Today: Ulam-stability... but first, a shorter—and much more reasonable—proof of Lemma 17.4.8.

Stabilizers done right

$\Theta : D \to \mathcal{B}(H)_{\leq 1}$

Recall $\Phi \in \mathrm{Aut}(\mathcal{Q}(H))$.

Lemma 17.4.8 *If* Φ *has a strongly continuous lifting* Θ *on* $\mathsf{D}[\mathsf{E}]$ *for some* $\mathsf{E} \in \mathsf{Part}_{\mathbb{N}}$*, then it has a lifting of product type on* $\mathsf{D}_{\mathsf{X}}[\mathsf{E}]$ *for some infinite* $\mathsf{X} \subseteq \mathbb{N}$*.*

To do this right, we'll need two useful lemmas.

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Lemma 17.4.8 *If Φ has a strongly continuous lifting Θ on $\mathrm{D}[\mathrm{E}]$ for some $\mathrm{E} \in \mathrm{Part}_{\mathbb{N}}$, then it has a lifting of product type on $\mathrm{D}_{\mathrm{X}}[\mathrm{E}]$ for some infinite $\mathrm{X} \subseteq \mathbb{N}$.*

To do this right, we'll need two useful lemmas.

Lemma A *If r is a projection, then for every a we have*

$$\|[a, r]\| = \|a - rar - (1 - r)a(1 - r)\|.$$

Lemma B *Suppose $a \in \mathcal{B}(H)$ and r_j, for $j \in \mathbb{N}$, is an increasing sequence of finite rank projections such that $r_j \to 1_{\mathcal{B}(H)}$ (in SOT) and $\sum_j \|[a, r_j]\| < \infty$. Then*

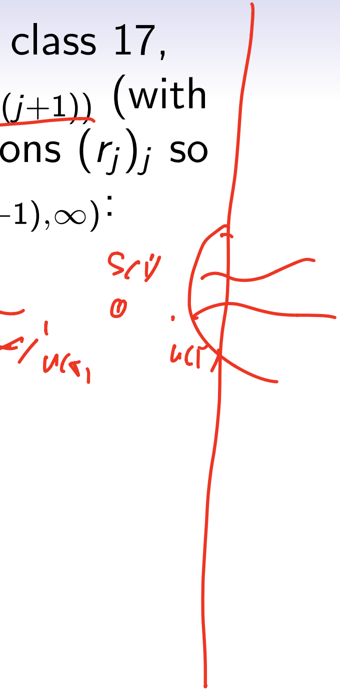
$$a - \sum_j (r_{j+1} - r_j)a(r_{j+1} - r_j)$$

is compact.

A good proof of Lemma 17.4.8: As in the last minutes of class 17, recursively find an increasing sequence $(n(j))_j$, $s(j) \in \mathrm{D}_{(n(j),n(j+1))}$ (with $n(0) := 0$), and an increasing sequence of finite-rank projections $(r_j)_j$ so that for all j, all a and b in $\mathrm{D}_{[0,n(j)]}$, and all c and d in $\mathrm{D}_{[n(j+1),\infty)}$:

1. $\|(\Theta(a+s(j)+c)-\Theta(b+s(j)+c))(1-r_j)\| \leq 2^{-j}$,
2. $\|(1-r_j)(\Theta(a+s(j)+c)-\Theta(b+s(j)+c))\| \leq 2^{-j}$,
3. $\|(\Theta(a+s(j)+c)-\Theta(a+s(j)+d))r_j\| \leq 2^{-j}$,
4. $\|r_j(\Theta(a+s(j)+c)-\Theta(a+s(j)+d))\| \leq 2^{-j}$.

Let $\mathrm{X} := \{n(j) | j \in \mathbb{N}\}$ and $s := \sum_j s(j)$.



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Let $\mathrm{X} := \{n(j) | j \in \mathbb{N}\}$ and $s := \sum_j s(j)$.
For every $x \in \mathrm{D}_\mathrm{X}$ and every j we have

$$\|[\Theta(x+s)-\Theta(x), r_j]\| \leq 2^{-j+2}.$$

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For $j \in \mathbb{N}$, define $\Xi_j(a)$ for $a \in \mathrm{D}_{\{n(j)\}}$ by

$$\Xi_j(a) := (r_{j+1}-r_j)(\Theta(x+s)-\Theta(x))(r_{j+1}-r_j).$$

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$$\Xi_j(a) := (r_{j+1}-r_j)(\Theta(x+s)-\Theta(\cancel{x}\,s))(r_{j+1}-r_j).$$

By Lemma B, the product type function Ξ determined by (Ξ_j) satisfies

$$\Xi(x) \approx^{\mathcal{K}(H)} \Theta(x+s)-\Theta(s) \approx^{\mathcal{K}(H)} \Phi_*(x),$$

completing the proof.

