Topological rigidity of the first Betti numbers and Ricci flow smoothing

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February 24, 2021

Gromov's almost flat manifold theorem

Theorem (Gromov, 1978; Ruh, 1982)

There is a dimensional constant $\varepsilon(n) \in (0,1)$ *such that if a closed n-dimensional Riemannian manifold* (M,g) *satisfies*

$$\operatorname{diam}(M,g)^2 \max_{\wedge^2 TM} |\mathbf{K}_g| \ \leq \ \varepsilon^2, \tag{1}$$

then M *is diffeomorphic to an almost flat manifold.*

- An *almost flat manifold* (a.k.a. *infranil manifold*) is by definition of the form N/Γ, where N is a simply connected nilpotent Lie group, Γ is a discrete subgroup of $N \rtimes Aut(N)$ with $[\Gamma : \Gamma \cap N] < w(n)$, the uniform bound here is part of Gromov's theorem.

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- The theorem is striking in that pure curvature and diameter information determines the diffeomorphism type of the manifold.
- Its importance is more apparent as the "collapsing fibers" in the collapsing geometry with bounded *sectional* curvature.

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- Equip these collections with the (pointed) Gromov-Hausdorff topology, then they become pre-compact among metric spaces.
- Due to the various curvature conditions, we expect to get better understanding of the weak limits and improve the regularity of the topology.

Cheeger's finiteness theorem

$\mathcal{M}_{Rm}(n, D)$ is a very large moduli space, but for any $k \leq n$, the moduli space $\mathcal{M}_{Rm}(k, D, v)$ is very small:

Theorem (Cheeger 1970)

The moduli space $\mathcal{M}_{Rm}(k, D, v)$ has only finitely many diffeomorphism classes. The Gromov-Hausdorff topology is the same as the $C^{1,\frac{1}{2}}$ Cheeger-Gromov topology.

- If $\{M_i\} \subset \mathcal{M}_{Rm}(k, D, v)$ and $M_i \to X$ in the Gromov-Hausdorff sense, then $X \in \mathcal{M}_{Rm}(k, D, v)$, and the convergence topology is improved to the $C^{1,\frac{1}{2}}$ Cheeger-Gromov sense.

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- Can think of the moduli spaces $\mathcal{M}_{Rm}(k, D, v)$ for $k \leq n$ as the "minimal models" of manifolds in $\mathcal{M}_{Rm}(n, D)$.
- Question: How can we relate a generic element in $\mathcal{M}_{Rm}(n, D)$ to one of the "minimal models"?

Fukaya's fiber bundle theorem

This question is answered by the Fukaya's fiber bundle theorem:

Theorem (Fukaya, 1987)

Given $D \geq 1$ and $v > 0$, there is some $\varepsilon(n, v) \in (0, 1)$ such that if $d_{GH}(M, N) < \varepsilon$ for some $M \in \mathcal{M}_{Rm}(n, D)$ and $N \in \mathcal{M}_{Rm}(k, D, v)$ with $k \leq n$, then there is a surjective C^1 map $f : M \to N$ which is an almost Riemannian submersion; moreover, the fibers of f are diffeomorphic to an almost flat manifold.

- We say that M *collapses* to N since $\dim N < \dim M$, and M looks, to the naked eyes, like the lower dimensional manifold N.

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- The curvature and metric information determines the diffeomorphism type of M according to the "minimal model" N: M can be thought of as a smooth family of almost flat manifolds parametrized over N.
- This theorem provides the basic picture of the collapsing geometry with bounded sectional curvature; whose structure are locally modeled on singular infranil fiber bundles over orbifolds with corners.

Collapsing geometry with Ricci curvature bounds

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- Motivation 3 - algebraic geometry: the abundance conjecture in Kähler geometry is about the long-time collapsing behavior of Kähler-Ricci flows with nef initial data.
- The general expectation is a singular fibration structure over a singular base (the base maybe topologically regular).

Collapsing fibers: Almost Ricci-flat manifolds

The following is an attempt to generalize Gromov's almost flat manifolds theorem to the setting of almost Ricci-flat manifolds.

Theorem (Dai-Wei-Ye, 1996)

There is a dimensional constant $\varepsilon(n) \in (0,1)$ such that if a closed n-dimensional Riemannian manifold (M, g) with conjugate radii bounded below by 1 satisfies

$$\mathrm{diam}(M,g)^2 \max_M |\mathbf{Rc}_g|_g \leq \varepsilon^2,$$

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- The almost flatness of a (new) mixed curvature condition is introduced by Kapovitch recently, and a similar result is obtained.

Collapsing fibers: Colding-Gromov gap theorem

Instead of the pointwise conjugate radii lower bound, Colding proved the following theorem which only assumes a very natural topological condition.

