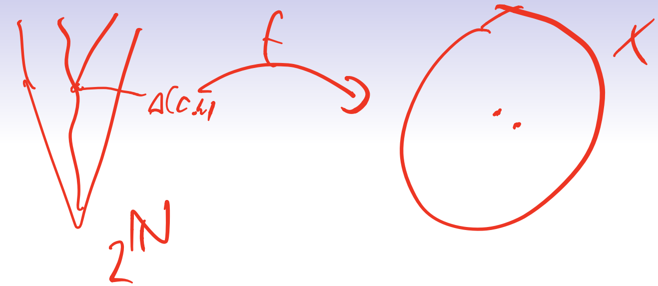
Massive C*-algebras, Winter 2021, I. Farah, Lecture 21

Today:

1. Continuing the proof that $\mathrm{OCA_T}$ implies Φ has a σ-narrow ε-approximation on $\mathrm{D_X[E]}$.
2. An example of an endomorphism of $\cancel{\ell_\infty/c_0}$ $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ with a σ-narrow lifting, but no continuous (Borel, C-measurable) lifting.



Our weapon of choice.

OCA$_\infty$ Whenever X is a separable metrizable space and $[X]^2 = L_0^n \sqcup L_1^n$, for $n \geq 0$, are open colourings such that $L_0^n \supseteq L_0^{n+1}$ for all n, one of the following alternatives applies.

0.1 There exists an uncountable $Z \subseteq \{0,1\}^{\mathbb{N}}$ and a continuous $f : Z \to X$ such that $\{f(a), f(b)\} \in L_0^{\Delta(a,b)}$ for all distinct a and b in Z.

0.2 There are $X_n \subseteq X$, for $n \in \mathbb{N}$, such that $X = \bigcup_n X_n$ and $[X_n]^2 \subseteq L_1^n$ for all n.

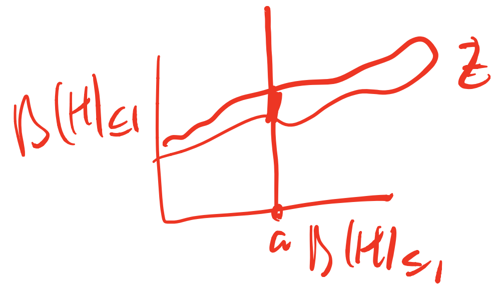
(Recall that OCA$_T$ implies OCA$_\infty$.)

Recall the following definitions (throughout §17.6–§17.7, $\mathcal{B}(H)_{\leq 1}$ is considered with respect to the WOT.)

Def 17.6.1 *A subset $\mathcal{Z}$ of $\mathcal{B}(H)^2_{\leq 1}$ is narrow if for all (a, b) and (a, c) in $\mathcal{Z}$ we have $b \approx^{\mathcal{K}} c$.*

$\|\pi(b-c)\| \leq \varepsilon$

It is ε-narrow if for all (a, b) and (a, c) in $\mathcal{Z}$ we have $b \approx^{\mathcal{K}}_{\varepsilon} c$.



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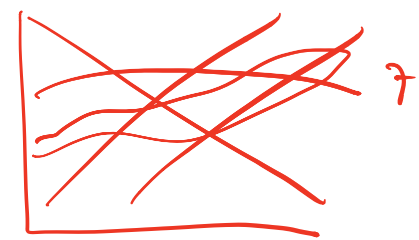
A function $f : \mathcal{B}(H)_{\leq 1} \to \mathcal{B}(H)_{\leq 1}$ is σ-narrow if its graph can be covered by a countable family of narrow Borel sets.
It is σ-ε-narrow if its graph can be covered by a countable family of ε-narrow Borel sets.



