. When \( \mathcal{F}_n \) is ordered by height:

1. The average of \( |\text{Cl}_2(K,\alpha)| \) over real \((K,\alpha)\) is \( \frac{5}{4} + \frac{1}{4}\sigma(k) \);
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The averages can change when we impose local conditions!

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When \((K,\alpha)\) \(\in \mathcal{F}_n\), unramified at all \( p | n \), are ordered by height:

1. The average of \( |\text{Cl}_2(K,\alpha)| \) over real \((K,\alpha)\) is \( \frac{3}{2} \) if \( n \) is a square and \( \frac{5}{4} \) else;
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(a) *The average of $|Cl_2(K)|$ over real $(K, \alpha)$ is $\frac{5}{4} + \frac{1}{4\sigma(k)}$;*

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The averages *can* change when we impose local conditions!

**Theorem (Bhargava–Hanke–S.)**

When $(K, \alpha) \in \mathcal{F}_n$, unramified at all $p \mid n$, are ordered by height:
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(b) The average of $|Cl_2(K)|$ over complex $(K, \alpha)$ is $2$ if $n$ is a square and $\frac{3}{2}$ else.
An $n$-monogenized cubic field $(K, \alpha)$ is sufficiently ramified at $p$ if either holds:

- $K_p := K \otimes \mathbb{Q}_p$ is a totally ramified extension of $\mathbb{Q}_p$;
- $K_p = \mathbb{Q}_p \times F$, where $F$ is ramified, and $\mathbb{Z}_p[\alpha_p] = \mathbb{Z}_p \times O$.

Fix $n = m^2 k$ and let $F \subset F_n$ be a subfamily defined by local conditions at finitely many primes. Define $\rho_p(F)$ to be the density of fields in $F_n$ that are sufficiently ramified at $p$.

Theorem (Bhargava–Hanke–S.)

When $(K, \alpha) \in F_n$ are ordered by height:

(a) The average of $|\text{Cl}_2(K)|$ over real $(K, \alpha)$ is $\frac{5}{4} + \frac{1}{4} \rho(F)$.

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When $(K, \alpha) \in F$ are ordered by height:

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The general $n$-monogenized theorem

An $n$-monogenized cubic field $(K, \alpha)$ is sufficiently ramified at $p$ if either holds:

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Arul Shankar

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When $(K, \alpha) \in \mathcal{F}$ are ordered by height:

(a) The average of $|\text{Cl}_2(K)|$ over real $(K, \alpha)$ is $\frac{5}{4} + \frac{1}{4} \rho(\mathcal{F})$.

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Arul Shankar

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The increase for nonsquare $n$ is within sufficiently ramified fields.
For each $n$, the average is computed using Magma over $\sim 1800$ different $n$-monogenized cubic fields of huge height ($\sim 10^{20}$).

<table>
<thead>
<tr>
<th>$n$</th>
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<th>Actual average</th>
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<tr>
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<td>2</td>
<td>1.3203</td>
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The first step is to parametrize the relevant arithmetic objects.

Let $U$ be the space of cubic polynomials, and let $U_n \subset U$ be the subspace with leading coefficient $n$. The group $G_a$ acts on $U$ via $\lambda \cdot f(x) = f(x + \lambda)$. This action preserves $U_n$.

The set of $n$-monogenized cubic fields naturally injects into the set of $\mathbb{Z}$-orbits on $U_n(\mathbb{Z})$.

Let $V = \mathbb{Z} \otimes \text{Sym}_2(3)$ be the space of pairs of ternary quadratic forms. Define the resolvent map $\text{Res} : V \to U$ sending $(A, B)$ to $4 \det(Ax + B)$.

The group $G_a \times \text{SL}_3(\mathbb{Z})$ acts on $V$ via $(\lambda, \gamma) \cdot (A, B) = (\gamma A \gamma^t, \gamma (B + \lambda A) \gamma^t)$. This action respects $\text{Res}$.

Theorem (Bhargava) There is a bijection between $\text{SL}_3(\mathbb{Z})$-orbits on $\text{Res}^{-1}(f)$ and index-2 subgroups of $\text{Cl}(K)$. 

Class groups of $n$-monogenic cubic fields
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Theorem (Bhargava) Let $f(x) \in U(\mathbb{Z})$ correspond to $\mathcal{O}_K$, for some cubic field $K$. There is a bijection between $\text{SL}_3(\mathbb{Z})$-orbits on $\text{Res}^{-1}(f)$ and index-2 subgroups of $\text{Cl}(K)$. 

