Today:

1. The story so far.
2. OCA_∞.

Important: There will be no classes on March 29 or April 2.
There will be classes on April 5 and April 9.

So far, we've proven the following ($\mathcal{B}(H)$, is considered with respect to the WOT).

Prop *Suppose* $\Phi \in \mathrm{Aut}(\mathcal{Q}(H))$ *is such that for every* $\varepsilon > 0$ *there are* $\mathsf{E} \in \mathrm{Part}_{\mathbb{N}}$, *with* $|E_n| \to \infty$, *and an infinite* $\mathsf{X} \subseteq \mathbb{N}$ *such that* Φ *has a C-measurable* ε*-approximation* Θ *on* $\mathsf{D}_{\mathsf{X}}[\mathsf{E}]$. *Then* $\mathrm{OCA}_{\mathrm{T}}$ *implies that* Φ *is inner.*

$$\|\pi(\Theta(a) - \phi_*(a))\| \leq \varepsilon, \ \forall a \in D_X$$

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$\forall F \in \mathrm{Part}_{\mathbb{N}}$

Proof: Lemma 17.5.3 (3): Φ has a C-measurable ε-approximation on $\mathsf{D}[\mathsf{E}]$ for every $\varepsilon > 0$.

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Proof: Lemma 17.5.3 (3): Φ has a C-measurable ε-approximation on D[E] for every $\varepsilon > 0$.
Lemma 17.4.5: Φ has a continuous lifting on $\mathsf{D}_{\mathsf{Y}}[\mathsf{E}]$ for some infinite Y.

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Proposition 17.5.4: it can be chosen to be of product type and such that its n-th component Ξ_n is a unital $1/n$-approximate $*$-homomorphism for every n.

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Theorem 17.2.6 (Corollary 17.5.5): for every $\mathsf{F} \in \mathsf{Part}_{\mathbb{N}}$, some *-homomorphism serves as a lifting of Φ on $\mathcal{D}[\mathsf{F}]$.
Proposition 17.5.7: it is implemented by a partial isometry for every $\mathsf{F} \in \mathsf{Part}_{\mathbb{N}}$.

$M_m(\mathbb{C}) \to M_n(\mathbb{C})$

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Proposition 17.5.7: it is implemented by a partial isometry for every $\mathsf{F} \in \mathsf{Part}_{\mathbb{N}}$. (But we want a unitary!)

From C-measurable approximations to innerness

Fix E and v such that $\mathrm{Ad}\,\dot{v}$ and Φ agree on $\mathcal{D}[E]$, and let $\Phi_1 := \mathrm{Ad}\,v^* \circ \Phi$

Then the restriction of Φ_1 to $\mathcal{D}[F]$ is implemented by a unitary for every $F \in \mathrm{Part}_{\mathbb{N}}$.

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Lemma 17.5.8 *Suppose that* $\mathsf{E} \in \mathrm{Part}_{\mathbb{N}}$, u_e *and* u_o *are in* $\mathrm{U}(\ell_\infty(\mathbb{N}))$, *and* $\mathrm{Ad}\,u_e$ *and* $\mathrm{Ad}\,u_o$ *agree on* $\mathcal{D}[\mathsf{E}]$ *modulo the compacts. Then there is* $w \in \mathrm{U}(\ell_\infty(\mathbb{N}))$ *such that* $\mathrm{Ad}\,w$ *agrees with* $\mathrm{Ad}\,u_e$ *on* $\mathcal{D}[\mathsf{E}^{\mathrm{even}}]$ *modulo the compacts and with* $\mathrm{Ad}\,u_o$ *on* $\mathcal{D}[\mathsf{E}^{\mathrm{odd}}]$ *modulo the compacts.*

The proof uses the same trick used at the end of the proof of Theorem 17.8.2.

$\mathcal{F}[F] = \{a \mid a = c_0 + c_1,\ c_0 \in \mathcal{D}[E^{even}],\ c_1 \in \mathcal{D}[E^{odd}]\}$

From C-measurable approximations to innerness

Fix E and v such that $\operatorname{Ad}\dot{v}$ and Φ agree on $\mathcal{D}[\mathrm{E}]$, and let
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$\mathrm{OCA_T}$ and Theorem 17.8.2: Φ_1 is inner, and therefore
$\Phi = \operatorname{Ad} v \circ \Phi_1$ is inner.

