Massive C^*-algebras, Winter 2021, I. Farah, Lecture 23

Today:

1. More ultrapowers and asymptotic sequence algebras (aka reduced powers).

Throughout this lecture, $\mathcal{U}$ stands for any nonprincipal ultrafilter on $\mathbb{N}$.

$c_{\mathcal{U}}(A) = \{(a_n) \mid \lim_{n \to \mathcal{U}} \|a_n\| = 0\}$

$A_{\mathcal{U}} = \ell_\infty(A)/c_{\mathcal{U}}(A)$ is the norm ultrapower.
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Thm (Connes, McDuff, Effros–Rosenberg,...) *For every unital separable* C^**-algebra* A *the following are equivalent (all embeddings are unital).*

$M_{2^\infty} = \bigotimes_{\mathbb{N}} M_2(\mathbb{C})$

1. $A \otimes M_{2^\infty} \cong A$.
2. $A \prec A \otimes M_{2^\infty}$ *(i.e,* $a \mapsto a \otimes 1_{M_{2^\infty}}$ *is an elementary embedding).*
3. $M_{2^\infty} \hookrightarrow A_{\mathcal{U}} \cap A'$.
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$M_2(A)$

To the list of equivalences one can also add the following

5. $A \prec A \otimes M_2(\mathbb{C})$ *(i.e.,* $a \mapsto a \otimes 1_2$ *is an elementary embedding).*
6. $M_2(\mathbb{C}) \hookrightarrow A_{\mathcal{U}} \cap A'$.

$a \to \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$

7. $M_2(\mathbb{C}) \hookrightarrow A_\infty \cap A'$.

(Note: There are Kirchberg algebras that satisfy $A \otimes M_2(\mathbb{C}) \cong A$ but fail all of the above statements.)

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We will prove some of the nontrivial implications.

$$A \hookrightarrow A$$

$$A \otimes M_{2^\infty} \subset A_{\mathcal{U}} \cap A'$$

Proof that $A \prec A \otimes M_{2^\infty}$ implies $M_{2^\infty} \hookrightarrow A_{\mathcal{U}} \cap A'$:

$$\begin{array}{ccc} A & \hookrightarrow & A_{\mathcal{U}} \\ \downarrow & \nearrow & \\ A \otimes M_{2^\infty} & & \end{array}$$

ctbly saturated

sep.

$$1_A \otimes M_{2^\infty}$$

Proof that $A \prec A \otimes M_{2^\infty}$ *implies* $M_{2^\infty} \hookrightarrow A_{\mathcal{U}} \cap A'$:
$A_{\mathcal{U}}$ is countably saturated and $A \prec A_{\mathcal{U}}$.

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Proof that $A \prec A \otimes M_{2^\infty}$ implies $M_{2^\infty} \hookrightarrow A_\infty \cap A'$:
A_∞ is countably saturated. Although $A \not\prec A_\infty$ in general, a sufficient amount of elementarity is preserved for the proof to go through.

Next, we will use the fact that $M_{2^\infty} \cong \bigotimes_{\mathbb{N}} M_2(\mathbb{C})$.

(this implies

$\bigotimes_{\mathbb{N}} M_{2^\infty} = M_{2^\infty}$

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Recall that $A \prec B$ if $A \subseteq B$ and for every formula $\psi(\bar{x})$ and all $\bar{a} \in A$ we have

$$\psi^A(\bar{a}) = \psi^B(\bar{a}).$$

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Thm (Tarski–Vaught test) *If $A \subseteq B$, then $A \prec B$ if and only if for every formula $\varphi(\bar{x}, y)$ and all $\bar{a} \in A$,*

$$\inf_{y \in A, \|y\| \leq 1} \varphi^B(\bar{a}, y) \leq \inf_{y \in B, \|y\| \leq 1} \varphi^B(\bar{a}, y).$$