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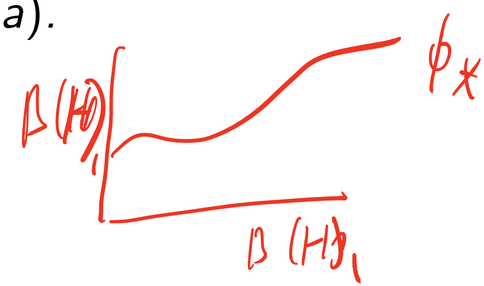
An endomorphism Φ of $\mathcal{Q}(H)$ has a σ-narrow lifting if its restriction to the unit ball has a lifting which is σ-narrow. It has a σ-narrow ε-approximation if there is a σ-ε-narrow function Θ such that every $a \in \mathcal{B}(H)_{\leq 1}$ satisfies $\Phi_*(a) \approx^{\mathcal{K}}_{\varepsilon} \Theta(a)$.



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A σ-narrow lifting on $\mathcal{D}$[E] or D[E] and a σ-narrow ε-approximation on $\mathcal{D}$[E] or D[E] are defined analogously.

I owe you an example. It is coming shortly, after the proof of Lemma 17.6.3.

We now complete the proof of the following, started last time.

$\forall E \in \mathrm{Part}_{\mathbb{N}}$

Lemma 17.6.3 *Assume* $\mathrm{OCA_T}$. *If* Φ *is an endomorphism of* $\mathcal{Q}(H)$ *and* $\varepsilon > 0$, *then* Φ *has a* σ*-narrow* ε*-approximation on* $\mathrm{D}_{\tilde{\mathrm{X}}}[\mathrm{E}]$ *for some infinite* $\tilde{\mathrm{X}}$.

$\Theta : \mathcal{B}(H)_{\leq 1} \to \mathcal{B}(H)_{\leq 1}$

I.e., there is a function Θ such that
(a) every $a \in \mathcal{B}(H)_{\leq 1}$ satisfies $\Phi_*(a) \approx^{\mathcal{K}}_{\varepsilon} \Theta(a)$ and
(b) the graph of Θ can be covered by a countable family of Borel sets $\mathcal{Z}_n$.

$\boxed{\varepsilon\text{-narrow}}$

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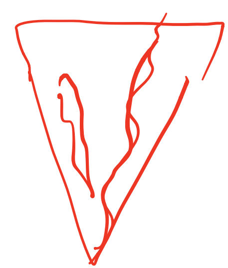
$2^{\mathbb{N}}$



A few conventions for the proof of Lemma 17.6.3:

1. We'll index the intervals in $\mathrm{E} \in \mathrm{Part}_{\mathbb{N}}$ by $\{0,1\}^{<\mathbb{N}}$: $\mathrm{E} = \langle E_s : s \in \{0,1\}^{<\mathbb{N}} \rangle$.
2. Fix $\mathrm{E} \in \mathrm{Part}_{\mathbb{N}}$ so that $\lim_n \min_{|s|=n} |E_s| = \infty$.
3. If $\mathrm{X} \subseteq \{0,1\}^{<\mathbb{N}}$ is infinite and included in a single branch of $\{0,1\}^{<\mathbb{N}}$, then this branch is denoted $\mathrm{B}(\mathrm{X})$.
4. Fix a discretization $\mathrm{D}[\mathrm{E}]$ of $\mathcal{D}[\mathrm{E}]$.

Proof of Lemma 17.6.3, that Φ has a σ-narrow ε-approximation on $\mathrm{D}_{\tilde{\mathrm{X}}}[\mathrm{E}]$ for some infinite $\tilde{\mathrm{X}}$.