If $\forall \varepsilon\ \exists E_\varepsilon\ X_\varepsilon$ On $D_{X_\varepsilon}[E_\varepsilon]$ ϕ has C-measurable ε-approx. $\Rightarrow$ ϕ is inner

OCA_∞

It only remains to prove that OCA_T implies Φ has a C-measurable ε-approximation on $\mathcal{D}[\mathrm{E}]$ for some E.

OCA_T Whenever X is a separable metrizable space and $[\mathrm{X}]^2 = L_0 \sqcup L_1$ is an open colouring, one of the following alternatives applies.

0.1 There exists an uncountable L_0-homogeneous $\mathrm{Y} \subseteq \mathrm{X}$. $[Y]^2 \subseteq L_0$

0.2 There are L_1-homogeneous sets X_n, for $n \in \mathbb{N}$, such that $\bigcup_n \mathrm{X}_n = \mathrm{X}$.

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OCA$_\infty$

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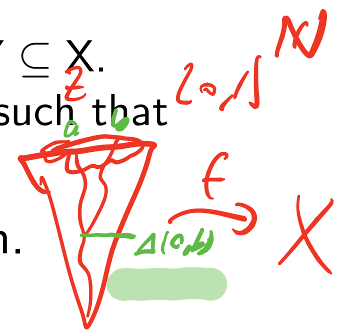

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OCA_∞ Whenever X is a separable metrizable space and $[\mathrm{X}]^2 = L_0^n \sqcup L_1^n$, for $n \geq 0$, are open colourings such that $L_0^n \supseteq L_0^{n+1}$ for all n, one of the following alternatives applies.

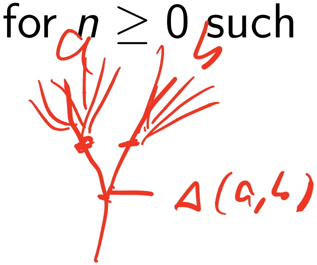
0.1 There exists an uncountable $\mathrm{Z} \subseteq \{0,1\}^{\mathbb{N}}$ and a continuous $f : \mathrm{Z} \to \mathrm{X}$ such that $\{f(a), f(b)\} \in L_0^{\Delta(a,b)}$ for all distinct a and b in Z.

$\bigcup_n X_n = X$

0.2 There are $\mathrm{X}_n \subseteq \mathrm{X}$, for $n \in \mathbb{N}$, such that $[\mathrm{X}_n]^2 \subseteq L_1^n$ for all n.

Thm 8.6.6 $\mathrm{OCA_T}$ *implies* OCA_∞.

Proof: Fix X and open colourings $[X]^2 = L_0^n \cup L_1^n$ for $n \geq 0$ such that $L_0^n \supseteq L_0^{n+1}$ for all n. Define a partition $[\{0,1\}^{\mathbb{N}} \times X]^2 = M_0 \cup M_1$ by



$$\{(a,x),(b,y)\} \in M_0 \text{ if and only if}$$

$$a \neq b,\ x \neq y, \text{ and } \{x,y\} \in L_0^{\Delta(a,b)}.$$

M_0 is open

Case 1 $\exists\, Y \subseteq \{0,1\}^{\mathbb{N}} \times X$, unctble,

$[Y]^2 \subseteq M_0$.

$(a, x), (b, y)$ in Y

$Z = \{a \mid \exists x\ (a, x) \in Y\}$ - analytic $\subseteq \{0,1\}^{\mathbb{N}}$

Y is the graph of $f: Z \to X$

If Case 1 fails, then OCA$_T$ $\Rightarrow$

Case 2 $\{0,1\}^{\mathbb{N}} \times X = \bigcup_n Y_n$, $[Y_n]^2 \subseteq M_1$.

W.l.o.g., Y_n is closed.

Fix $x \in X$

$Z_n(x) = \{a \mid (a, x) \in Y_n\}$.