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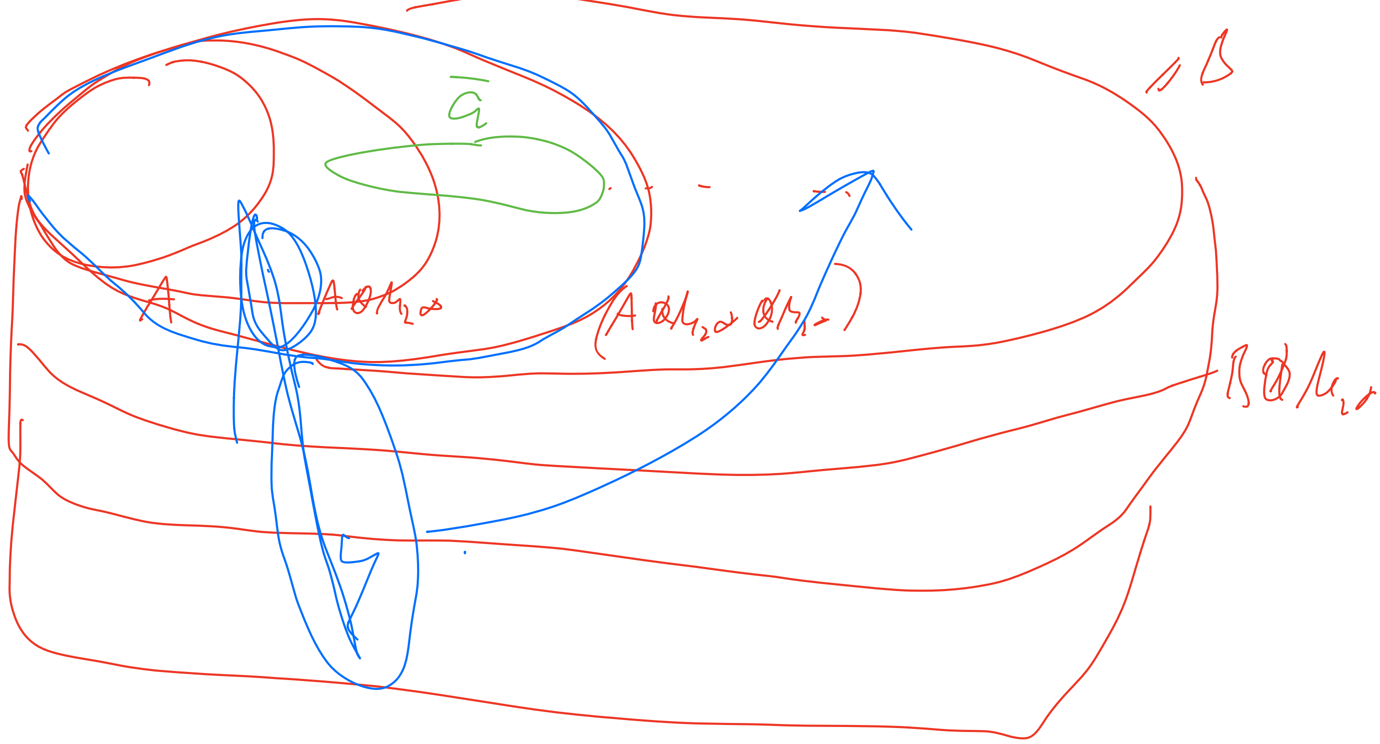
Back to the proof: It suffices to prove that $A \otimes M_{2^\infty} \prec (A \otimes M_{2^\infty}) \otimes M_{2^\infty}$.

$$\left(A \otimes \bigotimes_{\mathbb{N}} M_{2^\infty}\right) \prec \left(A \otimes \bigotimes_{\mathbb{N}} M_{2^\infty}\right) \otimes M_{2^\infty}$$



Fix φ. $\bar{a} \in A \otimes \bigotimes_{\mathbb{N}} M_{2^\infty}$

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unital

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Thm *Suppose A and C are separable, unital, and at least one of them is nuclear. Then $(D = \bigotimes_{\mathbb{N}} C)$ (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4).*

1. $A \cong A \otimes D$.
2. $A \prec A \otimes D$.
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Exercise. If D and A are separable and unital, then $D \hookrightarrow A_{\mathcal{U}} \cap A'$ if and only if $D \hookrightarrow A_{\infty} \cap A'$.

There are separable and unital A and C such that A is nuclear and $A \prec \bigotimes_{\mathbb{N}} C$ but $A \not\cong A \otimes \bigotimes_{\mathbb{N}} C$.

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Lemma *The 'flip' automorphism of $M_{2^\infty} \otimes M_{2^\infty}$ defined by $a \otimes b \to b \otimes a$ is approximately inner: there are unitaries u_n, for $n \in \mathbb{N}$, in $M_{2^\infty} \otimes M_{2^\infty}$ such that $\mathrm{Ad}\, u_n$ converges to the flip pointwise.*

C^*-algebras with this property are said to have an *approximately inner flip*.