Theorem (Colding, 1997; Cheeger-Colding, 1997)

There is a dimensional constant $\varepsilon(n) \in (0,1)$ such that if a closed n-dimensional Riemannian manifold (M, g) satisfies

$$\mathrm{diam}(M,g)^2 \mathbf{Rc}_g \ \geq \ -\varepsilon^2 g \quad \textit{and} \quad b_1(M) \ = \ n,$$

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- The diffeomorphism version of this theorem is proven by Cheeger and Colding based on the Reifenberg method.

Topological rigidity of the first Betti number

Recently, Bing Wang and I obtained the following torus bundle theorem for manifolds in $\mathcal{M}_{Rc}(n)$.

Theorem (H.-Wang, 2020)

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Some comments:

- This theorem is in the same spirit as Fukaya's fiber bundle theorem, which can be thought as a parametrized version of Gromov's almost flat manifolds theorem: our theorem could be though as a parametrized version of the Colding-Gromov gap theorem.

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- This theorem is in the same spirit as Fukaya's fiber bundle theorem, which can be thought as a parametrized version of Gromov's almost flat manifolds theorem: our theorem could be though as a parametrized version of the Colding-Gromov gap theorem.
- Notice that if M is an $(n-k)$-torus bundle over N and N is aspherical, then by the homotopy long exact sequence we have $b_1(M) - b_1(N) = n - k$. The theorem tells the inverse, whence serving as a topological rigidity theorem for manifolds in $\mathcal{M}_{Rc}(n)$ that are sufficiently close to the "minimal models" in the Gromov-Hausdorff sense.

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- We were later able to localize the Ricci flow smoothing technique and prove a singular fiber bundle theorem when the collapsing limit is actually an orbifold.
- In a forthcoming joint work with Xiaochun Rong, we are able to show that if the manifold (M, g) satisfying the assumptions of the fiber bundle theorem (esp. with $b_1(M) = b_1(N) + \dim M - \dim N$) and in additional has *almost non-negative* Ricci curvature $\mathbf{Rc}_g \geq -\delta g$, then it is homeomorphic to the product $N \times \mathbb{T}^{\dim M - \dim N}$.

Ricci flow smoothing: Existence and regularity

- On a closed manifold M of dimension m, a smooth family of metrics $g(t)$ is called a Ricci flow solution if it satisfies

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij},$$

where R_{ij} is the Ricci curvature. Hamilton (1982) showed the short time existence of the Ricci flow given any such initial data (M, g).

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- Shi's estimate (1989): the Riemannian curvature satisfies the following bound:

$$\sup_M |\mathbf{Rm}_{g(t)}| \ \leq \ Ct^{-1} \tag{2}$$

  whenever the flow exists up to time t.

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- However, notice that in Shi's estimate, the shorter the existence time of the flow is, the worse the curvature bound is (by the nature of the heat flow). So the key difficulty is to obtain a Ricci flow existence time lower bound, uniformly depending on the given info.: Ricci lower bound, collapsing to a lower dimensional "minimal model", and Betti number condition.

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- Another technical point is that the evolved metric should still be very collapsed, this follows from a distance distortion estimate.

Ricci flow smoothing: The pseudo-locality theorem

Theorem (Perelman, 2002; Tian-Wang, 2015)

For any $\alpha \in (0,1)$, there are positive constants $\varepsilon_P = \varepsilon_P(n,\alpha)$ and $\delta_P = \delta_P(n,\alpha)$ such that if (M,g) is a Ricci flow solution define for $t \in [0,T]$ with each time slice $(M,g(t))$ being a complete Riemannian manifold, and if one of the conditions holds for $p \in M$:

1. *(Perelman)* $\mathbf{R}_{g(0)} \geq -1$ *on* $B_{g(0)}(p,1)$ *and* $I_{B_{g(0)}(p,1)} \geq (1-\delta_P)I_m$, *or*
2. *(Tian-Wang)* $\mathbf{Rc}_{g(0)} \geq -\delta_P g(0)$ *on* $B_{g(0)}(p,1)$ *and* $\left|B_{g(0)}(p,1)\right|_{g(0)} \geq (1-\delta_P)\omega_n$,

where I_n and ω_n stands for the isoperimetric constant and volume of the n-Euclidean unit ball, respectively, and I_Ω denotes the isoperimetric constant for the domain $\Omega \subset M$, then

$$\forall t \in (0,\varepsilon_P^2], \qquad \sup_{B_{g(t)}(p,\varepsilon_P)} \left|\mathbf{Rm}_{g(t)}\right|_{g(t)} \leq \alpha t^{-1} + \varepsilon_P^{-2}. \tag{3}$$

Ricci flow smoothing: The distance distortion estimate

Lemma (Huang-Kong-Rong-Xu, 2018)