$\{0,1\}^{\mathbb{N}} \subseteq \bigcup Z_n(x)$

Baire Category $\Rightarrow \exists u_x \exists s_x \in \{0,1\}^{<\mathbb{N}}$

$Z_u(x) \supseteq [s] \; (= \{c \mid s \subseteq c\})$

There are ctbly many pairs (u_x, s_x).

$$\mathbb{X}_{(u,s)} = \{x \mid u_x = u,\ s_x = s\}$$

$$u_x = u_y,\ s_x = s_y,\ \Rightarrow \{x,y\} \in L_1^{|s|}$$

$$\text{So } \left[\mathbb{X}_{(u,s)}\right]^2 \subseteq L_1^{|s|}$$

since $L_1^n \supseteq L_1^{n+1}$, we can

reenumerate $(\mathbb{X}_{(u,s)})$ as $(\mathbb{X}_j)$

so that $[\mathbb{X}_j]^2 \subseteq L_1^j$

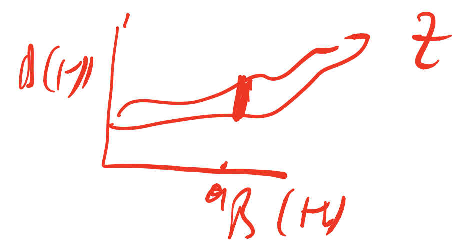
(add $\mathbb{X}_j = \emptyset$, if necessary)

ϕ_*

The following definition describes some intermediates between an arbitrary lifting and a Borel-measurable lifting. (Throughout §17.6–§17.7, $\mathcal{B}(H)_{\leq 1}$ is considered with respect to the WOT.)

Def 17.6.1 *A subset $\mathcal{Z}$ of $\mathcal{B}(H)^2_{\leq 1}$ is narrow if for all (a, b) and (a, c) in $\mathcal{Z}$ we have $b \approx^{\mathcal{K}} c$.* ($b - c \in \mathcal{K}(H)$ $\|\pi(b-c)\| \leq \varepsilon$
It is ε-narrow if for all (a, b) and (a, c) in $\mathcal{Z}$ we have $b \approx^{\mathcal{K}}_{\varepsilon} c$.

$$\mathcal{Z} = \{ (a, b) \mid b \approx^{\mathcal{K}} \phi_*(a) \}$$



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A function $f : \mathcal{B}(H)_{\leq 1} \to \mathcal{B}(H)_{\leq 1}$ is σ-narrow if its graph can be covered by a countable family of narrow Borel sets.
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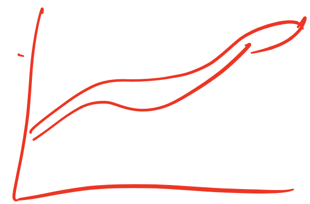

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An endomorphism Φ of $\mathcal{Q}(H)$ has a σ-narrow lifting if its restriction to the unit ball has a lifting which is σ-narrow. It has a σ-narrow ε-approximation if there is a σ-ε-narrow function Θ such that every $a \in \mathcal{B}(H)_{\leq 1}$ satisfies $\Phi_*(a) \approx^{\mathcal{K}}_{\varepsilon} \Theta(a)$.

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A σ-narrow lifting on $\mathcal{D}$[E] or D[E] and a σ-narrow ε-approximation on $\mathcal{D}$[E] or D[E] are defined analogously.

Example

There is an endomorphism Φ of ℓ_∞/c_0 with 2-narrow lifting, but no C-measurable (Borel, continuous,...) lifting.

$(\ell_\infty)_1 \rightsquigarrow w^*$-top, c_0^*

Let $\mathcal{U}_n$, $n \in \{0,1\}$ be nonprincipal ultrafilters on $\mathbb{N}$, $x_n \in \mathcal{U}_n$

Let $f : \ell_\infty \to \ell_\infty$ be

$$f(a)(n) = \lim_{j \to \mathcal{U}_0} a_j \qquad n \text{ even}$$

$$f(a)(n) = \lim_{j \to \mathcal{U}_1} a_j \qquad n \text{ odd}$$

$a - b \in c_0 \implies f(a) = f(b)$

f lifts a $*$-hom.

$$\phi : \ell_\infty / c_0 \to \ell_\infty / c_0$$

It has a Δ_2^1-Borel lifting but no Borel lifting

(HW: Fix this!)