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Let $f(x) \in U(\mathbb{Z})$ correspond to $O_K$, for some cubic field $K$. There is a bijection between $\text{SL}_3(\mathbb{Z})$-orbits on $\text{Res}^{-1}(f)$ and index-2 subgroups of $\text{Cl}(K)$. 

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*Arul Shankar*

*Class groups of $n$-monogenic cubic fields*
For the first result (where \( n \) varies), we need to count the relevant \( \mathbb{Z} \)-orbits on \( U(\mathbb{Z}) \), and the relevant \( \mathbb{Z} \times \text{SL}_3(\mathbb{Z}) \)-orbits on \( V(\mathbb{Z}) \), having bounded height and index.
Counting the relevant lattice points

For the first result (where \( n \) varies), we need to count the relevant \( \mathbb{Z} \)-orbits on \( U(\mathbb{Z}) \), and the relevant \( \mathbb{Z} \times \text{SL}_3(\mathbb{Z}) \)-orbits on \( V(\mathbb{Z}) \), having bounded height and index.

For the remaining results (where \( n \) is fixed), we need to count the relevant \( \mathbb{Z} \)-orbits on \( U_n(\mathbb{Z}) \), and the relevant \( \mathbb{Z} \times \text{SL}_3(\mathbb{Z}) \)-orbits on \( \text{Res}^{-1}(U_n(\mathbb{Z})) \), having bounded height.
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However, \( \text{Res}^{-1}(U_n(\mathbb{Z})) = \{(A, B) \in V(\mathbb{Z}) : \det(A) = \frac{n}{4} \} \) is not defined by a linear condition. Instead, we count \( \text{SO}_A(\mathbb{Z}) \)-orbits on integral ternary quadratic forms \( B \), such that \((A, B)\) has bounded height.

We then sum over a set of representatives \( A \) for the action of \( \text{SL}_3(\mathbb{Z}) \) on integral ternary quadratic forms \( A \) with \( \det(A) = \frac{n}{4} \).

In both cases, we perform the count using geometry–of–numbers methods along with a squarefree sieve. These methods yields answers in terms of products of local volumes of sets within \( U(\mathbb{Q}_v) \), \( U_n(\mathbb{Q}_v) \), \( V(\mathbb{Q}_v) \), and \( V_A(\mathbb{Q}_v) \).
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For the first result (where \( n \) varies), we need to count the relevant \( \mathbb{Z} \)-orbits on \( U(\mathbb{Z}) \), and the relevant \( \mathbb{Z} \times \text{SL}_3(\mathbb{Z}) \)-orbits on \( V(\mathbb{Z}) \), having bounded height and index.

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However, \( \text{Res}^{-1}(U_n(\mathbb{Z})) = \{(A, B) \in V(\mathbb{Z}) : \det(A) = \frac{n}{4}\} \) is not defined by a linear condition. Instead, we count \( \text{SO}_A(\mathbb{Z}) \)-orbits on integral ternary quadratic forms \( B \), such that \( (A, B) \) has bounded height.

We then sum over a set of representatives \( A \) for the action of \( \text{SL}_3(\mathbb{Z}) \) on integral ternary quadratic forms \( A \) with \( \det(A) = \frac{n}{4} \).

In both cases, we perform the count using geometry–of–numbers methods along with a squarefree sieve.

These methods yields answers in terms of products of local volumes of sets within \( U(\mathbb{Q}_v) \), \( U_n(\mathbb{Q}_v) \), \( V(\mathbb{Q}_v) \), and \( V_A(\mathbb{Q}_v) \).
For each $p$, pick a subset $S_p \subset U(\mathbb{Z}_p)$ of maximal forms. Our family $\mathcal{F}$ of cubic fields is in bijection with
\[
\mathcal{L} = U(\mathbb{R})^\pm \cap U(\mathbb{Z}) \cap \bigcap_p S_p.
\]
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$$\mathcal{L} = U(\mathbb{R})^\pm \cap U(\mathbb{Z}) \cap \bigcap_p S_p.$$ 