Examples

All UHF algebras, the Jiang–Su algebra $\mathcal{Z}$, $\mathcal{O}_2$, $\mathcal{O}_\infty$, every Kirchberg C^*-algebra in the Cuntz normal form, tensor products of these.

) Every automorphism of $M_n(\mathbb{C})$ is inner.

$$M_{2^\infty} = \bigotimes_{\mathbb{N}} M_2(\mathbb{C}) = \varinjlim M_{2^n}(\mathbb{C})$$

Proof that $M_{2^\infty} \hookrightarrow A_{\mathcal{U}} \cap A'$ (or $M_{2^\infty} \hookrightarrow A_\infty \cap A'$) implies $A \cong A \otimes M_{2^\infty}$.

The proof shows that if D has the approximately inner flip and $D \hookrightarrow A_{\mathcal{U}} \cap A'$ (or $D \hookrightarrow A_\infty \cap A'$) then $A \cong A \otimes \cancel{A}D$.

Proof that $M_{2^\infty} \hookrightarrow A_{\mathcal{U}} \cap A'$ (or $M_{2^\infty} \hookrightarrow A_\infty \cap A'$) implies $A \cong A \otimes M_{2^\infty}$.

The proof shows that if D has the approximately inner flip and $D \hookrightarrow A_{\mathcal{U}} \cap A'$ (or $D \hookrightarrow A_\infty \cap A'$) then $A \cong A \otimes A$.

The key ingredient is the following, proved by an intertwining argument.

Thm *Suppose $A \subseteq B$ are separable and there is a sequence of unitaries u_n in $B_{\mathcal{U}}$ such that:*

1. $\lim_n \operatorname{Ad} u_n(a) = a$ *for all* $a \in A$.
2. $\lim_n \operatorname{dist}(\operatorname{Ad} u_n(b), A) = 0$ *for all* $b \in B$.

Then $A \cong B$.

Fact If $u \in A_{\mathcal{U}}$ is a unitary,

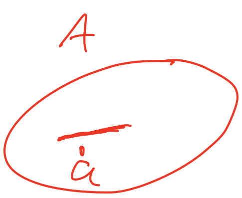
then u has a representing

Sequence (u_n) such that each u_n is a unitary.

(If The relation $\underbrace{\max(\|xx^*-1\|, \|x^*x-1\|)}_{R(x)} = 0$ is <u>weakly stable</u> (definable):

$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall A, \ \forall a \in A$

$$R^A(a) < \delta \implies \exists b \in A, \ R^A(b) = 0$$
and $\|a-b\| < \varepsilon$



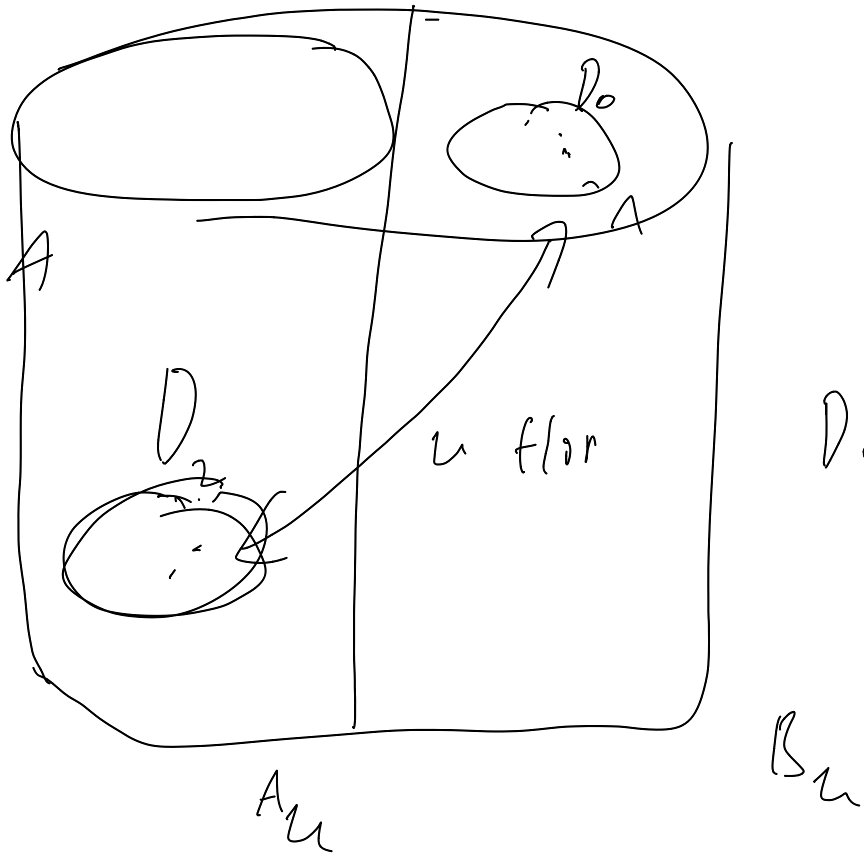
W.S. $\iff \forall a \in A_\mathcal{U}$, $R^{A_\mathcal{U}}(a) = 0$
$\iff a = (a_n)/\mathcal{U}$, $R^A(a_n) = 0, \ \forall n$.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A \arrow[d] & \cdots & A \arrow[d, "\cong"] \\ B \arrow[r] \arrow[ur] & B \arrow[r] \arrow[ur] & B & \cdots & B \end{tikzcd}$$