Lemma 17.6.3 *Assume* $\mathsf{OCA_T}$. *If* Φ *is an endomorphism of* $\mathcal{Q}(H)$ *and* $\varepsilon > 0$, *then* Φ *has a* σ*-narrow* ε*-approximation on* $\mathsf{D}_{\tilde{\mathsf{X}}}[\mathsf{E}]$ *for some infinite* $\tilde{\mathsf{X}}$.

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I.e., there is a function Θ such that
(a) every $a \in \mathcal{B}(H)_{\leq 1}$ satisfies $\Phi_*(a) \approx^{\mathcal{K}}_{\varepsilon} \Theta(a)$ and
(b) the graph of Θ can be covered by a countable family of Borel sets $\mathcal{Z}_n$.

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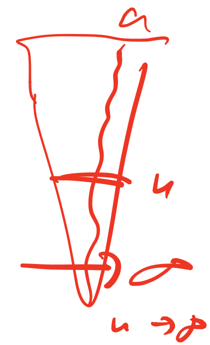
A few conventions for the proof of Lemma 17.6.3:

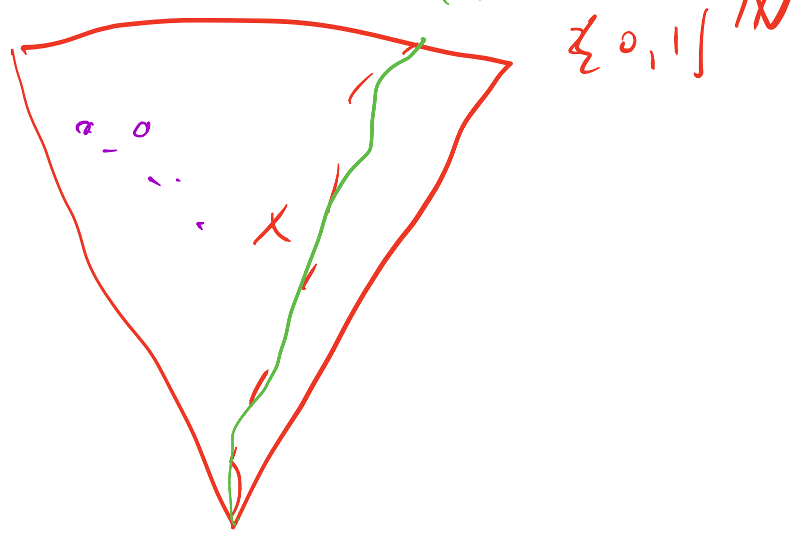
1. We'll index the intervals in $\mathrm{E} \in \mathrm{Part}_{\mathbb{N}}$ by $\{0,1\}^{<\mathbb{N}}$:
   $\mathrm{E} = \langle E_s : s \in \{0,1\}^{<\mathbb{N}} \rangle$.
2. Fix $\mathrm{E} \in \mathrm{Part}_{\mathbb{N}}$ so that $\lim_n \min_{|s|=n} |E_s| = \infty$.
3. If $\mathrm{X} \subseteq \{0,1\}^{<\mathbb{N}}$ is infinite and included in a single branch of $\{0,1\}^{<\mathbb{N}}$, then this branch is denoted $\mathrm{B(X)}$.
4. Fix a discretization $\mathrm{D[E]}$ of $\mathcal{D}[\mathrm{E}]$.

$|E_{a|n}|$



$E_s = (;]$

B(X)


X
{0,1}^N

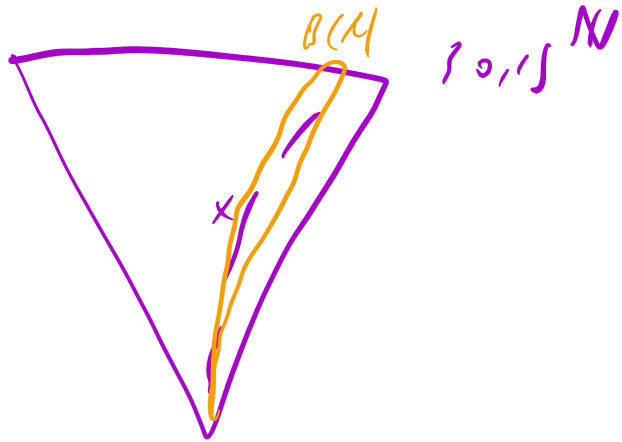
Proof of Lemma 17.6.3, that Φ has a σ-narrow ε-approximation on $D_{\tilde{X}}[E]$ for some infinite $\tilde{X}$.

