Solitons in Bryant's G_2-Laplacian flow.

Mark Haskins
Duke University

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Rowan Juneman & Johannes Nordström (Bath)

The Lie group G_2 and 3-forms in 7 dimensions

In 1886 Engel suggested to Killing a way to construct a 14-dimensional simple Lie group using *generic* 3*-forms in* 7 *variables*; details not settled until 1907.

A complex k-form on $\mathbb{C}^n$ is **generic** if its $GL(n, \mathbb{C})$-orbit is *open* in $\Lambda^k(\mathbb{C}^n)^*$.

Given any k-form ϕ on $\mathbb{C}^n$ define a Lie subgroup of $GL(n, \mathbb{C})$ by

$$G_\phi := \{A \in GL(n, \mathbb{C}) : A^*\phi = \phi\}.$$

For a generic k-form ϕ we need $\dim GL(n, \mathbb{C}) - \dim G_\phi = \dim \Lambda^k(\mathbb{C}^n)^*$.

For $k = 3$, $n = 7$ the dimension of G_ϕ for a generic form must be

$$dim\, G_\phi = dim\, GL(7, \mathbb{C}) - dim\, \Lambda^3(\mathbb{C}^7)^* = 49 - 35 = 14$$

which is the dimension of G_2!

Generic 3-forms in $\mathbb{R}^7$, G_2 and the octonions

In 1900 Engel showed:

- there is exactly one $GL(7, \mathbb{C})$ orbit of generic 3-forms in $\mathbb{C}^7$
- For every generic 3-form ϕ the isotropy group G_ϕ is isomorphic to $G_2(\mathbb{C})$.

In 1907 Reichel (Engel's student) considered generic real 3-forms in $\mathbb{R}^7$.

- There are 2 types of generic (real) 3-forms in $\mathbb{R}^7$.
  - Case 1: $\exists$ an invariant symmetric bilinear form of signature $(7, 0)$;
    G_ϕ is isomorphic to the compact real simple Lie group $G_2 \subset SO(7)$.
    We will call such a generic real 3-form φ *positive* and write $\varphi \in \mathcal{P}^3(\mathbb{R}^7)$.
  - Case 2: $\exists$ invariant symmetric bilinear form of signature $(4, 3)$.

Define a vector cross-product and a 3-form on $\mathbb{R}^7 = \mathrm{Im}(\mathbb{O})$ using octonionic multiplication and the norm

$$u \times v := \mathrm{Im}(uv),$$
$$\varphi_0(u, v, w) := \langle u \times v, w \rangle = \langle uv, w \rangle.$$

Then $\varphi_0 \in \mathcal{P}^3(\mathbb{R}^7)$ so $G_2 = Aut(\mathbb{O}) = \{A \in \mathrm{GL}(7, \mathbb{R}) |\ A^*\varphi_0 = \varphi_0\}$.

Positive 3-forms & G_2-structures on 7-manifolds

For an oriented smooth 7-manifold M and $p \in M$

$$\mathcal{P}_p(M) := \{\varphi \in \Lambda^3 T_p^* M \mid \iota^* \varphi_0 = \varphi \text{ for } \iota : T_p M \to \mathbb{R}^7\}$$

where ι is any orientation preserving isomorphism. $\mathcal{P}(M)$ denotes the bundle over M with fibre $\mathcal{P}_p(M)$.

A 3-form φ on M is *positive* if φ is a section of $\mathcal{P}(M)$, i.e. $\varphi_p \in \mathcal{P}_p(M)\ \forall p$.

Each positive 3-form on M defines a reduction of the frame bundle $\mathcal{F}M$ to a principal subbundle of $\mathcal{F}M$ with fibre G_2, i.e. a G_2-structure on M that induces the given orientation on M.

Positive 3-forms on M $\leftrightsquigarrow$ (oriented) G_2-structures on M.

Existence of a positive 3-form on M^7 is a topological question: M needs to be orientable and spinnable.

G_2 as a Riemannian holonomy group

Possible holonomy groups of Riemannian manifolds[1] are extremely limited:

- 5 possible infinite families and
- 2 exceptional cases, the Lie groups G_2 and $Spin_7$ (in dims 7 and 8)

Both exceptional cases and 3 of the infinite families constitute the special holonomy metrics. (The other 2 are generic Riemannian/Kähler metrics).

Special (exceptional) holonomy metrics are necessarily Einstein (Ricci-flat).