Fix $d \geq (2\varepsilon)^{-1}$ ~~and $n \geq 1$~~. Let

$$\mathcal{X} := \{(\mathrm{X}, a) : \mathrm{B}(\mathrm{X}) \text{ is defined and } a \in \mathrm{D}_{\mathrm{X}}\}.$$

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In order to topologize $\mathcal{X}$, identify $(\mathrm{X}, a) \in \mathcal{X}$ with

$$(\underline{\mathrm{B}(\mathrm{X}), \mathrm{X}, a}, \underbrace{q_{\mathrm{X}}}_{=}, \underline{\Phi_*}(a)) \in \underline{\{0,1\}^{\mathbb{N}} \times \mathcal{P}(\{0,1\}^{<\mathbb{N}}) \times \mathrm{D} \times \mathcal{B}(H)^2_{\leq 1}}$$

$\phi_*(p_X)$

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Let $\{(\mathrm{X}, a), (\mathrm{Y}, b)\} \in M_0^{d,n}$ if the following conditions are satisfied:

($M_0^d 1$) $\mathrm{B}(\mathrm{X}) \neq \mathrm{B}(\mathrm{Y})$,

($M_0^d 2$) $p_{\mathrm{X}} b = p_{\mathrm{Y}} a$, and

($M_0^{d,n} 3$) $\|p_{[n,\infty)}(\Phi_*(a) q_{\mathrm{Y}} - q_{\mathrm{X}} \Phi_*(b))\| > 1/d$ or
$\|p_{[n,\infty)}(q_{\mathrm{Y}} \Phi_*(a) - \Phi_*(b) q_{\mathrm{X}})\| > 1/d$.

$\{(\mathsf{X},a),(\mathsf{Y},b)\}\in M_0^{d,n}$ iff $(M_0^d 1)$ $\mathsf{B}(\mathsf{X})\neq\mathsf{B}(\mathsf{Y})$, $(M_0^d 2)$ $p_{\mathsf{X}}b=p_{\mathsf{Y}}a$, and $(M_0^{d,n}3)$ $\|p_{[n,\infty)}(\Phi_*(a)q_{\mathsf{Y}}-q_{\mathsf{X}}\Phi_*(b))\|>\frac{1}{d}$ or $\|p_{[n,\infty)}(q_{\mathsf{Y}}\Phi_*(a)-\Phi_*(b)q_{\mathsf{X}})\|>\frac{1}{d}$.

From the last time:

Claim. *For every n, the partition $[\mathcal{X}]^2=M_0^{d,n}\cup M_1^{d,n}$ is open.*

Claim. *There is no uncountable $\mathsf{Z}\subseteq\{0,1\}^{\mathbb{N}}$ such that some continuous $f:\mathsf{Z}\to\mathcal{X}$ satisfies $\{f(a),f(b)\}\in M_0^{d,\Delta(a,b)}$ for all distinct a and b in Z.*

$\{(\mathrm{X},a),(\mathrm{Y},b)\}\in M_0^{d,n}$ iff (M_0^d1) $\mathrm{B}(\mathrm{X})\neq\mathrm{B}(\mathrm{Y})$, ($M_0^d2$) $p_{\mathrm{X}}b=p_{\mathrm{Y}}a$, and
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By OCA_∞, there are $M_1^{d,n}$-homogeneous sets $\mathcal{X}_n^d$, for $n\in\mathbb{N}$, such that $\mathcal{X}\subseteq\bigcup_n\mathcal{X}_n^d$.

$\{(\mathsf{X}, a), (\mathsf{Y}, b)\} \in M_1^{d,n}$ iff $(M_1^d 1)$ $\mathsf{B}(\mathsf{X}) = \mathsf{B}(\mathsf{Y})$ or $(M_1^d 2)$ $p_{\mathsf{X}} b \cancel{\leq} p_{\mathsf{Y}} a$, or $(M_1^{d,n} 3)$ $\|p_{[n,\infty)}(\Phi_*(a) q_{\mathsf{Y}} - q_{\mathsf{X}} \Phi_*(b))\| \leq \frac{1}{d}$ and $\|p_{[n,\infty)}(q_{\mathsf{Y}} \Phi_*(a) - \Phi_*(b) q_{\mathsf{X}})\| \leq \frac{1}{d}$.

Fix $M_1^{d,n}$-homogeneous sets $\mathcal{X}_n^d$, for $n \in \mathbb{N}$, such that $\mathcal{X} \subseteq \bigcup_n \mathcal{X}_n^d$.