Fix $d \geq (2\varepsilon)^{-1}$ and $n \geq 1$. Let

$$\mathcal{X} := \{(X, a) : B(X) \text{ is defined and } a \in D_X\}.$$

$D_X \subseteq D$

$P_X a = a \restriction_X = a$



$P_X = \mathrm{Proj}_{\mathrm{span}\{e_i \mid i \in \bigcup E_n,\ n \in X\}}$

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$$\mathcal{X} := \{(\mathrm{X}, a) : \mathrm{B}(\mathrm{X}) \text{ is defined and } a \in \mathrm{D}_{\mathrm{X}}\}.$$

In order to topologize $\mathcal{X}$, identify $(\mathrm{X}, a) \in \mathcal{X}$ with

$$(\mathrm{B}(\mathrm{X}), \mathrm{X}, a, q_{\mathrm{X}}, \Phi_*(a)) \in \{0,1\}^{\mathbb{N}} \times \mathcal{P}(\{0,1\}^{<\mathbb{N}}) \times \mathrm{D} \times \mathcal{B}(H)^2_{\leq 1}$$

$\phi_*(\rho_x)$

Proof of Lemma 17.6.3, that Φ has a σ-narrow ε-approximation on $\mathrm{D}_{\tilde{\mathrm{X}}}[\mathrm{E}]$ for some infinite $\tilde{\mathrm{X}}$.

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$p_X a = a$

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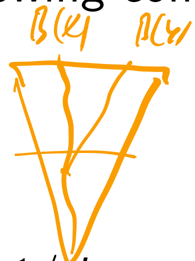
Let $\{(\mathrm{X}, a), (\mathrm{Y}, b)\} \in M_0^{d,n}$ if the following conditions are satisfied:

$(M_0^d 1)$ $\mathrm{B}(\mathrm{X}) \neq \mathrm{B}(\mathrm{Y})$,

$(M_0^d 2)$ $p_{\mathrm{X}} b = p_{\mathrm{Y}} a$, and

$\phi_*(p_Y)$

$(M_0^{d,n} 3)$ $\|p_{[n,\infty)}(\Phi_*(a) q_{\mathrm{Y}} - q_{\mathrm{X}} \Phi_*(b))\| > 1/d$ or
$\|p_{[n,\infty)}(q_{\mathrm{Y}} \Phi_*(a) - \Phi_*(b) q_{\mathrm{X}})\| > 1/d$.



$\{(X, a), (Y, b)\} \in M_0^{d,n}$ iff $(M_0^d 1)$ $B(X) \neq B(Y)$, $(M_0^d 2)$ $p_X b = p_Y a$, and $(M_0^{d,n} 3)$ $\|p_{[n,\infty)}(\Phi_*(a) q_Y - q_X \Phi_*(b))\| > \frac{1}{d}$ or $\|p_{[n,\infty)}(q_Y \Phi_*(a) - \Phi_*(b) q_X)\| > \frac{1}{d}$.

Claim. *For every n, the partition $[\mathcal{X}]^2 = M_0^{d,n} \cup M_1^{d,n}$ is open.*

$M_0^{d,n} \supseteq M_0^{d,n+1}$

$\{(\mathsf{X}, a), (\mathsf{Y}, b)\} \in M_0^{d,n}$ iff $(M_0^d 1)$ $\mathsf{B}(\mathsf{X}) \neq \mathsf{B}(\mathsf{Y})$, $(M_0^d 2)$ $p_{\mathsf{X}} b = p_{\mathsf{Y}} a$, and $(M_0^{d,n} 3)$ $\|p_{[n,\infty)}(\Phi_*(a) q_{\mathsf{Y}} - q_{\mathsf{X}} \Phi_*(b))\| > \frac{1}{d}$ or $\|p_{[n,\infty)}(q_{\mathsf{Y}} \Phi_*(a) - \Phi_*(b) q_{\mathsf{X}})\| > \frac{1}{d}$.

