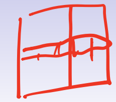
Massive C^*-algebras, Winter 2021, I. Farah, Lecture 22

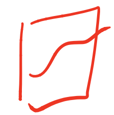
Today:

1. Completing the proof that $\mathsf{OCA_T}$ implies all automorphisms of $\mathcal{Q}(H)$ are inner.
2. Generalizations.

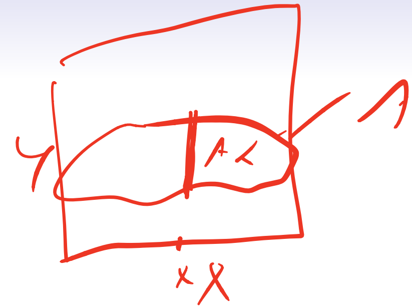


$\mathcal{Z}$

Recall the following definitions (throughout §17.6–§17.7, $\mathcal{B}(H)_{\leq 1}$ is considered with respect to the WOT.)



Def 17.6.1 *A subset $\mathcal{Z}$ of $\mathcal{B}(H)^2_{\leq 1}$ is narrow if for all (a,b) and (a,c) in $\mathcal{Z}$ we have $b \approx^{\mathcal{K}} c$.*
It is ε-narrow if for all (a,b) and (a,c) in $\mathcal{Z}$ we have $b \approx^{\mathcal{K}}_{\varepsilon} c$.
A function $f : \mathcal{B}(H)_{\leq 1} \to \mathcal{B}(H)_{\leq 1}$ is σ-narrow if its graph can be covered by a countable family of narrow Borel sets.
It is σ-ε-narrow if its graph can be covered by a countable family of ε-narrow Borel sets.
An endomorphism Φ of $\mathcal{Q}(H)$ has a σ-narrow lifting if its restriction to the unit ball has a lifting which is σ-narrow. It has a σ-narrow ε-approximation if there is a σ-ε-narrow function Θ such that every $a \in \mathcal{B}(H)_{\leq 1}$ satisfies $\Phi_*(a) \approx^{\mathcal{K}}_{\varepsilon} \Theta(a)$.
A σ-narrow lifting on $\mathcal{D}$[E] or D[E] and a σ-narrow ε-approximation on $\mathcal{D}$[E] or D[E] are defined analogously.



We'll need another result from the classical descriptive set theory.

P.S.

Thm B.2.14 (Novikov) *If* X *and* Y *are Polish spaces and* $A \subseteq X \times Y$ *is analytic, then the set* $\{x \in X : A_x \text{ is nonmeager}\}$ *is analytic.*   s.p ≠ P.S

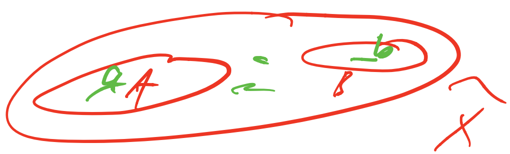
$A_x = \{y \mid (x,y) \in A\}$

Lemma 17.7.1 *Suppose Φ is an endomorphism of $\mathcal{Q}(H)$, $d \geq 1$, $\mathsf{E} \in \mathsf{Part}_{\mathbb{N}}$, and there exists a $1/d$-narrow analytic set $\mathcal{Z} \subseteq \mathsf{D}_{\tilde{\mathsf{X}}} \times \mathcal{B}(H)_{\leq 1}$. Then for every $\mathsf{A} \subseteq \tilde{\mathsf{X}}$ such that both A and $\tilde{\mathsf{X}} \setminus \mathsf{A}$ are infinite at least one of the following applies.*

1. *There is a C-measurable $3/d$-approximation of Φ on D_A.*
2. *There are $\mathsf{B} \subseteq \tilde{\mathsf{X}} \setminus \mathsf{A}$, $a \in \mathsf{D}_\mathsf{A}$, and $b \in \mathsf{D}_\mathsf{B}$ such that both B and $\tilde{\mathsf{X}} \setminus (\mathsf{A} \cup \mathsf{B})$ are infinite and for every uniformization Ξ of $\mathcal{Z}$ and even $c \in \mathsf{D}_{\tilde{\mathsf{X}} \setminus (\mathsf{A} \cup \mathsf{B})}$ such that $a + b + c \in \mathrm{dom}(\Xi)$ satisfy $\Xi(a + b + c)q_\mathsf{A} \not\approx^{\mathcal{K}}_{1/d} \Phi_*(a)$.*