Proving existence of metrics of special holonomy (locally; complete metrics on noncompact and on compact spaces) took many years (1955–1997), and involved many deep developments in geometry and geometric analysis:

- Yau's proof of the Calabi conjecture (1978) settled affirmatively the cases with holonomy $SU(n)$ and $Sp(n)$. Yau used analytic methods to prove existence of solutions to a complex Monge-Ampère equation – a *fully nonlinear scalar elliptic equation*.
- For the two exceptional holonomy cases we can no longer reduce to a scalar equation as in the $SU(n)$ case. The best one can do involves systems of nonlinear first-order PDEs.

[1] Berger 1955: simply connected irreducible non-locally-symmetric case

1st-order PDE system for G_2 holonomy metrics

Important fact: The holonomy group $Hol_g(M)$ determines the parallel tensors on (M, g). In particular $Hol_g(M) \subseteq G_2 \subset \mathrm{SO}(7)$ implies

M^7 admits a g-parallel positive 3-form φ.

Converse: *How to get a G_2-holonomy metric from a G_2-structure?*

Theorem

Let (M, φ, g_φ) be a G_2-structure; the following are equivalent

1. $\mathrm{Hol}(g_\varphi) \subseteq G_2$ *and* φ *is the induced* 3*-form*
2. $d\varphi = d^*\varphi = 0$, *where* d^* *is defined using Hodge star* $*$ *w.r.t.* g_φ.

Call such a G_2-structure a torsion-free G_2 structure.

2 is *nonlinear* in φ: g_φ depends nonlinearly on φ and d^* depends on g_φ.

By writing equation for 3-form φ (not metric g directly) and allowing $\mathrm{Hol}(g_\varphi) \subseteq G_2$ we obtain *differential* (not integro-differential) equations.

2 is a 1st-order system of 49=(35+21-7) equations on the 35 coeffs of φ!

Bryant's Laplacian flow

There is a natural flow on *closed* G_2-structures. Solve

$$\frac{d\varphi_t}{dt} = \Delta_{\varphi_t}\varphi_t \qquad \text{(LF)}$$

with initial condition φ_0 satisfying $d\varphi_0 = 0$. (Then $d\varphi_t = 0$ for all t.)

- Induced metric g_t evolves under (LF) by

$$\frac{dg_t}{dt} = -2\mathrm{Ric}(g_t) + \text{ terms quadratic in torsion of } \varphi_t$$

- Stationary points of (LF) are exactly torsion-free G_2-structures.
- (LF) is the (upward) gradient flow for Hitchin's volume functional

$$\mathrm{vol}(\varphi) := \frac{1}{7}\int_M \varphi \wedge *\varphi$$

  when restricted to cohomology class of φ_0.

  Critical points of $\mathrm{vol}(\varphi)$ in $[\varphi]$ are maxima (strict modulo diffeos).
- On a compact manifold $\mathrm{vol}(\varphi_t)$ is increasing along (LF).
- Bryant-Xu : Short-time existence & uniqueness of solutions to (LF)
- Lotay–Wei: Torsion-free G_2-structures are stable under (LF).

Solitons in the Laplacian flow

G_2-structure φ, vector field X, $\lambda \in \mathbb{R}$ satisfying

$$\begin{cases} d\varphi &= \quad 0, \\ \Delta_\varphi \varphi = & \lambda\varphi + \mathcal{L}_X \varphi. \end{cases}$$

$\Leftrightarrow$ self-similar solution of Laplacian flow

$$\varphi_t = k(t)^3 f^* \varphi, \qquad \frac{df}{dt} = k(t)^{-2} X, \qquad k(t) = \frac{3 + 2\lambda t}{3}$$

$\lambda > 0$: *expanders* (immortal solutions, i.e. exist up to $t = +\infty$)
$\lambda = 0$: *steady* solitons (eternal solutions, i.e. exist for all time $t \in \mathbb{R}$)
$\lambda < 0$: *shrinkers* (ancient solutions, i.e. exist backwards to $t = -\infty$)

- Non-steady soliton $\Rightarrow \varphi$ exact
- Solitons on a *compact* manifold are stationary or expanders
- Scaling behaviour:
  (φ, X) is a λ-soliton $\Leftrightarrow (k^3\varphi, k^{-2}X)$ is a $k^{-2}\lambda$-soliton.

Our motivation and overview of results

Motivation: in most geometric flows solitons provide the models for singularity formation. So we look for (symmetric) solitons of Laplacian flow.

Goal: Find asymptotically conical (AC) G_2 solitons with cohomogeneity one: $SU(3)$-invariant ones on $\Lambda^2_+\mathbb{C}P^2$ and $Sp(2)$-invariant ones on $\Lambda^2_+\mathbb{S}^4$.