Ulam-stability of approximate *-homorphisms

The following definition and theorem are used in order to set the stage.

Def 17.2.1 *An ε-representation of a group G in a unital C^*-algebra A is a function $\Theta\colon G \to \mathrm{U}(A)$ such that $\sup_{x,y\in G} \|\Theta(xy) - \Theta(x)\Theta(y)\| \leq \varepsilon$ and $\Theta(1) = 1$.*

$\mathcal{B}(H)$

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Thm 17.2.2 (Kazhdan, Grove–Karcher–Roh, Alekseev–Glebsky–Gordon) *Assume G is a compact* amenable *group and A is a von Neumann algebra. If $\varepsilon < 1/10$ then for every Borel-measurable ε-representation $\Theta\colon G \to \mathrm{U}(A)$ there exists a unitary representation $\Lambda\colon G \to \mathrm{U}(A)$ such that $\|\Lambda - \Theta\| \leq 2\varepsilon$.* $\square$

(A proof of Thm 17.2.2 can be extracted from the proof of the following theorem.)

Thm 17.2.3 (Burger–Ozawa–Thom) *Assume A and B are* unital *C^*-algebras, A is finite-dimensional, $\varepsilon < 1/28$, and $\Theta\colon A_1 \to B_2$ is a uniformly bounded, Borel-measurable function that satisfies $\Theta[\mathrm{U}(A)] \subseteq \mathrm{U}(B)$ and*

$$\|\Theta(ga) - \Theta(g)\Theta(a)\| \leq \varepsilon$$

for all $g \in \mathrm{U}(A)$ and all $a \in A_1$, and $\Theta(1) = 1$. Then there exists a uniformly bounded, Borel-measurable function $\Lambda\colon A_1 \to B_2$ which satisfies $\|\Lambda - \Theta\| \leq 4\varepsilon$ and

$$\Lambda(ga) - \Lambda(g)\Lambda(a) = 0$$

for all $g \in \mathrm{U}(A)$ and all $a \in A_1$.

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$$\Lambda(ga) - \Lambda(g)\Lambda(a) = 0$$

for all $g \in \mathrm{U}(A)$ and all $a \in A_1$.

Proof: Let μ denote the ~~(right)~~ Haar measure on $\mathrm{U}(A)$ and let (Bochner integral)

$x \in U(A)$

$$\Theta'(a) := \int \Theta(x)^*\Theta(xa)\,d\mu(x).$$

Thm 17.2.3: Assuming $\Theta[\mathrm{U}(A)] \subseteq \mathrm{U}(B)$ *and*

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for all $g \in \mathrm{U}(A)$ *and all* $a \in A_1$, *and* $\Theta(1) = 1$. *Let* g *and* x *range over* $\mathrm{U}(A)$ *and* $a \in A_1$. *Define*

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$$\Theta'(a) := \int \underline{\Theta(x)^*\Theta(xa)}\, d\mu(x).$$

$x \to xg$

Then

$$\int \Theta(xg^{-1})^*\Theta(xa)\,d\mu(x) = \Theta'(ga).$$

$= \int \theta(x)^* \theta(xga)\, d\mu(x)$

hence $\boxed{\Theta'(g^{-1}) = \Theta'(g)^*.}$

$a \to 1$

$g \to g^{-1}$

RHS $\theta'(ga) = \theta'(g^{-1})$

LHS: $\int \underline{\theta(xg)^* \theta(x)}\, d\mu(x)$

$*$

$= \Theta(g)$

Thm 17.2.3: Assuming $\Theta[\mathrm{U}(A)] \subseteq \mathrm{U}(B)$ *and*

$$\|\Theta(ga) - \Theta(g)\Theta(a)\| \leq \varepsilon$$

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$$\Theta'(a) := \int \Theta(x)^*\Theta(xa)d\mu(x).$$