(otherwise, $a \mapsto \Xi(a+b+c)\, \varepsilon_A$ is a lift of ϕ on D_A

$\varepsilon_A = \phi_*(P_A)$)



$D[E]$ $\quad |E_n| \to \infty$

$D_Y[E]$

Lemma 17.7.1 *Suppose Φ is an endomorphism of $\mathcal{Q}(H)$, $d \geq 1$, $\mathsf{E} \in \mathrm{Part}_{\mathbb{N}}$, and there exists a $1/d$-narrow analytic set $\mathcal{Z} \subseteq \mathsf{D}_{\tilde{\mathsf{X}}} \times \mathcal{B}(H)_{\leq 1}$. Then for every $\mathsf{A} \subseteq \tilde{\mathsf{X}}$ such that both A and $\tilde{\mathsf{X}} \setminus \mathsf{A}$ are infinite at least one of the following applies.*

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Proof: Let

Is $c =_{\frac{1}{d}} \phi_*(a)$?

Borel!

$$\mathcal{V} := \{(a, b, c) \in \mathsf{D}_\mathsf{A} \times \mathsf{D}_{\tilde{\mathsf{X}} \setminus \mathsf{A}} \times \mathcal{B}(H)_{\leq 1} :$$
$$(\exists c' \in \mathcal{B}(H)_{\leq 1})(a + b, c') \in \mathcal{Z}, c \approx^{\mathcal{K}}_{1/d} c' q_\mathsf{A}\}.$$

$$\mathcal{W}(a) := \{b \in \mathsf{D}_{\tilde{\mathsf{X}} \setminus \mathsf{A}} : (a, b, \Phi_*(a)) \in \mathcal{V}\}, \text{ for } a \in \mathsf{D}_\mathsf{A}.$$

U is a projection of Z, hence analytic

analytic $\supseteq \{(a,b,c,c') \mid (a+b,c') \in Z,\ c \simeq^{k}_{1/b} c' \leq_A\}$

$\boxed{\text{Case 1} \quad \forall a \in D_A,\ W(a)}$ is relatively comeager in $D_{\hat{X}} \setminus A$.

$$Y = \{(a,c) \in D_A \times B(H)_{\leq 1} \mid \{b \in D_{\hat{X}} \setminus A : (a,b,c) \in U\} \text{ is relatively comeager in } D_{X \setminus A}\}$$

Y is analytic.

Let θ be a c-measurable uniformization of Y.

Then, $\forall a \in \mathrm{dom}(Y)$ $\exists b \in W(a)$ and $(a, b, \theta(a)) \in U$.

Then there is c', $(a+b, c') \in Z$

$\theta(a) \simeq^{x}_{1/d} c' \leq_{\hat{X}}$, also $\exists c''$

$(a+b, c'') \in Z$ $\quad \phi_X(a) \simeq^{k}_{1/d} c'' \leq_{\hat{X}}$.

$a \to \theta(a)$ is $\frac{2}{1}$-approx, c-measurable.

<u>Case 2</u> $\exists a \in D_A$, $w(a)$ is not comeager. So there is a basic open U, $w(a) \cap U$ is relatively meager in U.

$(w(a), U \subseteq D_{X \setminus A})$.

Then $\exists J_n \subset\subset X \setminus A$ disjoint, $\exists s(n) \in D_{J_n}$ so that

$$U \cap \{ y \in D_{X\setminus A} \mid \exists^\infty n \ y \restriction J_n = s(n) \} \cap W(a) = \emptyset.$$

$(U = [J, t],\ J \subset\subset X \setminus A)$

$B = \bigcup_n J_{2n}$, $L = \sum_n s(2n)$

Lemma 17.7.2 *Suppose* Φ *is an endomorphism of* $\mathcal{Q}(H)$, $d \geq 1$, $\mathsf{E} \in \mathsf{Part}_{\mathbb{N}}$, *and* Φ *has a* σ*-narrow* $1/d$*-approximation on* $\mathsf{D}_{\tilde{\mathsf{X}}}[\mathsf{E}]$. *Then the following holds.*

1. *There are an infinite* $\mathsf{A} \subseteq \tilde{\mathsf{X}}$ *and a C-measurable* $3/d$*-approximation to* Φ *on* $\mathsf{D}_{\mathsf{A}}[\mathsf{E}]$.
2. *There is a C-measurable* $3/d$*-approximation of* Φ *on* $\mathsf{D}[\mathsf{F}]$ *for all* $\mathsf{F} \in \mathsf{Part}_{\mathbb{N}}$.