The average size of $|\text{Cl}(K)[2]|$ over $K \in \mathcal{F}$ is

$$\frac{\#\text{Res}^{-1}(\mathcal{L}_X)}{\#\mathcal{L}_X} = 1 + \tau(\text{SL}_3(\mathbb{Q})) \text{Mass}_\pm \frac{\prod_p \int_{f \in S_p} \text{Mass}_p(f) df}{\prod_p \int_{f \in S_p} df}.$$ 

Above, for a form $f$ corresponding to $K_p/\mathbb{Q}_p$, $\text{Mass}_p(f)$ is

$$\sum_{(A, B) \in \text{Res}^{-1}(f)} \frac{1}{\#\text{Stab}(A, B)} = \sum_{L_p \in \text{Res}^{-1}(K_p)} \frac{1}{\#\text{Aut}(L_p, K_p)} = 1.$$ 

The necessary result follows from the computation of $\text{Mass}_\pm$, which is $\frac{1}{4}$ for $+$ and $\frac{1}{2}$ for $-$. 

Arul Shankar

Class groups of $n$-monogenic cubic fields
For each $p$, pick a subset $S_p \subset U(\mathbb{Z}_p)$ of maximal forms. Our family $\mathcal{F}$ of cubic fields is in bijection with

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The average size of $|\text{Cl}(K)[2]|$ over $K \in \mathcal{F}$ is

$$\frac{\#\text{Res}^{-1}(\mathcal{L}_x)}{\#\mathcal{L}_x} = 1 + \tau(\text{SL}_3(\mathbb{Q})) \text{Mass}_\infty \frac{\prod_p \int_{f \in S_p} \text{Mass}_p(f) df}{\prod_p \int_{f \in S_p} df}.$$ 

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For each $p$, pick a subset $S_p \subset U(\mathbb{Z}_p)$ of maximal forms. Our family $\mathcal{F}$ of cubic fields is in bijection with

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Above, for a form $f$ corresponding to $K_p/\mathbb{Q}_p$, $\text{Mass}_p(f)$ is

$$\sum_{(A,B) \in \text{Res}^{-1}(f) / \text{SL}_3(\mathbb{Z}_p)} \frac{1}{\#\text{Stab}(A,B)} = \sum_{L_p \in \text{Res}^{-1}(K_p)} \frac{1}{\#\text{Aut}(L_p, K_p)}.$$
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Above, for a form $f$ corresponding to $K_p/\mathbb{Q}_p$, $\text{Mass}_p(f)$ is

$$\sum_{(A,B) \in \text{Res}^{-1}(f) \backslash \text{SL}_3(\mathbb{Z}_p)} \frac{1}{\#\text{Stab}(A,B)} = \sum_{L_p \in \text{Res}^{-1}(K_p)} \frac{1}{\#\text{Aut}(L_p, K_p)} = 1.$$
Computing local volumes: varying \( n \)

For each \( p \), pick a subset \( S_p \subset U(\mathbb{Z}_p) \) of maximal forms. Our family \( \mathcal{F} \) of cubic fields is in bijection with

\[
\mathcal{L} = U(\mathbb{R})^\pm \cap U(\mathbb{Z}) \cap \bigcap_p S_p.
\]

The average size of \( |\text{Cl}(K)[2]| \) over \( K \in \mathcal{F} \) is

\[
\frac{\#\text{Res}^{-1}(\mathcal{L}_X)}{\#\mathcal{L}_X} = 1 + \tau(\text{SL}_3(\mathbb{Q}))\text{Mass}_{\infty}^{\pm} \prod_p \int_{f \in S_p} \text{Mass}_p(f) df / \prod_p \int_{f \in S_p} df.
\]

Above, for a form \( f \) corresponding to \( K_p/\mathbb{Q}_p \), \( \text{Mass}_p(f) \) is

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\sum_{(A,B) \in \text{Res}^{-1}(f)/\text{SL}_3(\mathbb{Z}_p)} \frac{1}{\#\text{Stab}(A,B)} = \sum_{L_p \in \text{Res}^{-1}(K_p)} \frac{1}{\#\text{Aut}(L_p, K_p)} = 1.
\]

The necessary result follows from the computation of \( \text{Mass}_{\infty}^{\pm} \), which is \( \frac{1}{4} \) for \( + \) and \( \frac{1}{2} \) for \( - \).
Fixing $n$ introduces important changes: specifically the theory of quadratic forms now plays a crucial role.
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where the sum is over genera \( A \) having determinant \( n \).
Computing local volumes: fixed $n$