Theorem (A)

$\exists$ a 1-parameter family of steady solitons on $\Lambda^2_+\mathbb{C}P^2$ asymptotic with rate -1 to torsion-free cone (deformations of the Bryant-Salamon AC G_2-metric).

- **AC** steady solitons a new feature (compared to Ricci/Kähler-Ricci flow).

Theorem (B)

$\exists$ an explicit AC shrinker with rate -2 on $\Lambda^2_+\mathbb{S}^4$ and $\Lambda^2_+\mathbb{C}P^2$.

- Shrinkers are rare! Possible models for *formation of conical singularities*.

Theorem (C)

$\exists$ a 1-parameter family of complete expanders on $\Lambda^2_+\mathbb{S}^4$ and on $\Lambda^2_+\mathbb{C}P^2$.

- Models for how Laplacian flow can *smooth out certain conical singularities*.

Closed invariant G_2-structures on $\Lambda^2_+ M^4 \setminus M$

For $M = \mathbb{C}P^2$ or $\mathbb{S}^4$, $\Lambda^2_+ M$ has a cohomogeneity one action by $G = SU(3)$ or $Sp(2)$. $\Lambda^2_+ M \setminus M$ is diffeomorphic to $\mathbb{R}_+ \times \Sigma$, for $\Sigma = SU(3)/T^2$ or $\mathbb{C}P^3$.

There are G-invariant forms $\omega_1, \omega_2, \omega_3 \in \Omega^2(\Sigma)$ and $\alpha \in \Omega^3(\Sigma)$ such that any closed G-invariant G_2-structure on $\mathbb{R}_+ \times \Sigma$ with $\|\frac{\partial}{\partial t}\| = 1$ can be written as

$$\varphi = (f_1^2\omega_1 + f_2^2\omega_2 + f_3^2\omega_3) \wedge dt + f_1 f_2 f_3 \alpha, \qquad f_i : \mathbb{R}_+ \to \mathbb{R}_+$$

with

$$\frac{d(f_1 f_2 f_3)}{dt} = \tfrac{1}{2}(f_1^2 + f_2^2 + f_3^2). \qquad (\#)$$

$$(\#) \Rightarrow \frac{d}{dt}(f_1 f_2 f_3)^{1/3} \geq \frac{1}{2}.$$

For $Sp(2)$-invariance in addition require $f_2 = f_3$. Structure equations for ω_i, α the same in both cases $\Rightarrow$ $\Lambda^2_+\mathbb{S}^4$ case can be treated as a special case of $\Lambda^2_+\mathbb{C}P^2$ case where $f_2 = f_3$.

Closed invariant G_2 cones

Helpful to analyse invariant G_2-structures on $\mathbb{R}_+ \times \Sigma$ in terms of scale and homothety class of invariant metrics on Σ:

$$\text{scale} \quad g := \sqrt[3]{f_1 f_2 f_3} = \sqrt[6]{\mathrm{vol}(\Sigma)}$$
$$\text{homethety class} \quad \frac{f_1}{g}, \frac{f_2}{g}, \frac{f_3}{g}$$

φ closed and homothety class constant implies g linear and φ conical:

$$d\varphi = 0 \;\Rightarrow\; \frac{dg}{dt} = \frac{1}{6}\left(\frac{f_1^2}{g^2} + \frac{f_2^2}{g^2} + \frac{f_3^2}{g^2}\right) \;\Rightarrow\; f_i = c_i t$$

with

$$6c_1c_2c_3 = c_1^2 + c_2^2 + c_3^2. \qquad (*)$$

Note: *any* positive triple (c_1, c_2, c_3) can be uniquely rescaled to satisfy $(*)$
$\rightsquigarrow$ 2-parameter family of closed conical G_2-structures on $\mathbb{R}_+ \times SU(3)/T^2$.

In other words, given homothety class on Σ, there is a unique choice of "cone angle" that makes it a closed cone.

Evolution equations

On the face of it, the soliton condition for

$$\varphi = (f_1^2\omega_1 + f_2^2\omega_2 + f_3^2\omega_3) \wedge dt + f_1 f_2 f_3 \alpha, \qquad X = u\frac{\partial}{\partial t}$$

is 2nd-order ODE system for (f_1,f_2,f_3,u) (with some constraints).

Can rewrite as a *1st-order system in 5 variables*: the 3 f_i and 2 variables determining the torsion 2-form τ of φ.

$\Rightarrow$ there is a 4-parameter family of local $SU(3)$-invariant solitons and a 2-parameter family of Sp_2-invariant solitons.

Q: Which of these invariant solitons extends to a complete AC solution?