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$$\Delta((\mathrm{X}, a), (\mathrm{Y}, b)) := \min\{k : (\exists s \in \{0,1\}^k)(s \in \mathrm{X} \Delta \mathrm{Y} \text{ or } (s \in \mathrm{X} \cap \mathrm{Y} \text{ and } a(s) \neq b(s)))\}.$$

E_s $\qquad$ $P_s\, B(H)\, P_s$

For $k \in \mathbb{N}$ let $e_k := p_{[0,k)}$.

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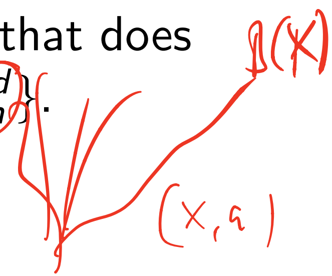
For $k \in \mathbb{N}$ let $e_k := p_{[0,k)}$.

For every n, fix a countable dense $\mathcal{E}_n^d \subseteq \mathcal{X}_n^d$. The closure of each $\mathcal{E}_n^d$ is $M_1^{d,n}$-homogeneous. Fix a branch $\hat{\mathsf{B}}$ of $\{0,1\}^{<\mathbb{N}}$ that does not belong to the countable set $\{\mathsf{B}(\mathsf{X}) : (\mathsf{X},a) \in \bigcup_n \mathcal{E}_n^d\}$.



Note: If $\hat{B} = B(X)$, $(X,a) \in \mathcal{X}_n^d$, $(Y,b) \in \mathcal{E}_n^d$, then $B(X) \neq B(Y)$

$\{(\mathrm{X}, a), (\mathrm{Y}, b)\} \in M_1^{d,n}$ iff $(M_1^d 1)$ $\mathrm{B}(\mathrm{X}) = \mathrm{B}(\mathrm{Y})$ or $(M_1^d 2)$ $p_{\mathrm{X}} b \not\leq p_{\mathrm{Y}} a$, or $(M_1^{d,n} 3)$ $\|p_{[n,\infty)}(\Phi_*(a) q_{\mathrm{Y}} - q_{\mathrm{X}} \Phi_*(b))\| \leq \frac{1}{d}$ and $\|p_{[n,\infty)}(q_{\mathrm{Y}} \Phi_*(a) - \Phi_*(b) q_{\mathrm{X}})\| \leq \frac{1}{d}$.

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$e_k := p_{[0,k)}$. $\quad \mathcal{E}_n^d \subseteq \mathcal{X}_n^d$ countable dense.

$\tilde{\mathrm{B}} \in \{0,1\}^{\mathbb{N}} \setminus \{\mathrm{B}(\mathrm{X}) : (\mathrm{X}, a) \in \bigcup_n \mathcal{E}_n^d\}$.

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Fix k

$\tilde{\mathsf{B}} \in \{0,1\}^{\mathbb{N}} \setminus \{\mathsf{B}(\mathsf{X}) : (\mathsf{X}, a) \in \bigcup_n \mathcal{E}_n^d\}$.

$\forall k$ Choose $\mathsf{F}_{k,n} \Subset \mathcal{E}_n^d$ so that for every $(\mathsf{X}, a) \in \mathcal{E}_n^d$ there is $(\mathsf{Y}, b) \in \mathsf{F}_{k,n}$ such that $\Delta((\mathsf{X}, a), (\mathsf{Y}, b)) > k$ and

$$\max(\|e_k(\Phi_*(p_{\mathsf{X}}) - \Phi_*(p_{\mathsf{Y}}))e_k\|, \|e_k(\Phi_*(a) - \Phi_*(b))e_k\|) < 1/k.$$

$\varepsilon_X \qquad \varepsilon_Y \qquad Y \qquad X$

The same holds $\forall (X,a) \in \mathcal{X}^d_n$.