Let $(M, g(t))$ be a smooth Ricci flow solution such that

1. *the initial data $g(0)$ satisfies $\mathbf{Rc}_{g(0)} \geq -(n-1)g(0)$ and $|B_{g(0)(p,1)}|_{g(0)} \geq v > 0$ for all $p \in M$,*
2. *the space-time curvature is bounded as $|Rm_{g(t)}|_{g(t)} \leq \alpha t^{-1}$,*

then for any $x, y \in M$ satisfying $d_{g(0)}(x, y) \leq \sqrt{t}$, we have

$$|d_{g(0)}(x, y) - d_{g(t)}(x, y)| \leq \Psi_D(\alpha|v)\sqrt{t}.$$

- $\Psi_D(\alpha|v) \to 0$ as $\alpha \to 0$. Here we can take $\alpha \in (0, 1)$ arbitrarily small.

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- $\Psi_D(\alpha|v) \to 0$ as $\alpha \to 0$. Here we can take $\alpha \in (0,1)$ arbitrarily small.
- The space-time curvature bound is natural.
- Here the uniform estimate depends on the volume lower bound (i.e. non-collapsing) of the initial data — same issue for the pseudo-locality theorem: the initial data should locally look almost like the n-dimensional Euclidean space!

Key difficulty: collapsing initial condition

- In our situation, the n-dimensional manifold $(M, g) \in \mathcal{M}_{Rc}(n)$ may be very collapsed, since it is Gromov-Hausdorff close to a lower dimensional manifold N — the locally almost n-Euclidean conditions may drastically fail.

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- Once we have started the flow, we can pull it back to the universal covering space $\widetilde{M}$ of M. Here we let $\pi : \widetilde{M} \to M$ denote the covering map, and let $\tilde{g}(t) := \pi^* g(t)$ denote the covering flow.

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- Instead of showing uniform estimate for the Ricci flow $(M, g(t))$, we will consider the covering flow $(\widetilde{M}, \tilde{g}(t))$.

Key difficulty: collapsing initial condition

- In our situation, the n-dimensional manifold $(M, g) \in \mathcal{M}_{Rc}(n)$ may be very collapsed, since it is Gromov-Hausdorff close to a lower dimensional manifold N — the locally almost n-Euclidean conditions may drastically fail.
- While we can start the Ricci flow with initial data (M, g), the possible collapsing may destroy the uniform estimates in the pseudo-locality theorem, or the distance distortion estimate.
- Once we have started the flow, we can pull it back to the universal covering space $\widetilde{M}$ of M. Here we let $\pi : \widetilde{M} \to M$ denote the covering map, and let $\tilde{g}(t) := \pi^* g(t)$ denote the covering flow.
- Instead of showing uniform estimate for the Ricci flow $(M, g(t))$, we will consider the covering flow $(\widetilde{M}, \tilde{g}(t))$.
- In view of the pseudo-locality theorem and the distance distortion estimate, we will need to prove the almost locally Euclidean condition for the initial data $(\widetilde{M}, \tilde{g})$.

Almost locally Euclidean universal covering spaces

The key lemma is the following almost locally Euclidean condition for the universal covering of $(M, g) \in \mathcal{M}_{Rc}(n)$, provided the topological information encoded in the first Betti numbers.

Lemma (H.-Wang, 2020)

For any $\varepsilon \in (0,1)$ fixed, there are $\delta_{ALE} \in (0,1)$ and $r_{ALE} \in (0,1)$, solely determined by ε, n, D and v, to the following effect: if $(M, g) \in \mathcal{M}_{Rc}(n)$ and $(N, h) \in \mathcal{M}_{Rm}(k, D, v)$ with $k \leq n$ satisfy

1. *$d_{GH}(M, N) < \delta$ for some $\delta \leq \delta_{ALE}$, and*
2. *$b_1(M) - b_1(N) = n - k$,*

then for any $r \in (0, r_{ALE}]$ and $\tilde{p} \in \widetilde{M}$ we have

$$\left|B_{\tilde{g}}(\tilde{p}, r)\right|_{\tilde{g}} \ \geq \ (1-\varepsilon)\omega_n r^n. \tag{4}$$

Pseudo-local fundamental group

- This lemma is based on the ε-regularity theorem due to Naber and Zhang for Ricci curvature. To state their result let us first define the pseudo-local fundamental group.

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- Given $(M, g) \in \mathcal{M}_{Rc}(n)$, for any $\delta \in (0, 1)$ and any $p \in M$, the *pseudo-local fundamental group* at p, denoted by $\tilde{\Gamma}_\delta(p)$, is defined as $\tilde{\Gamma}_\delta(p) := Image\,[\pi_1(B_g(p, \delta), p) \to \pi_1(M, p)]$.