Tendency: if $\frac{f_1}{g}, \frac{f_2}{g}, \frac{f_3}{g}$ remain bounded as $t \to \infty$ then asymptotic to closed cone.

Rough strategy for finding AC solitons on $\Lambda^2_+ M = M \sqcup \mathbb{R}_+ \times \Sigma$.

1. Solutions on $(0, \epsilon) \times \Sigma$ that extend smoothly across M at $t = 0$?
2. Solutions for large t asymptotic to prescribed closed cone (c_1, c_2, c_3)?
3. Do solutions from **1** and **2** fit together?

Initial value problem near zero section of $\Lambda^2_+ M^4$

Understand solutions near zero section of $\Lambda^2_+ M$ à la Eschenburg-Wang.

$$\varphi = (f_1^2\omega_1 + f_2^2\omega_2 + f_3^2\omega_3) \wedge dt + f_1 f_2 f_3 \alpha$$

on $\mathbb{R}_+ \times \Sigma$ extends to smooth G_2-structure on $\Lambda^2_+ M$ iff

f_1 is odd with $f_1'(0) = 1$, and f_2 and f_3 are even with $m := f_2(0) = f_3(0) \neq 0$.

Resulting singular initial value problem has formal power series solutions that are convergent. (It is a **regular singular point** of 1st-order ODE system).

Proposition

For each $\lambda \in \mathbb{R}$, there is

- *a 2-parameter family $\varphi_{m,c}$ of solutions defined for small t that extend smoothly to a λ-soliton on (nhd of zero section in) $\Lambda^2_+ \mathbb{C}P^2$;*
- *1-parameter subfamily $\varphi_m = \varphi_{m,0}$ also defines λ-solitons on $\Lambda^2_+ \mathbb{S}^4$.*

Two scale-invariant parameters: λm^2 and c.

So up to scale: *2-parameter families of local expanders/shrinkers* on $\Lambda^2_+ \mathbb{C}P^2$

a 1-parameter family of local steady solitons on $\Lambda^2_+ \mathbb{C}P^2$

Expanders

Theorem (C)

For $\lambda > 0$, each φ_m extends to a complete solution with $f_2 = f_3$, and

$$\frac{f_i}{t} \to c_i$$

for (c_1, c_2, c_2) a closed cone with $c_1 \leq c_2$.

This gives 1-parameter families of expanders on both $\Lambda^2_+\mathbb{S}^4$ and $\Lambda^2_+\mathbb{C}P^2$.

Strong Expectations

- These solitons are all AC, with rate -2
- 1-1 correspondence with closed cones such that $c_1 < c_2$:
  any closed cone with $c_2 = c_3$ on "one side" of the torsion-free cone $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is the AC end of a unique expander

Conjecture

For $\lambda > 0$, an open subfamily of $\varphi_{m,c}$ (but not all) extend to complete solutions, defining a 2-parameter family of AC solitons on $\Lambda^2_+\mathbb{C}P^2$.

Rigidity/stability dichotomy of $\lambda \neq 0$ AC ends

For any $\lambda \neq 0$, $\exists$ a ! *formal power series* solution $\mathcal{P}$ in t^{-1} determined by the cone and a solution of the ODE system that is smooth in a nhd of $t = +\infty$ whose Taylor series is $\mathcal{P}$.

For $\lambda < 0$, for each closed cone (c_1, c_2, c_3) there is a **unique** solution defined for large t asymptotic to the given cone; so AC shrinker ends are rigid.

Given $\lambda > 0$ and any closed cone (c_1, c_2, c_3), we expect:

- $\exists$ a 2-parameter family of AC ends asymptotic to the given cone.
- Difference between two solutions is of order $\exp(-\frac{\lambda}{6}t^2) * \text{polynomial}$.
- If $c_2 = c_3$, then a 1-parameter subfamily has $f_2 = f_3$.

Flow lines of this 4=(2+2)-parameter family of solutions fill open subset of 5-dimensional phase space, so **AC expander ends are stable**.

“Explanation” for shrinker/expander dichtomy: ODEs for expanders/ shrinkers asymptotic to given cone has an **irregular singularity** at $t = +\infty$.

• When $\lambda > 0$ other smooth solutions also with Taylor series $\mathcal{P}$ at $t = \infty$ exist because of exponentially small corrections of form $\exp\left(-\frac{1}{6}\lambda t^2\right) * p(t)$.

• When $\lambda < 0$ the exponential terms blow-up as $t \to \infty$.