Lemma 17.7.2 *Suppose Φ is an endomorphism of $\mathcal{Q}(H)$, $d \geq 1$, $\mathrm{E} \in \mathrm{Part}_{\mathbb{N}}$, and Φ has a σ-narrow $1/d$-approximation on $\mathrm{D}_{\tilde{\mathrm{X}}}[\mathrm{E}]$. Then the following holds.*

1. *There are an infinite $\mathrm{A} \subseteq \tilde{\mathrm{X}}$ and a C-measurable $3/d$-approximation to Φ on $\mathrm{D}_{\mathrm{A}}[\mathrm{E}]$.*
2. *There is a C-measurable $3/d$-approximation of Φ on $\mathrm{D}[\mathrm{F}]$ for all $\mathrm{F} \in \mathrm{Part}_{\mathbb{N}}$.*

so, $\forall A \subseteq X$ $\forall n$ $D_A \to D(H)/\varepsilon_1$
$a \to \Xi_n(a)$ Σ_A is not $3/d$-ap.

Proof: 1

Assume otherwise. Fix $1/d$-narrow analytic sets $(\mathcal{Z}_n)$ that cover the graph of a $1/d$-approximation of Φ on $\mathrm{D}_{\tilde{\mathrm{X}}}$. Fix a C-measurable uniformization Ξ_n of $\mathcal{Z}_n$. We will find $\tilde{\mathrm{X}} = \bigsqcup_n \mathrm{A}(n) \sqcup \bigsqcup_n \mathrm{B}(n)$, $a(n) \in \mathrm{D}_{\mathrm{A}(n)}$, and $b(n) \in \mathrm{D}_{\mathrm{B}(n)}$, so that $a := \sum_n a(n)$ and $b := \sum_n b(n)$ satisfy $\Xi_m(a+b) \not\approx^{\mathcal{K}}_{1/d} \Phi_*(a+b)$ for all m.

$\mathrm{OCA_T}$ implies all automorphisms of $\mathcal{Q}(H)$ are inner

We have finally proved that $\mathrm{OCA_T}$ implies that for every $\mathrm{E} \in \mathrm{Part}_{\mathbb{N}}$ and every $\varepsilon > 0$, Φ has a C-measurable ε-approximation on $\mathcal{D}_{\mathrm{X}}[\mathrm{E}]$ for some infinite X.
Let's quickly take a look at the remaining part of the proof

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OCA$_T$

1. Lemma 17.5.3 (3): $\forall\,\mathrm{E}$, Φ has a C-measurable ε-approximation on D[E] for all $\varepsilon > 0$.
2. Lemma 17.4.5: $\forall\,\mathrm{E}$, Φ has a continuous lifting on $\mathrm{D_Y}[\mathrm{E}]$ for some infinite Y.
3. Proposition 17.5.4: $\forall\,\mathrm{E}$, Φ has a product type lifting Ξ on $\mathcal{D}[\mathrm{E}]$ such that each Ξ_n is a unital $1/n$-approximate *-homomorphism.
4. Theorem 17.2.6 (Corollary 17.5.5): $\forall\,\mathrm{E}$, Φ has a lifting on $\mathcal{D}[\mathrm{E}]$ that is a *-homomorphism.
5. Proposition 17.5.7: $\forall\,\mathrm{E}$, Φ has a lifting on $\mathcal{D}[\mathrm{E}]$ of the form $a \mapsto vav^*$.
6. Lemma 17.5.8: For some Fredholm partial isometry w, $\forall\,\mathrm{E}$, $\mathrm{Ad}\,w \circ \Phi$ has a lifting on $\mathcal{D}[\mathrm{E}]$ of the form $a \mapsto uau^*$ for a unitary u.

OCA$_T$

7. Theorem 17.8.2 the 'coherent family of unitaries' implementing $\mathrm{Ad}\,w \circ \Phi$ can be uniformized by a single unitary.

Thm (McKenney, McKenney–Vignati, Vignati) $\mathrm{OCA_T}+MA$ *imply that every isomorphism between coronas of separable* C^**-algebras has a Borel-measurable lifting.*

McKenney, P. and Vignati, A. Forcing axioms and coronas of nuclear C^*-algebras. J. Math. Logic, to appear.