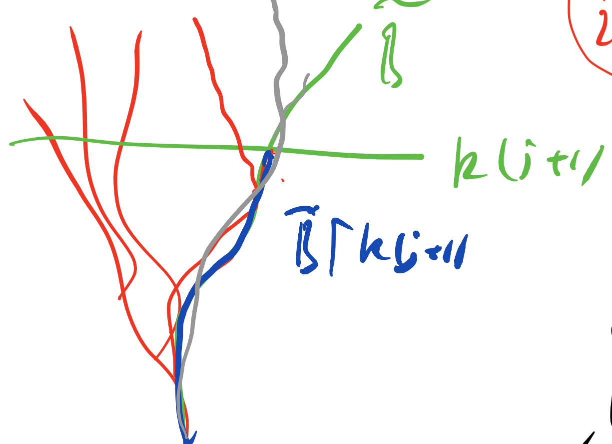
Note: $\bigcup_{k\in\mathbb{Z}} F_{k,n} = \mathcal{X}_n^d$, $\forall z \in \mathbb{N}_\infty$

Defin. $k(j)$, $j\in\mathbb{N}$:

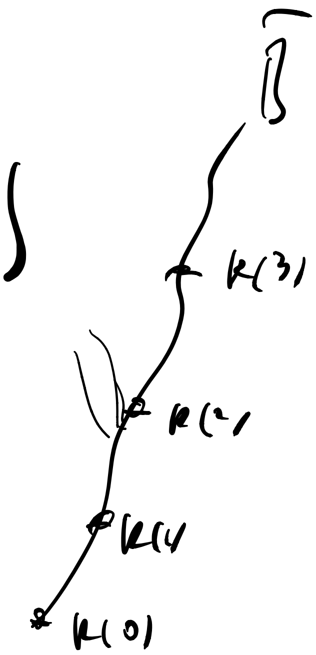
$k(0)=0$, $k(j+1)>k(j)$, and

$k(j+1)$ is the minimal such that

$\tilde{\beta}\restriction k(j+1) \neq \beta(y)\restriction k(j+1)$ $\forall (y,b)\in \bigcup_{j\le k(j)} F_{j,n}$ fix



Let $\hat{X} = \{\tilde{\beta}\restriction k(j+1) \mid j\in\mathbb{N}\}$



Fix $(y,b)\in\mathcal{X}_n^d$, and $k\in\mathbb{N}$,

$\phi_x(a)$?

$W(y,b,k) := \{(a,c)\in D_{\hat{X}} \times B(H)_{\le 1} \mid$

$(\hat{X}, a) \in \mathcal{X}^d_n$, $\boxed{\Delta((\hat{X}, a), (y, b)) > h}$

and $\max(\|e_k(\phi_*(b) - c) e_k\|,$

$\|e_k(\Sigma_{\hat{X}} - \Sigma_y) e_k\|) \leq \frac{1}{k} \}$

This set is closed.

Let $Z_n := \bigcap_{m} \bigcup_{j \geq m} \{ W(y, b, k(j)) : (y, b) \in F_{k(j), n} \}$

Borel

$Z_n \subseteq D_{\hat{X}} \times B(H)_{\leq 1}$.

<u>claim</u> $\forall n$ $(\hat{X}, a) \in \mathcal{X}^d_n$ then

$(a, \phi_*(a)) \in Z_n$.

<u>Pf</u> $(\Leftarrow)$ $\forall j$ $(a, \phi_*(a)) \in W(y, b, k(j))$

for some $(y, b) \in F_{k(j), n}$.

<u>claim</u> $(a, c) \in Z_n \Rightarrow \Sigma_{\hat{X}} c \approx^{\mathcal{K}}_{1/d} \phi_*(a)$.