Shrinkers: consequences of AC end rigidity

Heuristic for $\lambda < 0$:

Invariant shrinkers on $\mathbb{R}_+ \times SU(3)/T^2$ are flow lines in 5-dim phase space.
In 4-dimensional space of flow lines

- 2-dimensional submanifold extends across zero section $\mathbb{C}P^2 \subset \Lambda^2_+\mathbb{C}P^2$
- 2-dimensional submanifold has AC behaviour

Expect transverse intersections $\rightsquigarrow$ *finitely* many AC shrinkers on $\Lambda^2_+\mathbb{C}P^2$.

Similarly, restricting attention to solutions with $f_2 = f_3$:

2-dimensional space of flow lines; 1-dim submanifold extends over special orbit; 1-dim submanifold has AC behaviour. Expect transverse intersections $\rightsquigarrow$ *finitely* many AC shrinkers on $\Lambda^2_+\mathbb{S}^4$.
In fact, can spot one explicit solution!

Theorem B: For $\lambda = -1$

$$f_1 = t, \quad f_2^2 = f_3^2 = \tfrac{9}{4} + \tfrac{1}{4}t^2, \quad u = \frac{t}{3} + \frac{4t}{9+t^2}$$

is an AC shrinker with rate -2 asymptotic to cone $(1, \frac{1}{2}, \frac{1}{2})$.

Conjecture: *This is the unique Sp_2-invariant AC shrinker on $\Lambda^2_+\mathbb{S}^4$.*

Steady solitons

Significant qualitative differences from $\lambda \neq 0$:

Near special orbit $\mathbb{C}P^2$, only a 1-parameter family of solutions up to scale.
Unique one with $f_2 = f_3$: static soliton from Bryant–Salamon AC G_2-mfd.

Theorem

No non-stationary steady solitons on $\Lambda^2_+ \mathbb{S}^4$.

Decoupling

- For $\lambda = 0$, the flow can be separated into evolution of *scale* g and evolution of 4 scale-normalised variables.
- Unique fixed point for the scale-normalised flow is the torsion-free cone; It is a *stable* fixed point.

Theorem (A)

There exists a 1-parameter family (up to scale) of AC steady solitons on $\Lambda^2_+ \mathbb{C}P^2$ all asymptotic to the torsion-free cone over $SU(3)/T^2$; the family includes steady solitons with arbitrarily small torsion.

Comparison with other flows: the steady case

• All known *steady* solitons in Ricci flow have *sub-Euclidean* volume growth:
◦ the Bryant soliton; Appleton's resolutions of (some of) its quotients.
◦ Bryant soliton known to appear in a finite-time singularity of RF.
◦ known Kähler examples have at most half-dimensional volume growth (Cao, Conlon–Deruelle). *Not* seen in finite-time singular behaviour of KRF.

• Our steady AC G_2 solitons most closely resemble Joyce-Lee-Tsui's (JLT) *translating solitons* in Lagrangian mean curvature flow (LMCF).
◦ Joyce conjectures JLT translating solitons *can* appear in finite-time singularities of LMCF if Floer homology is obstructed.
◦ Speculate that our steady G_2 solitons can also arise as finite-time singularities of Laplacian flow on a compact 7-manifold.

(Our 2-parameter family of AC G_2 *expanders* on $\Lambda^2_+\mathbb{C}P^2$ resembles JLT's family of exact Maslov-zero LMCF expanders asymptotic to pairs of transverse Lagrangian 3-planes).

Comparison with other flows: shrinkers

Ricci flow: One obvious significant difference: absence of *compact* shrinkers in G_2 flow; associated with positive curvature in RF, whereas scalar curvature is non-positive for closed G_2-structures.

General theory for *noncompact complete shrinkers* in RF is well-developed:
◦ their properties are a hybrid of those of positively curved Einstein manifolds and spaces with non-negative Ricci, e.g. at most Euclidean volume growth.
◦ AC (gradient) shrinkers are extremely rigid–manifestation of parabolic backwards uniqueness phenomenon, also seen in MCF.
◦ AC end behaviour of our (highly symmetric) G_2 shrinkers some indication such strong rigidity also holds for AC G_2 (gradient?) shrinkers.

LMCF: self-shrinkers exist and do occur but *not* in the Maslov-zero (graded) setting. **Q:** Is there any natural condition to impose in the G_2 setting that would rule out our AC shrinkers on $\Lambda^2_+\mathbb{S}^4$ and $\Lambda^2_+\mathbb{C}P^2$?

KRF: Feldman-Ilmanen-Knopf (FIK) constructed symmetric ALE Kähler shrinkers; simplest FIK shrinker does appear as a finite-time blowup of KRF on 1-point blowup of $\mathbb{C}P^2$ and is associated with blowing down the point.

Thanks for your attention!