Then

$$\int \Theta(xg^{-1})^*\Theta(xa)d\mu(x) = \Theta'(ga).$$

hence $\Theta'(g^{-1}) = \Theta'(g)^*$. Consider

$$\mathcal{I} := \int (\underbrace{\Theta(xg^{-1}) - \Theta(x)\Theta(g^{-1})}_{\|\cdot\| \leq \varepsilon})^*(\underbrace{\Theta(xa) - \Theta(x)\Theta(a)}_{\|\cdot\| \leq \varepsilon})d\mu(x)$$

$\|I\| \leq \varepsilon^2$

$$I = \int \theta(xg^{-1})^* \theta(xa)\, d\mu(x) - \int \theta(xg^{-1})^* \theta(x)\theta(a)\, d\mu(x)$$

$$- \int \theta(g^{-1})^* \theta(x)^* \theta(xa)\, d\mu(x) + \int \theta(g^{-1})^* \theta(x)^* \theta(x) \theta(a)\, d\mu(x)$$

$$= \theta'(ga) - \theta'(g)\theta(a) - \theta(g^{-1})^* \theta'(a) + \theta(g^{-1})^*\theta(a)$$

$$= \theta'(ga) - \theta'(g)\theta'(a) + \theta'(g)(\theta'(a) - \theta(a)) - \theta(g^{-1})^*(\theta'(a) - \theta(a))$$

$$= \boxed{\theta'(ga) - \theta'(g)\theta'(a)} + (\theta'(g) - \theta(g^{-1})^*) \cdot (\theta'(a) - \theta(a))$$

$\leq \varepsilon^2$

Claim $\|\theta - \theta'\| \leq \underline{\varepsilon}$

$\underline{\text{If}}$ $\|\theta(x)^* \theta(xa) - \theta(a)\|$

$= \|\theta(xa) - \theta(x)\theta(a)\| \leq \varepsilon.$

Therefore, $\|\theta'(xa) - \theta'(x)\theta(a)\| \leq \underline{2\varepsilon^2}$

$\forall x \in U(A), \ \forall a \in A.$

θ θ'.

Q: $\theta'[U(A)] \subseteq U(B)$?

Not quite, but almost.

Fix $g \in U(A)$.

$\|\theta'(ga) - \theta'(g)\theta'(a)\| \leq 2\varepsilon^2$

$a \to g^{-1}$

$\theta'(g)^*$

$\|1 - \theta'(g)\theta'(g^{-1})\| \leq 2\varepsilon^2$

$\|1 - \theta'(g)\theta'(g)^*\| \leq 2\varepsilon^2$

$1 - 2\varepsilon^2 \leq \theta'(g)\theta'(g)^* \leq 1$

$\|1 - |\theta'(g)|\| \leq 2\varepsilon^2$ $\qquad 2\varepsilon^2 < 1$

Let $\begin{cases} \theta''(g) = \theta'(g) \cdot |\theta'(g)|^{-1} \\ \theta''(a) = \theta'(a), \quad a \notin U(A). \end{cases}$

Then $\|\theta''(ga) - \theta''(g)\theta''(a)\| \leq 14\varepsilon^2$

$\|\theta'' - \theta\| \leq 2\varepsilon$

Define $\theta^{(1)}$, using the same trick

$$\|\theta^{(1)} - \theta\| \leq 2\cdot(2\varepsilon^2)$$

$$\|\theta^{(1)}(ga) - \theta^{(1)}(g)\theta^{(1)}(a)\| \leq 2(2\varepsilon^2)^2 = 8\varepsilon^4$$

etc., find $\theta^{(n)}$

$$\lim_{n\to\infty} \theta^{(n)} \to \Lambda$$, as desired.

Given unital operator algebras A and B, when can a group homomorphism from $\mathcal{U}(A)$ into $\mathcal{U}(B)$ be extended to a $*$-homomorphism from A into B?

Doy

Exercise 1.11.16: Every $a \in A$ unital can be written as a linear combination of four unitaries.

Fact: $0 \le a \le 1$ then $a = \frac{1}{2}(u + v)$,

for unitaries u, v.

$u = a + i(1-a^2)^{1/2}$

$v = a - i(1-a^2)^{1/2}$

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Exercise 1.11.16: Every $a \in A$ can be written as a linear combination of four unitaries

Lemma 17.2.4 *Suppose A and B are unital C^*-algebras and $\Lambda\colon \mathrm{U}(A) \to \mathrm{U}(B)$ is a group homomorphism. If A has a faithful tracial state τ, B has a faithful tracial state σ, and $\sigma(\Lambda(u)) = \tau(u)$, then Λ has a unique extension to a *-homomorphism.*

$\tau(a^*a) = 0 \iff a = 0$

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Proof: We need to prove that the obvious map

$$\Phi\Big(\sum_{j<n} \lambda_j u_j\Big) := \sum_{j<n} \lambda_j \Lambda(u_j)$$

is well-defined.