Pf we'll prove

$$\Rightarrow \quad \| p_{[n,\infty)} \left(\phi_*(a) \Sigma_{\hat{X}} - \Sigma_{\hat{X}} c \right) \| \leq \frac{1}{d}$$

otherwise, fix $\delta > 0$ s. that

$$\| p_{[n,\infty)} \left(\phi_*(a) \Sigma_{\hat{X}} - \Sigma_{\hat{X}} c \right) \| > \frac{1}{d} + \delta$$

and $j \geq \max(n, 2/\delta)$ and

$(y, b) \in F_{k(j),n}$ s. that

$$(a, c) \in W(y, b, k(j)).$$

$W(y,b,k) := \{ (a,c) \in D_{\hat{X}} \times B(H)_{\leq 1} \mid$

$(\hat{X}, a) \in \mathcal{X}_n^d$, $\Delta((\hat{X}, a), (y,b)) < k$

and $\max(\| e_k(\phi_*(b) - c) e_k \|,$
$\| e_k(\Sigma_{\hat{X}} - \Sigma_y) e_k \|) \leq \frac{1}{k} \}$

Then $\max (\| p_{[n,\infty)} (\phi_*(b) - c) e_{k(j)} \|,$

$$\| p_{[n,\infty)} (\Sigma_{\hat{X}} - \Sigma_y) e_{k(j)} \|) \leq \frac{1}{k(j)} < \delta$$

$$\{ (\hat{X}, a), (y, b) \} \in M_1^{d,n}$$

hence $\| p_{[n,\infty)} (\phi_*(a) \Sigma_y - \Sigma_{\hat{X}} \phi_*(b)) \| \leq \frac{1}{d}$

(also symmetric ...)

write $x \approx_{\delta, k} y$ if $\| \rho x - y e_k \| < \delta$

$P_{[u,\infty)} \phi_x(0) \varepsilon_{\tilde{X}} \approx_{\delta, k(i)} P_{[u,\infty)} \phi_x(0) \varepsilon_y$

$\approx_{1/d} P_{[u,\infty)} \varepsilon_{\tilde{X}} \phi_x(u) \approx_{\delta, k(i)} P_{[u,\infty)} \varepsilon_{\tilde{X}} C$

So $\phi_x(0) \underset{1/d}{\overset{\not\approx}{\approx}} \varepsilon_{\tilde{X}} C$ $k(i)$

<u>Claim</u> $Z_u' = \{ (a, \varepsilon_{\tilde{X}} C) \mid (a, c) \in Z_u \}^{D_{\tilde{X}}}$

is $2/d$ - narrow.

$(a, \phi_x(0))$

Moreover, $\underline{\bigcup Z_u'} \supseteq \{ (a, \phi(a)) : a \in D_{\tilde{X}} \}$

$\forall a \in D_{\tilde{X}}$

$\exists u$ $\underline{(\tilde{X}, a)} \in \mathcal{X}_u$ D

$\{(\mathsf{X}, a), (\mathsf{Y}, b)\} \in M_1^{d,n}$ iff ($M_1^d 1$) $\mathsf{B}(\mathsf{X}) = \mathsf{B}(\mathsf{Y})$ or ($M_1^d 2$) $p_{\mathsf{X}} b \leq p_{\mathsf{Y}} a$, or ($M_1^{d,n} 3$) $\|p_{[n,\infty)}(\Phi_*(a) q_{\mathsf{Y}} - q_{\mathsf{X}} \Phi_*(b))\| \leq \frac{1}{d}$ and $\|p_{[n,\infty)}(q_{\mathsf{Y}} \Phi_*(a) - \Phi_*(b) q_{\mathsf{X}})\| \leq \frac{1}{d}$.

$$\Delta((\mathsf{X}, a), (\mathsf{Y}, b)) := \min\{k : (\exists s \in \{0,1\}^k)(s \in \mathsf{X} \Delta \mathsf{Y} \text{ or } (s \in \mathsf{X} \cap \mathsf{Y} \text{ and } a(s) \neq b(s)))\}.$$

$e_k := p_{[0,k)}$. $\mathcal{E}_n^d \subseteq \mathcal{X}_n^d$ countable dense.
$\tilde{\mathsf{B}} \in \{0,1\}^{\mathbb{N}} \setminus \{\mathsf{B}(\mathsf{X}) : (\mathsf{X}, a) \in \bigcup_n \mathcal{E}_n^d\}$.