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- If $\pi : (\widetilde{M}, \tilde{g}, \tilde{p}) \to (M, g, p)$ is the pointed Riemannian universal covering, then the restriction $\pi : \pi^{-1}(B_g(p, 2))_0 \to B_g(p, 2)$ is a normal covering whose deck transformation group G is a sub-group of $\pi_1(M, p)$.

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- For this normal covering we can consider the following group $\widehat{G}_\delta(p) := \langle \gamma \in \pi_1(M,p) : \ d_{\tilde{g}}(\gamma.\tilde{p}, \tilde{p}) < 2\delta \rangle$.

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- It can be easily seen that $\widehat{G}_\delta(p) = \tilde{\Gamma}_\delta(p)$.

ε-Regularity for the Ricci curvature

Theorem (Naber-Zhang, 2016)

For any $\varepsilon > 0$ there is a $\delta(n, \varepsilon) \in (0, 1)$ such that if (M, g, p) is a pointed Riemannian n-manifold with $\mathbf{Rc}_g \geq -(n-1)g$, and $B_g(p, 2) \Subset B_g(p, 4)$, then for any normal covering $\hat{\pi} : (W, \hat{p}) \to (B_g(p, 2), p)$ with $\hat{\pi}(\hat{p}) = p$, covering metric $\hat{g}$ and deck transformation group G, if $d_{GH}\left(B_g(p, 2), \mathbb{B}^k(2)\right) < \delta$, then the group $\widehat{G}_\delta(p)$ is almost nilpotent with nilpotency rank not exceeding $m - k$; moreover, if it rank $\widehat{G}_\delta(p) = n - k$, then for some $r \in (\delta, 1)$ it holds that $d_{GH}\left(B_{\hat{g}}(\hat{p}, r), \mathbb{B}^n(r)\right) < \varepsilon r$.

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- In this case we have rank $\tilde{\Gamma}_\delta(p) \leq n-k$ and the equality gives for any $\tilde{p} \in \widetilde{M}$ that

$$d_{GH}(B_{\tilde{g}}(\tilde{p}, r), \mathbb{B}^n(r)) < \varepsilon r.$$

Almost nilpotent groups

- If Γ is an almost nilpotent group, then it admits a polycyclic decomposition

$$\Gamma \equiv \Gamma_l \triangleright \Gamma_{l-1} \triangleright \cdots \triangleright \Gamma_1 \triangleright \Gamma_0 = \{e\}.$$

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- The nilpotency rank of an almost nilpotent group is independent of the polycyclic decomposition.
- Intuitively, for $\tilde{\Gamma}_\delta(p)$, its nilpotency rank counts how many independent and non-trivial circles are packet in $B_g(p, \delta)$.

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- Besides the pseudo-local fundamental group at every point, we also consider the group $H_1^{\varepsilon}(M;\mathbb{Z})$, generated by singular homology classes with a representation by a geodesic loop of length not exceeding 10ε. Under the assumption $d_{GH}(M,N)<\varepsilon$, it is then shown that

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$$b_1(M) - b_1(N) = \text{rank } H_1^{\varepsilon}(M;\mathbb{Z}).$$

- For $\varepsilon < \varepsilon(m)$ sufficiently small, generalizing the work of Colding and Naber, it is shown that

$$\text{rank } H_1^{\varepsilon}(M;\mathbb{Z}) \leq \text{rank } \tilde{\Gamma}_{\delta(\varepsilon)}(p) \leq n - k$$

for any $p \in M$.

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- The Riemannian metric $\varepsilon_P^{-2} g(\varepsilon_P^2)$ is sufficiently collapsed by the distance distortion estimate, and has bounded sectional curvature. Therefore, by Fukaya's fiber bundle theorem we obtain the bundle structure.
- We are yet to check that each fiber F_p is a flat torus, rather than a generic almost flat manifold.

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  This forces the almost nilpotent group $\tilde{\Gamma}_{\delta(\varepsilon)}(p)$ to be a finitely generated abelian group.
- Each fiber F_p has abelian fundamental group, and thus it has to be a finite quotient of the $(n-k)$-torus.

Outline of the proof—torus bundle structure

- If F_p is not diffeomorphic to $\mathbb{T}^{n-k}$, that means there is a finite order action of some element in $\pi_1(F_p, p)$. But since $\pi_1(F_p, p) \cong \tilde{\Gamma}_\varepsilon(p)$ and M is a smooth manifold, there must be certain finite order action by some $\gamma \in \tilde{\Gamma}_\varepsilon(p)$ on an invariant neighborhood U of $p \in M$ that fixes the central fiber, i.e. the fibration $F|_U : U \to F(U) \subset N$ is $\tilde{\Gamma}_\varepsilon(p)$ equivariant with $\gamma.F_p = F_p$. However, this will leave $F(p) \in N$ a singular orbifold point, contradicting our assumption that (N, h) is a smooth manifold.

Thank you!