This means:

$$\sum_{j<n} \lambda_j u_j = 0 \Rightarrow \sum_{j<n} \lambda_j \ldots$$

$$\Updownarrow$$

$$\underbrace{\Big(\sum_{j<n} \lambda_j u_j\Big)^* \Big(\sum_{j<n} \lambda_j u_j\Big)} = 0$$

$$\Updownarrow$$

$$\tau\Big(\qquad \Big) = 0$$

$$\Leftrightarrow \sum_{j<n} \sum_{i<n} \bar{\lambda}_j \lambda_i \tau(u_j^* u_i) = 0$$

$$\Leftrightarrow \sum_{j<n} \sum_{i<n} \bar{\lambda}_j \lambda_i \delta(\Lambda(u_j^* u_i)) = 0$$

$$\vdots$$

$$\Leftrightarrow \sum_{j<n} \lambda_j \Lambda(u_j) = 0.$$

We will need *Stone's one-parameter group theorem*:
If $\mathbb{R} \to \mathrm{U}(B) \colon t \mapsto u_t$ is a norm-continuous group homomorphism, then there exists a self-adjoint $b \in B$ such that $u_t = \exp(itb)$ for all t.

Recall:

Def 17.2.5 *Given $\varepsilon > 0$ and C^*-algebras A and B, some $\Theta\colon A_1 \to B_1$ is an ε-*-homomorphism if for all x, y in A_1 and $\lambda \in \mathbb{C}$, $|\lambda| \leq 1$, each one of $\Theta(x^*) - \Theta(x)^*$, $\Theta(x + y) - \Theta(x) - \Theta(y)$, $\Theta(xy) - \Theta(x)\Theta(y)$, and $\Theta(\lambda x) - \lambda\Theta(x)$ has norm not greater than ε. It is unital if in addition $\Theta[\mathrm{U}(A)] \subseteq \mathrm{U}(B)$ and $\Theta(1) = 1$.*

Thm 17.2.6 *Suppose* $\varepsilon < 1/28$, $m \geq 1$, A *is a* unital C^**-algebra with a faithful tracial state* σ*, and* $\Theta\colon M_m(\mathbb{C}) \to A$ *is a unital* ε-**-homomorphism. Then there exists a* **-homomorphism* $\Phi\colon M_m(\mathbb{C}) \to A$ *such that* $\underline{\|\Theta - \Phi\|} \leq \underline{16\varepsilon}$.

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Proof: By Theorem 17.2.3, there is a Borel-measurable $\Lambda \colon \mathrm{U}(M_m(\mathbb{C})) \to A$ such that $\Lambda(uv) = \Lambda(u)\Lambda(v)$ for all u and v, and $\|\Lambda - \Theta\| \leq 4\varepsilon$.

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Using Stone, there is $\tilde{\Lambda}\colon A_{\mathrm{sa}} \to B_{\mathrm{sa}}$ such that

$$\Lambda(\exp(ira)) = \exp(ir\tilde{\Lambda}(a))$$

$a \in A_{sa}$

for all $r \in \mathbb{R}$.

Pettis $\Rightarrow$ Λ is ctu.

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for all $r \in \mathbb{R}$.

1. $\tilde{\Lambda}(1) = 1$.
2. If p is a projection, then $\tilde{\Lambda}(p)$ is a projection.

$\exp(ir)$ $\qquad\qquad\qquad b = b^*$

(1) Let $\underline{b := \tilde{\Lambda}(1)}$. Suppose $\exists r \in Sp(b) \setminus \{1\}$.

Fix $t \in \mathbb{R}$, $\boxed{t(r-1) = \pi}$ $r \in \mathbb{R}$

$Sp(\exp(itb)) \ni -1$ $|e^{itr} - e^{it}| = 2$

$\wr \delta'$

$\Lambda(it)$

$\wr \delta$

$\underline{\underline{it}}$ $\theta(it)$ ↖ scalar.