Choose $\mathsf{F}_{k,n} \Subset \mathcal{E}_n^d$ so that for every $(\mathsf{X}, a) \in \mathcal{E}_n^d$ there is $(\mathsf{Y}, b) \in \mathsf{F}_{k,n}$ such that $\Delta((\mathsf{X}, a), (\mathsf{Y}, b)) > k$ and

$$\max(\|(\Phi_*(p_{\mathsf{X}}) - \Phi_*(p_{\mathsf{Y}}))e_k\|, \|(\Phi_*(a) - \Phi_*(b))e_k\|) < 1/k.$$

Example

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2-narrow
with a σ-narrow lifting, but no C-measurable (Borel, continuous,. . .) lifting.

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Proof: Fix a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Let $\Phi_*(A) := \mathbb{N}$ if $A \in \mathcal{U}$ and $\Phi_*(A) := \emptyset$ if $A \notin \mathcal{U}$.

Then Φ has a lifting whose graph is covered by two constant functions.

ϕ_*

To prove that Φ does not have a continuous lifting, one uses the fact that for every nonempty basic open subset V on $\mathcal{P}(\mathbb{N})$, both $V \cap \mathcal{U}$ and $V \setminus \mathcal{U}$ are nonmeager.

$\phi_* : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$

$\{\emptyset, \mathbb{N}\}$

ϕ_* is a homo, hom $(\phi / 2\,\mathrm{Fin}$.

$\phi : \mathcal{P}(\mathbb{N})/_{Fin} \to \mathcal{P}(\mathbb{N})/_{Fin}$

$A \to \mathbb{N} \setminus A$ homeo,

$\mathcal{U} \to \mathcal{P}(\mathbb{N}) \setminus \mathcal{U}$

$\cancel{\mathcal{U}}$ $\quad Fin \cong \bigoplus_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$

$Fin \curvearrowright \mathcal{P}(\mathbb{N})$

$B . A = B \Delta A$

$\mathcal{U}$ is Fin-invariant

$\Rightarrow \forall n \ \forall s, t \in \{0,1\}^n$

$\mathcal{U} \cap [s]$ is n.m.

$\Leftrightarrow \mathcal{U} \cap [t]$ is n.m.

Example

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Question *Is there an endomorphism of ℓ_∞/c_0 with a σ-narrow lifting but no continuous (Borel, C-measurable) lifting?*

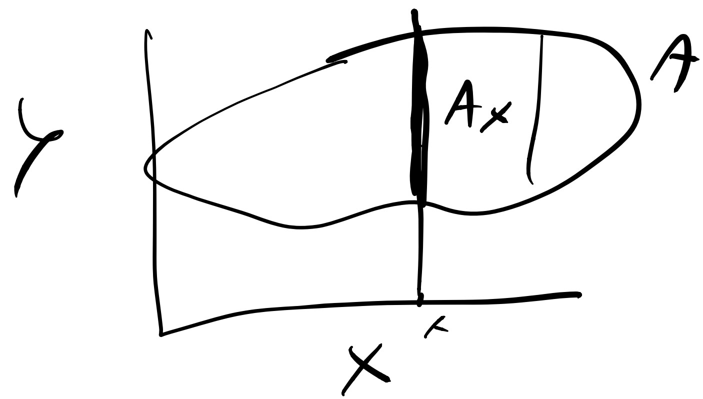
Note: ϕ ω^* gives

$\psi : \ell_\infty \to \ell_\infty$, $\psi(f) = \lim_{n \to \mathcal{U}} f(n)$

We'll need another result from the classical descriptive set theory.