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$$= 1 + 2 \sum_A \text{Mass}_\infty \frac{\prod_p \int_{f \in \mathcal{S}_p} \text{Mass}_p(f, A) df}{\prod_p \int_{f \in \mathcal{S}_p} df},$$

where the sum is over genera $A$ having determinant $n$. Evaluating this sum using the theory of quadratic forms yields the result.
For an integer $d \geq 0$, let $\mathcal{F}(d)$ denote the family of monogenized degree-$d$ number fields. For integers $r_1, r_2$ with $r_1 + 2r_2 = d$, let $\mathcal{F}(d, r_1, r_2) \subset \mathcal{F}(d)$ be the set of fields with signature $(r_1, r_2)$. 

Theorem (Siad) 
Let $d \geq 3$ be an odd integer. The average of $|\text{Cl}_2(K)|$ over $(K, \alpha) \in \mathcal{F}(d, r_1, r_2)$ is bounded above by $1 + 2^{2r_1 + 2r_2}$, with equality conditional on a widely expected tail estimate.

Let $d \geq 4$ be an even integer. The average of $|\text{Cl}_2(K)|$ over $(K, \alpha) \in \mathcal{F}(d, 0, r_2)$ is bounded by $\prod_p (1 + \rho_p)(1 + 2^{2r_2}) + 2^{2 - r_2}$, with equality conditional on a widely expected tail estimate.

In the even degree case, the product over $p$ is the contribution from genus theory.

Siad also proves the result for all signatures. These results use a generalization of Bhargava’s parametrization due to Wood.
For an integer $d \geq 0$, let $\mathcal{F}(d)$ denote the family of monogenized degree-$d$ number fields. For integers $r_1, r_2$ with $r_1 + 2r_2 = d$, let $\mathcal{F}(d, r_1, r_2) \subset \mathcal{F}(d)$ be the set of fields with signature $(r_1, r_2)$.

**Theorem (Siad)**

Let $d \geq 3$ be an odd integer. The average of $|\text{Cl}_2(K)|$ over $(K, \alpha) \in \mathcal{F}(d, r_1, r_2)$ is bounded above by $1 + 2^{2-r_1-r_2}$, with equality conditional on a widely expected tail estimate.
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In the even degree case, the product over $p$ is the contribution from genus theory.
Generalizations to higher degrees: monogenic case

For an integer \( d \geq 0 \), let \( \mathcal{F}(d) \) denote the family of monogenized degree-\( d \) number fields. For integers \( r_1, r_2 \) with \( r_1 + 2r_2 = d \), let \( \mathcal{F}(d, r_1, r_2) \subset \mathcal{F}(d) \) be the set of fields with signature \((r_1, r_2)\).

**Theorem (Siad)**

Let \( d \geq 3 \) be an odd integer. The average of \( |Cl_2(K)| \) over \((K, \alpha) \in \mathcal{F}(d, r_1, r_2)\) is bounded above by \( 1 + 2^{2-r_1-r_2} \), with equality conditional on a widely expected tail estimate.

Let \( d \geq 4 \) be an even integer. The average of \( |Cl_2(K)| \) over \((K, \alpha) \in \mathcal{F}(d, 0, r_2)\) is bounded by \( \prod_p (1 + \rho_p)(1 + 2^{2-r_2}) + 2^{-r_2} \), with equality conditional on a widely expected tail estimate.

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Siad also proves the result for all signatures. These results use a generalization of Bhargava’s parametrization due to Wood.
For positive integers $d$ and $n$, let $\mathcal{F}(d, n)$ denote the family of degree-$d$ number fields corresponding to the family of degree-$d$ polynomials whose first coefficient is $n$. 

Theorem (Swaminathan) Let $d \geq 3$ be an odd integer. The average of $|Cl_2(K)|$ over $(K, \alpha) \in \mathcal{F}(d, n; r_1, r_2)$ is bounded above by $1 + 2^{1 - r_1 - r_2} \left(1 + \frac{1}{k} \frac{n - 3}{2} \sigma(k)\right)$, with equality conditional on a widely expected tail estimate. Swaminathan also proves the analogous (much more complicated) result in the even degree case. Both Siad's and Swaminathan's work have a host of applications towards finding fields with odd class number, various unit signatures, and much more!
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$$1 + 2^{1-r_1-r_2} \left(1 + \frac{1}{k^{(n-3)/2} \sigma(k)}\right),$$

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Thank you!