Claim. *There is no uncountable $\mathsf{Z} \subseteq \{0,1\}^{\mathbb{N}}$ such that some continuous $f : \mathsf{Z} \to \mathcal{X}$ satisfies $\{f(a), f(b)\} \in M_0^{d,\Delta(a,b)}$ for all distinct a and b in Z.*

Pf Assume otherwise, fix Z, f.

i.e., $\forall x, y$ in Z, $x \neq y \Rightarrow \{f(x), f(y)\} \in M_0^{d, \Delta(x,y)}$

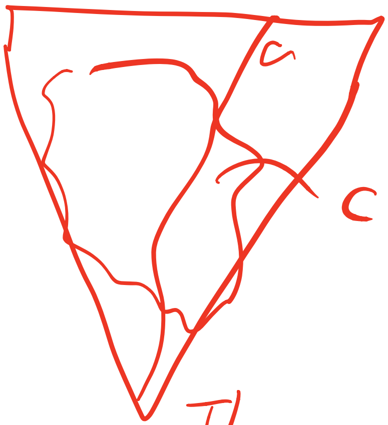
$f(x) = (X, a)$ $\quad f(y) = (Y, b)$

Define $c \in D[E]$ so that $\forall s \in \{0,1\}^{<\mathbb{N}}$

$\underline{c_s = a_s}$ for some $\underline{(X, a) \in f[Z]}$, if such (X, a) exists.

(o/w $c_s = 0$).

Then $c_s = a_s$ $\forall s$ $\forall (X, a) \in f[Z]$ (if exists)



$\{0,1\}^{<\mathbb{N}}$

Then, $\forall (X, a) \in f[Z]$

$P_X c = a$, $\underline{\underline{c P_X = a}}$

Then $\phi_*(P_X c) =^K \phi_*(a)$

$\mathcal{E}_X \phi_*(c) - \phi_*(a) \in K(H)$.

Find $n = n(X, a)$ so that

$$\boxed{\| P_{[n, \infty)} (\mathcal{E}_X \phi_*(c) - \phi_*(a)) \| < \frac{1}{d}}$$

w/og, $\exists n$ $\forall (X, c) \in f[Z]$

$n = n(X, c)$. ?

Find x, y in Z, $\Delta(x, y) > 2n$

Then $\| P_{[n, \infty)} (\mathcal{E}_X \phi_*(b) - \phi_*(c) \mathcal{E}_Y) \| > \frac{1}{2n}$

$(f(x) = (X, a)$, $f(y) = (Y, b))$.

or $\| P_{[n, \infty)} (\phi_*(b) \mathcal{E}_X - \mathcal{E}_Y \phi_*(a)) \| > \frac{1}{2n}$

(X,c)

$$\boxed{\| P_{[u,\infty)} (\Sigma_X \phi_*(c) - \phi_*(a)) \| < \frac{1}{d}}$$

also

$$\| P_{[u,\infty)} (\phi_*(c) \Sigma_X - \phi_*(a)) \| < \frac{1}{d}.$$

Fix (X,a), (Y,b) in $f[7]$.

Then (write $d =^u_\varepsilon e \iff \| P_{[u,\infty)} (d-e) \| < \varepsilon$):

$$\Sigma_Y \phi_*(a) \simeq^u_{1/d} \Sigma_Y \phi_*(c) \Sigma_X \simeq^u_{1/d} \phi_*(b) \Sigma_X$$

so

$$\Sigma_Y \phi_*(c) \simeq^u_{2/d} \phi_*(b) \Sigma_X$$

So

$$\| P_{[u,\infty)} (\Sigma_Y \phi_*(c) - \phi_*(b) \Sigma_X) \| > \frac{2}{d}$$

Similarly,

$$\| P_{[u,\infty)} (\phi_*(a) \Sigma_Y - \Sigma_X \phi_*(b)) \| > \frac{2}{d}$$

therefore, since $\{f(x), f(y)\} \in M_0^{d, \Delta(x,y)}$, where $f(x) = (X,a)$, $f(y) = (Y,b)$,

we have $\Delta(x,y) < \frac{d}{2}$.

But t is unstable!