Thm B.2.14 (Novikov) *If* X *and* Y *are Polish spaces and* $A \subseteq X \times Y$ *is analytic, then the set* $\{x \in X : A_x$ *is nonmeager*$\}$ *is analytic.*

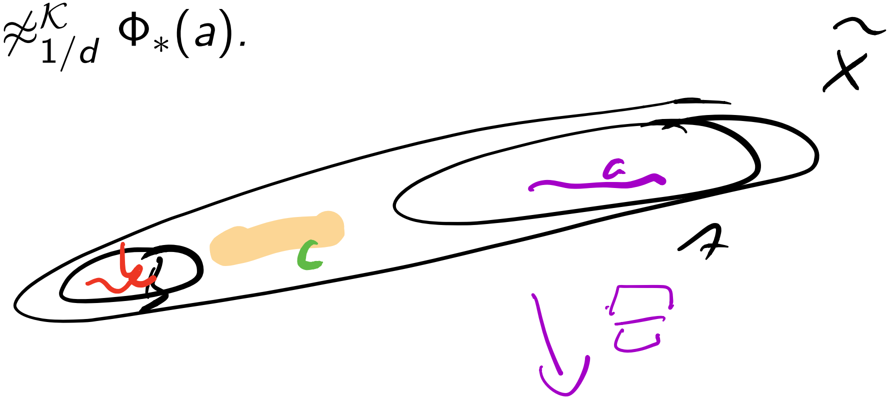
$$A_x = \{ y \mid (x, y) \in A \}$$



Lemma 17.7.1 *Suppose Φ is an endomorphism of $\mathcal{Q}(H)$, $d \geq 1$, $\mathsf{E} \in \mathrm{Part}_{\mathbb{N}}$, and there exists a $1/d$-narrow analytic set $\mathcal{Z} \subseteq \mathsf{D}_{\tilde{\mathsf{X}}} \times \mathcal{B}(H)_{\leq 1}$. Then for every $\mathsf{A} \subseteq \tilde{\mathsf{X}}$ such that both A and $\tilde{\mathsf{X}} \setminus \mathsf{A}$ are infinite at least one of the following applies.*

1. *There is a C-measurable $3/d$-approximation of Φ on D_{A}.*
2. *There are $\mathsf{B} \subseteq \tilde{\mathsf{X}} \setminus \mathsf{A}$, $a \in \mathsf{D}_{\mathsf{A}}$, and $b \in \mathsf{D}_{\mathsf{B}}$ such that both B and $\tilde{\mathsf{X}} \setminus (\mathsf{A} \cup \mathsf{B})$ are infinite and every uniformization Ξ of $\mathcal{Z}$ and $c \in \mathsf{D}_{\tilde{\mathsf{X}} \setminus (\mathsf{A} \cup \mathsf{B})}$ such that $a + b + c \in \mathrm{dom}(\Xi)$ satisfy $\Xi(a+b+c)q_{\mathsf{A}} \not\approx^{\mathcal{K}}_{1/d} \Phi_*(a)$.*

$$\underset{\phi_*(a)}{\overset{\Xi(a+b+c)}{\shortparallel}}$$



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$\mathcal{X}$

Proof: Let

$$\mathcal{V} := \{(a, b, c) \in \mathsf{D}_{\mathsf{A}} \times \mathsf{D}_{\tilde{\mathsf{X}} \setminus \mathsf{A}} \times \mathcal{B}(H)_{\leq 1} : (\exists c' \in \mathcal{B}(H)_{\leq 1})(a + b, c') \in \mathcal{Z}, c \approx^{\mathcal{K}}_{1/d} c' q_{\mathsf{A}}\}.$$

$$\mathcal{W}(a) := \{b \in \mathsf{D}_{\tilde{\mathsf{X}} \setminus \mathsf{A}} : (a, b, \Phi_*(a)) \in \mathcal{V}\}, \text{ for } a \in \mathsf{D}_{\mathsf{A}}.$$