Vignati, Rigidity conjectures. Ann. Inst. Fourier, to appear.

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Σ^1_2 Π^1_2 $\forall \dots \exists$

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In order to improve the conclusion to 'every isomorphism between coronas of separable C^*-algebras is trivial' we need

(1) A stronger Ulam-stability result (cf. Kadison–Kastler stability)

(2) The right definition of 'trivial'.

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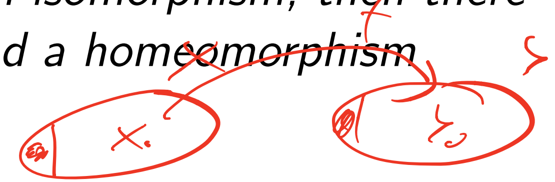
(1) A stronger Ulam-stability result (cf. Kadison–Kastler stability)
(2) The right definition of 'trivial'.
Vignati did both in the abelian case (using results of Šemrl for (1)).

Thm (Vignati) *Assume* $\mathrm{OCA_T}$ *and MA If X and Y are locally compact, noncompact, Polish spaces and* $\Phi\colon C_b(X)/C_0(X) \to C_b(Y)/C_0(Y)$ *is an isomorphism, then there are co-compact $X_0 \subseteq X$ and $Y_0 \subseteq Y$ and a homeomorphism $f\colon Y_0 \to X_0$ such that $a \mapsto a \circ f$ lifts Φ.*

(Parovičenko)



CH: If X is ctble locally cpct, then $C_b(X)/C_0(X) \cong \ell_\infty/c_0$

$$\ell_\infty/c_0 = C_b(\mathbb{N})/C_0(\mathbb{N})$$

s_n , $s_n^* s_n = 1$ $s_n s_n^* \, s_m s_m^* = 0$

$n \neq m$

+MA

Thm (Vignati) $\mathrm{OCA_T}$ *implies* $\mathcal{Q}(H) \ncong \mathcal{M}(\mathcal{O}_\infty \otimes \mathcal{K})/\mathcal{O}_\infty \otimes \mathcal{K}$.

Thm (Vignati) $\mathrm{OCA_T}$ *implies* $\boxed{\mathcal{Q}(H) \not\cong \mathcal{M}(\mathcal{O}_\infty \otimes \mathcal{K})/\mathcal{O}_\infty \otimes \mathcal{K}}.$

Fact

$\mathcal{M}(\mathcal{O}_\infty \otimes \mathcal{K})/\mathcal{O}_\infty \otimes \mathcal{K}$ *has a K-theory reversing automorphism.*

$K_1 = \mathbb{Z}, \quad K_0 = 0$

Thm (Vignati) $\mathrm{OCA_T}$ *implies* $\mathcal{Q}(H) \not\cong \mathcal{M}(\mathcal{O}_\infty \otimes \mathcal{K})/\mathcal{O}_\infty \otimes \mathcal{K}$.

Fact
$\mathcal{M}(\mathcal{O}_\infty \otimes \mathcal{K})/\mathcal{O}_\infty \otimes \mathcal{K}$ *has a K-theory reversing automorphism.*

Question *Are there examples of simple separable* C^**-algebras* A *and* B *such that the assertion '*$\mathcal{M}(A)/A \cong \mathcal{M}(B)/B$ *is independent from ZFC?*

$X \subseteq \mathbb{N}$

$A, B - \bigoplus_{n \in X} M_n(\mathbb{C})$

Endomorphisms

$OCA_T + MA$

Conjecture (F., 1997) PFA implies that every endomorphism of ℓ_∞/c_0 lifts to an endomorphism (not necessarily w*-continuous) of ℓ_∞.

$$\begin{tikzcd}
\ell_\infty \arrow[r] \arrow[d] & \ell_\infty \arrow[d] \\
\ell_\infty/c_0 \arrow[r, "\phi"'] & \ell_\infty/c_0
\end{tikzcd}$$

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Thm (Dow) *There is an endomorphism of ℓ_∞/c_0 that does not lift to an endomorphism of ℓ_∞.*

Dow, A. A non-trivial copy of $\beta\mathbb{N} \setminus \mathbb{N}$, Proc. AMS 142.8 (2014): 2907-2913.