~~$\theta(it) \longrightarrow \theta$~~

Theorem 17.2.6: $\Theta \colon M_m(\mathbb{C}) \to A$ *unital* ε*-***-homo. So far, we have:*
(i) A homomorphism $\Lambda \colon \mathrm{U}(M_m(\mathbb{C})) \to \mathrm{U}(A)$ *such that* $\|\Lambda - \Theta\| \leq 4\varepsilon$.
(ii) $\tilde{\Lambda} \colon A_{\mathrm{sa}} \to B_{\mathrm{sa}}$ *such that* $\Lambda(\exp(ira)) = \exp(ir\tilde{\Lambda}(a))$ *for* $r \in \mathbb{R}$.

Claim. *Suppose p and q are projections.*

1. *We have* $\tilde{\Lambda}(p) = \frac{1}{2}(1 - \Lambda(u_p))$.

$u_p = 1 - 2p$
$= (1-p) \cdot e^{0} + p \cdot e^{i\pi}$

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Claim. *Suppose p and q are projections.*

1. *We have* $\tilde{\Lambda}(p) = \frac{1}{2}(1 - \Lambda(u_p))$.
2. *If p and q are Murray–von Neumann equivalent,* then

$\underline{(\exists v \quad v^*v = p, \quad vv^* = q)} \Leftarrow \exists \underline{u} \in \mathcal{U}(M_m(\mathbb{C}))$

$\tilde{\Lambda}(p) \sim_{MvN} \tilde{\Lambda}(q)$ $\qquad \underline{upu^* = q}$

$\Lambda(1-2q)$

$\Lambda(u(1-2p)u^*)$

$= \Lambda(u)\Lambda(1-2p)\,\Lambda(u^*)$

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3. *If p and q commute, then so do* $\tilde{\Lambda}(p)$ *and* $\tilde{\Lambda}(q)$.

$\Updownarrow$

$p+q-pq$ is self-adjoint

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3. *If p and q commute, then so do* $\tilde{\Lambda}(p)$ *and* $\tilde{\Lambda}(q)$.
4. *If* $pq = 0$, *then* $\tilde{\Lambda}(p)\tilde{\Lambda}(q) = 0$ *and* $\tilde{\Lambda}(p+q) = \tilde{\Lambda}(p) + \tilde{\Lambda}(q)$.

$1-2p \quad , 1-2q$

$p+q$ is a proj. $\quad 1-2(p+q)$

Theorem 17.2.6: $\Theta\colon M_m(\mathbb{C}) \to A$ *unital* ε*-***-homo. So far, we have:*
(i) A homomorphism $\Lambda\colon \mathrm{U}(M_m(\mathbb{C})) \to \mathrm{U}(A)$ *such that* $\|\Lambda - \Theta\| \leq 4\varepsilon$.
(ii) $\tilde{\Lambda}\colon A_{\mathrm{sa}} \to B_{\mathrm{sa}}$ *such that* $\Lambda(\exp(ira)) = \exp(ir\tilde{\Lambda}(a))$ *for* $r \in \mathbb{R}$.

Claim. *Suppose p and q are projections.*

1. *We have* $\tilde{\Lambda}(p) = \frac{1}{2}(1 - \Lambda(u_p))$.
2. *If p and q are Murray–von Neumann equivalent,then so are* $\tilde{\Lambda}(p)$ *and* $\tilde{\Lambda}(q)$.
3. *If p and q commute, then so do* $\tilde{\Lambda}(p)$ *and* $\tilde{\Lambda}(q)$.
4. *If* $pq = 0$, *then* $\tilde{\Lambda}(p)\tilde{\Lambda}(q) = 0$ *and* $\tilde{\Lambda}(p+q) = \tilde{\Lambda}(p) + \tilde{\Lambda}(q)$.
5. *If* $\sum_{j<m} p_j = 1$ *for projections* p_j, *for* $j < m$, *then* $\sum_{j<m} \tilde{\Lambda}(p_j) = 1$.

$M_m(\mathbb{C})$

Recall that A has ~~a~~ faithful tracial state σ. By Lemma 17.2.4, suffices to prove $\tau(u) = \sigma(\Lambda(u))$ for every $u \in \mathrm{U}(M_m(\mathbb{C}))$. By the Spectral Theorem,

$$u = \sum_{j<m} \exp(i\lambda_j)p_j = \boxed{\prod_{j<m} \exp(i\lambda_j p_j)},$$

and $p_i \sim_{MvN} p_j$ for all i, j.

$\tau(u) = \sigma(\Lambda(u))$

$= \sum_j i\lambda_j$

$\sum_j \hat{\Lambda}(p_j) = 1$

$\Lambda(u) = \prod_{j<m} \Lambda(\exp(i\lambda_j p_j))$