Suppose $D \hookrightarrow A_\mathcal{U} \cap A'$.

Then $\bigotimes_\mathbb{N} D \hookrightarrow A_\mathcal{U} \cap A'$.

A, $B = A \otimes \bigotimes_\mathbb{N} D$. Note $\bigotimes_\mathbb{N} D \hookrightarrow B_\mathcal{U} \cap B'$



$D_2 \cong D_0 \cong D_1 \cong D$

$D_2 \hookrightarrow \underline{A_u} \cap A'$

$D_0, D_2 \leq A^1$

Ultrapowers vs. the asymptotic sequence algebras

Thm *Suppose the Continuum Hypothesis. Then there exists a nonprincipal ultrafilter $\mathcal{V}$ on $\mathbb{N}$ such that for every separable A the quotient map*

$$\pi_{\mathcal{U}} \colon A_\infty \to A_{\mathcal{U}}$$

has a right inverse.

$$\begin{tikzcd}
 & A_\infty \arrow[d, shift left, "\pi_{\mathcal{U}}"] & \ell_\infty(A)/c_0(A) \\
A \arrow[ur, "\iota_\infty"] \arrow[dr, "\iota_\infty"'] & & \\
 & A_{\mathcal{U}} \arrow[u, shift left, "?"] & \ell_\infty(A)/c_{\mathcal{U}}(A)
\end{tikzcd}$$

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In the following K is the Cantor space, $\{0,1\}^{\mathbb{N}}$:

Thm *For every separable A there is an elementary embedding $A \otimes C(K) \prec A_\infty$ which commutes with the diagonal embedding of A.*

$$C(K,A) = A \otimes C(K) < A_\infty$$

$$\downarrow \qquad \cong$$

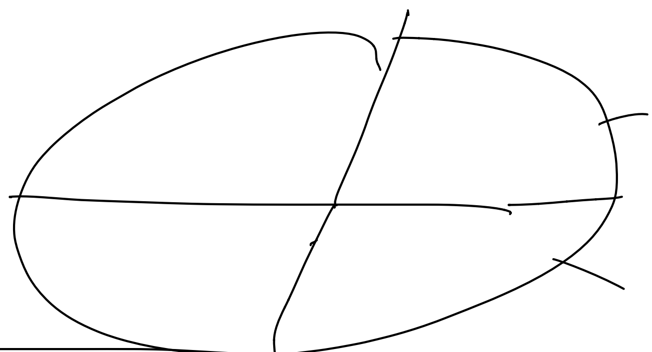
$$(A \otimes \mathbb{C}(K))_u$$

$$C(K) \hookrightarrow A_\infty \cap A' \quad ?$$

$$C(K) \hookrightarrow \ell_\infty / c_0$$

$$C(K) \cong \varinjlim \mathbb{C}^{2^n}$$

$$\mathbb{C}^{2^n} \hookrightarrow \ell_\infty / c_0$$



Consider $(A \otimes C(K), A, \pi_0, \phi)$

Fix $0 \in K$, $\pi_0 : (A \otimes C(K)) \to A$

$$A \otimes C(K) = C(K,A)$$

$$\pi_0(f) = f(0)$$

Let $\phi : A \to C(K, A)$

$$\phi(a) \longrightarrow f, \quad f(x) = a, \ \forall x \in K$$

$$\begin{tikzcd} C(K,A) \arrow[d, bend right, "\pi_0"'] \\ A \arrow[u, bend right, "\phi"'] \end{tikzcd} \qquad \pi_0 \circ \phi = id_A$$