Endomorphisms of $\mathcal{Q}(H)$

Fix an isomorphism $\Phi_n \colon \mathcal{Q}(H) \otimes M_n(\mathbb{C}) \to \mathcal{Q}(H)$, for every $n \geq 1$.

$\shortparallel$

$M_n(Q(H)) \cong Q(H)$

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Thm (Vaccaro, 2019) $\mathrm{OCA_T}$ *implies that every endomorphism of* $\mathcal{Q}(H)$ *is unitarily equivalent to* $\Phi_n \circ (\mathrm{id}_{\mathcal{Q}(H)} \otimes 1_n)$ *for some* $n \geq 1$. *Therefore* $\mathrm{OCA_T}$ *implies* $\underline{End(\mathcal{Q}(H), \circ)}/ \underline{\sim_u} \cong \underline{(\mathbb{N} \setminus \{0\}, \cdot)}$.

Vaccaro, A. Trivial Endomorphisms of the Calkin Algebra. arXiv:1910.07230 (2019).

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$\mathrm{OCA_T} \Rightarrow$

Coro *There are* C^**-algebras* A *and* B *such that* $A \hookrightarrow \mathcal{Q}(H)$, $B \hookrightarrow \mathcal{Q}(H)$, *but* $A \otimes_\alpha B \not\hookrightarrow \mathcal{Q}(H)$ *for any tensor product* $\otimes_\alpha$. *There is a countable inductive system* (A_n) *such that* $A_n \hookrightarrow \mathcal{Q}(H)$ *for all n, but* $\lim_n A_n \not\hookrightarrow \mathcal{Q}(H)$.

$\{A \mid A \hookrightarrow Q(H)\}$

$Q(H) \otimes Q(H) \not\cong Q(H)$

Endomorphisms of $\mathcal{Q}(H)$

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F.–Hirshberg–Vignati: CH implies the negation of the conclusions of the Corollary. <u>and Thm.</u>

Farah, I., Hirshberg, I. and Vignati, A. The Calkin algebra is $\aleph_1$-universal. Israel J. Math. 237 (2020): 287-309.

<u>CH</u> $\{A \mid A \hookrightarrow \mathcal{Q}(H)\} = \{A \mid \chi(A) \leq \aleph_1\}$

Embedding separable C^*-algebras into massive C^*-algebras

Two *-homomorphisms Φ, Ψ from A into B are

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[dr, "\psi"'] & B \\ & B \arrow[u, "\mathrm{Ad}\, u"'] \end{tikzcd}$$

1. unitarily equivalent if there is $u \in \mathrm{U}(B)$ such that $\mathrm{Ad}\, u \circ \Psi = \Phi$.
2. approximately unitarily equivalent if there is a net $u_\lambda \in \mathrm{U}(B)$ such that $\mathrm{Ad}\, u_\lambda \circ \Psi(a) \to \Phi(a)$ for all $a \in A$.

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Exercise. If B is countably (quantifier-free) saturated and A is separable, then $\Phi\colon A \to B$ and $\Psi\colon A \to B$ are unitarily equivalent if and only if they are approximately unitarily equivalent.

Embedding separable C^*-algebras into massive C^*-algebras

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Exercise. If B is countably (quantifier-free) saturated and A is separable, then $\Phi\colon A \to B$ and $\Psi\colon A \to B$ are unitarily equivalent if and only if they are approximately unitarily equivalent.

Degree-1 saturation does not suffice; the conclusion is false for $A = M_{2^\infty}$ and $B = \mathcal{Q}(H)$.

Recall that $B_\infty := \ell_\infty(B)/c_0(B)$.
If $f : \mathbb{N} \to \mathbb{N}$ is an injection, define $f_* : B_\infty \to B_\infty$ by its action on the representing sequences

$f_* : \ell_\infty(B) \to \ell_\infty(B)$

$$f_*((b_n)) = (b_{f(n)}).$$

$B \hookrightarrow \ell_\infty(B)/c_0(B)$ $\qquad$ $f_* \restriction B = \mathrm{id}_B$

Recall that $B_\infty := \ell_\infty(B)/c_0(B)$.
If $f\colon \mathbb{N} \to \mathbb{N}$ is an injection, define $f_*\colon B_\infty \to B_\infty$ by its action on the representing sequences

$$f_*((b_n)) = (b_{f(n)}).$$

Exercise. If $\mathcal{U}$ is an ultrafilter, $f\colon \mathbb{N} \to \mathbb{N}$ is an injection, then the f_* defines an endomorphism of $B_\mathcal{U}$ iff $f(n) = n$ for $\mathcal{U}$-many n.

Thm (Kirchberg, Phillips, Gabe) *Suppose that A is separable and $\Phi\colon A \to B_\infty$ is a *-homomorphism. TFAE*

1. $\Phi \sim_u \Psi$ *for some* $\Psi\colon A \to B$ *(the diagonal copy of B in B_∞).*
2. *For every injection $f\colon \mathbb{N} \to \mathbb{N}$, $f_* \circ \Phi \sim_u \Phi$.*

Pf Note, for $\Phi, \Psi : A \to B_\infty$

$\phi \sim_u \psi \iff \phi \sim_{a.u} \psi$

(B_∞ is ctbly saturated).

① $\Rightarrow$ ② Assume ①. Fix $u \in U(B_\infty)$

$\psi := \mathrm{Ad}\, u \circ \phi$, has $\psi[A] \subseteq B$.

Then $\forall f : \mathbb{N} \to \mathbb{N}$, inj. $f_* \circ \psi = \psi$.

Fix f. Then

$$\begin{aligned} f_* \circ \phi &= f_* \circ \mathrm{Ad}\, u^* \circ \psi \\ &= \mathrm{Ad}\, f_*(u^*) \circ \psi = \mathrm{Ad}\, f_*(u^*) \circ \mathrm{Ad}\, u \circ \phi \\ &= \mathrm{Ad}\, f_*(u^* u) \circ \phi. \end{aligned}$$

② ⟹ ① Assume ②..

Fix a lift of ϕ, ϕ_*

$A \to \ell_\infty(B)$

$\phi_*(a) = (\varphi_n(a))_{n\in\mathbb{N}}$, $\varphi_n : A \to B$

<u>claim</u> $\forall F \Subset A$, $\forall \varepsilon > 0$, $\exists m \in \mathbb{N}$

$\forall n \geq m$ $\exists v \in U(B)$

$$\|\varphi_n(a) - v\varphi_m(a)v^*\| < \varepsilon, \forall a \in F.$$

Pf Assume otherwise. Fix F, ε.

$\forall m$ $\exists f(m) > m$, $(> f(i), \forall i < m)$

s.t. $\nexists v \in U(B)$ $\|\varphi_n(a) - v\varphi_m(a)v^*\| < \varepsilon$ $\forall a \in F$.

Then $f : \mathbb{N} \to \mathbb{N}$, injection.

$f_* \circ \phi \sim_u \phi$. (by ②).

Fix $u \in U(B_\infty)$, then $u \in U(B)^{\mathbb{N}}$

$\forall a \in A$ $\phi(a) = \mathrm{Ad}\, u \circ f_* \circ \phi(a)$.

Therefore, $\forall a \in A$

$$\| \varphi_n(a) - u_n \varphi_{f(n)}(a) u_n^* \| \to 0$$

$$\| u_n^* \varphi_n(a) u_n - \varphi_{f(n)}(a) \| \to 0$$

$a \in F$

Write $A = \bigcup_n^{\uparrow} F_n$, $F_n \subset\subset A$,

~~fix~~ $\varepsilon_n = 2^{-n}$.

Find $m_0 < m_1 < m_2 < \cdots$

and $v_0, v_1 \ldots$ in $\mathcal{U}(B)$

so that $\forall j$, $\forall a \in F_j$,

$$\| v_j \varphi_{m_j}(a) v_j^* - \varphi_{m_{j-1}}(a) \| < 2^{-j} \quad (*)$$

Then let $u_n = v_n v_{n-1} \cdots v_0$.

$\lim_{n \to \infty} u_n^* \varphi_{m_n}(a) u_n$ exists, $\forall a \in A$.

(follows from $*$)

$$\psi(a) := \lim_{n \to \infty} u_n^* \varphi_{m_n}(a) u_n$$

is a $*$-homo, $\psi : A \to B$.

If $f(n) = m_n$, then

$v := (v_n)_n \in U(B_\infty)$ gives

$$\mathrm{Ad}\, v \circ \psi = f_* \circ \phi, \quad \text{and } f_* \circ \phi \sim_u \phi,$$

so ① follows. □