$\underline{(A \otimes C(K), A, \pi_0, \phi)}$

Take an ultrapower.

$$\begin{tikzcd} C(K,A) \arrow[d, bend right, "\pi_0"'] \arrow[rr, phantom, "\xrightarrow{\ \mathcal{U}\ }"] & & C(K,A)_{\mathcal{U}} \cong A_\infty \arrow[d, bend right, "(\pi_0)_{\mathcal{U}}"'] \\ A \arrow[u, bend right, "\phi"'] & & A_{\mathcal{U}} \arrow[u, bend right, "\phi_{\mathcal{U}}"'] \end{tikzcd}$$

<u>Fact</u> $\boxed{(A_\infty, A_{\mathcal{U}}, \pi_{\mathcal{U}})}$ is countably saturated $\Leftarrow$ $\mathcal{U}$

is a P-Point. (i.e., if $X_n \in \mathcal{U}$, $n \in \mathbb{N}$, then $\exists X \in \mathcal{U}$, $X \setminus X_n$ is finite, for all n.)

$A_{\mathcal{U}}$ ctbly sat.

uses Łoś

$\ell_\infty(M)$

A_∞ ctbly sat.

Feferman-Vaught-Ghasemi

$C(K, M)$, $A_{\mathcal{U}}$

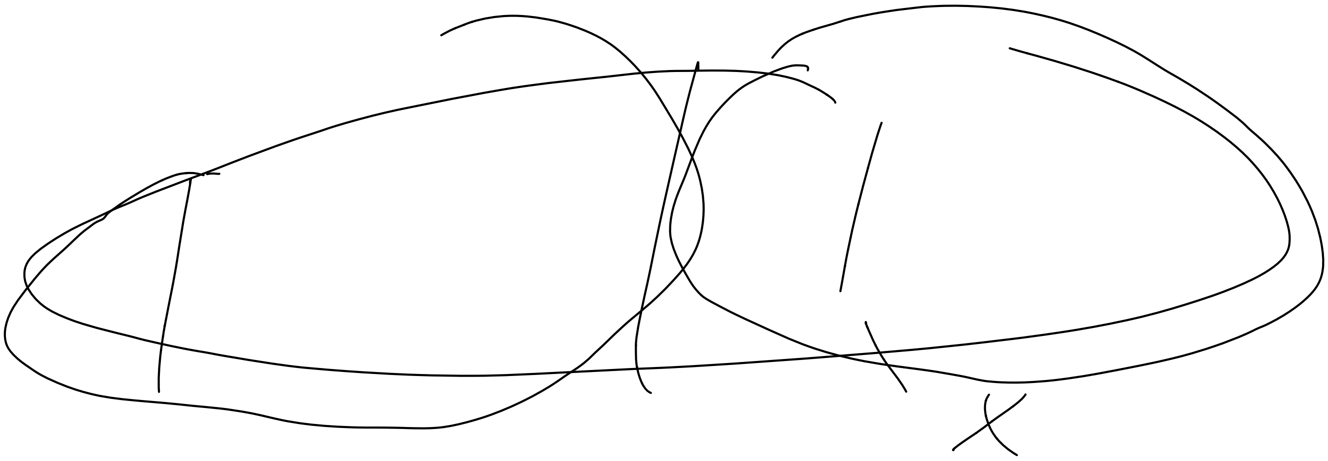
$(A_\infty, A_{\mathcal{U}}, \pi_{\mathcal{U}})$ ctbly sat.

Feferman-Vaught-Ghasemi

$t(\bar{x})$

$\ell_\infty(A)$ $\varphi_i(\bar{x})$, $i \in \mathbb{N}$

$\bar{a}$ $\varphi_i^A(\bar{a}) < \varepsilon$



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Thm *For every separable A there is an elementary embedding $A \otimes C(K) \prec A_\infty$ which commutes with the diagonal embedding of A. Therefore CH implies $A_\infty \cong (A \otimes C(K))_{\mathcal{U}}$.*

I. Farah, 'Between reduced powers and ultrapowers', arXiv:1904.11776.