. . . still proving that $\mathrm{OCA}_{\mathrm{T}}$ implies all automorphisms of $\mathcal{Q}(H)$ are inner.

$\mathrm{D}(n) \subseteq (M_{m(n)}(\mathbb{C}))_1$ finite, 2^{-n}-dense

$\prod_n \mathrm{D}(n)$, $=$

From the last time:

Def 17.4.6 *A function* $\Xi \colon \mathrm{D} \to \mathcal{B}(H)_{\leq 1}$ *is of a product type if there are orthogonal projections* $r_n \in \mathcal{B}(H)$ *and* $\Xi_n \colon \mathrm{D}(n) \to r_n(\mathcal{B}(H)_{\leq 1})r_n$ *for* $n \in \mathbb{N}$ *such that (with the SOT-convergent series)* $\Xi(a) = \sum_n \Xi_n(a_n)$ *for all* $a \in \mathrm{D}$.

$a = (a_n)$, $a_n \in \mathrm{D}(n)$

Lemma 17.4.8 *If* Φ *has a continuous lifting* Θ *on* $D[E]$ *for some* $E \in \mathrm{Part}_{\mathbb{N}}$*, then it has a lifting of product type on* $D_X[E]$ *for some infinite* $X \subseteq \mathbb{N}$.

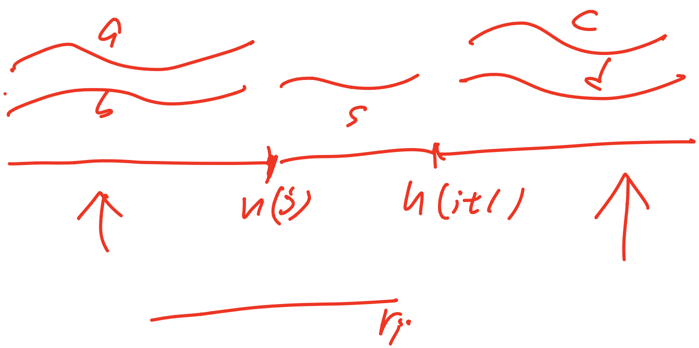
$D_X[E] = \prod_{n \in X} D(n)$

Lemma 17.4.8 *If Φ has a continuous lifting Θ on* $\mathrm{D}[\mathrm{E}]$ *for some* $\mathrm{E} \in \mathrm{Part}_{\mathbb{N}}$, *then it has a lifting of product type on* $\mathrm{D}_{\mathrm{X}}[\mathrm{E}]$ *for some infinite* $\mathrm{X} \subseteq \mathbb{N}$.

Proof: Find an increasing sequence $(n(j))_j$, $s(j) \in \mathrm{D}_{(n(j),n(j+1))}$ (with $n(0) := 0$), and an increasing sequence of finite-rank projections $(r_j)_j$ so that for all j, all a and b in $\mathrm{D}_{[0,n(j)]}$, and all c and d in $\mathrm{D}_{[n(j+1),\infty)}$:

s(j) is a stabilizer

1. $\|(\Theta(a+s(j)+c) - \Theta(b+s(j)+c))(1-r_j)\| \leq 2^{-j}$,
2. $\|(1-r_j)(\Theta(a+s(j)+c) - \Theta(b+s(j)+c))\| \leq 2^{-j}$,
3. $\|(\Theta(a+s(j)+c) - \Theta(a+s(j)+d))r_j\| \leq 2^{-j}$,
4. $\|r_j(\Theta(a+s(j)+c) - \Theta(a+s(j)+d))\| \leq 2^{-j}$.



Lemma 17.4.8 *If Φ has a continuous lifting Θ on $\mathrm{D}[\mathrm{E}]$ for some $\mathrm{E} \in \mathrm{Part}_{\mathbb{N}}$, then it has a lifting of product type on $\mathrm{D}_{\mathrm{X}}[\mathrm{E}]$ for some infinite $\mathrm{X} \subseteq \mathbb{N}$.*

Proof: Find an increasing sequence $(n(j))_j$, $\boxed{s(j) \in \mathrm{D}_{(n(j),n(j+1))}}$ (with $n(0) := 0$), and an increasing sequence of finite-rank projections $(r_j)_j$ so that for all j, all a and b in $\mathrm{D}_{[0,n(j)]}$, and all c and d in $\mathrm{D}_{[n(j+1),\infty)}$:

1. $\|(\Theta(a+s(j)+c) - \Theta(b+s(j)+c))(1-r_j)\| \leq 2^{-j}$,
2. $\|(1-r_j)(\Theta(a+s(j)+c) - \Theta(b+s(j)+c))\| \leq 2^{-j}$,
3. $\|(\Theta(a+s(j)+c) - \Theta(a+s(j)+d))r_j\| \leq 2^{-j}$,
4. $\|r_j(\Theta(a+s(j)+c) - \Theta(a+s(j)+d))\| \leq 2^{-j}$.

Let $\mathrm{X} := \{n(j) : j \in \mathbb{N}\}$ and $s := \sum_j s(j)$ (an element of $\mathrm{D}_{\mathbb{N}\setminus\mathrm{X}}$).
For each j define $\tilde{\Xi}_j : \mathrm{D}_{n(j)} \to (r_{j+1} - r_j)\mathcal{B}(H)_{\leq 1}(r_{j+1} - r_j)$ by

$$\tilde{\Xi}_j(x) := (r_{j+1} - r_j)\Theta(s+x)(r_{j+1} - r_j).$$

$$\|(r_{j+1}-r_j)(\tilde{\Xi}_j(x) - \Theta(x+s))(r_{j+1}-r_j)\| < 2^{-j+1}$$

Lemma 17.4.8 *If Φ has a continuous lifting Θ on $\mathsf{D}[\mathsf{E}]$ for some $\mathsf{E} \in \mathrm{Part}_{\mathbb{N}}$, then it has a lifting of product type on $\mathsf{D}_{\mathsf{X}}[\mathsf{E}]$ for some infinite $\mathsf{X} \subseteq \mathbb{N}$.*

Proof: Find an increasing sequence $(n(j))_j$, $s(j) \in \mathsf{D}_{(n(j),n(j+1))}$ (with $n(0) := 0$), and an increasing sequence of finite-rank projections $(r_j)_j$ so that for all j, all a and b in $\mathsf{D}_{[0,n(j)]}$, and all c and d in $\mathsf{D}_{[n(j+1),\infty)}$:

1. $\|(\Theta(a + s(j) + c) - \Theta(b + s(j) + c))(1 - r_j)\| \leq 2^{-j}$,
2. $\|(1 - r_j)(\Theta(a + s(j) + c) - \Theta(b + s(j) + c))\| \leq 2^{-j}$,
3. $\|(\Theta(a + s(j) + c) - \Theta(a + s(j) + d))r_j\| \leq 2^{-j}$,
4. $\|r_j(\Theta(a + s(j) + c) - \Theta(a + s(j) + d))\| \leq 2^{-j}$.

Let $\mathsf{X} := \{n(j) : j \in \mathbb{N}\}$ and $s := \sum_j s(j)$ (an element of $\mathsf{D}_{\mathbb{N}\setminus\mathsf{X}}$).
For each j define $\tilde{\Xi}_j : \mathsf{D}_{n(j)} \to (r_{j+1} - r_j)\mathcal{B}(H)_{\leq 1}(r_{j+1} - r_j)$ by

$$\tilde{\Xi}_j(x) := (r_{j+1} - r_j)\Theta(s + x)(r_{j+1} - r_j).$$

The function $\mathsf{D}_{\mathsf{X}} \xrightarrow{\tilde{\Xi}} \mathcal{B}(H) \colon x \mapsto \sum_{j \in \mathsf{X}} \tilde{\Xi}_j(x_{n(j)})$ is of product type, but probably not a lifting.

$$\Theta(x) - \Theta(x+\underline{\underline{s}}) \in K(H), \quad \forall x$$

$\hat{\Xi}(x) - \Theta(x+s) \in K(H), \ \forall x$

$\xi_x = \phi_x(P_x), \quad P_x = \mathrm{Proj}_{\overline{\mathrm{span}}\{\xi_j \mid j \in \bigcup_{n \in X} E_n\}}$

~~Let $\Xi_j^0(x) := q_X \tilde{\Xi}(x) q_X$ and $\Xi^0(x) = \sum_{j \in X} \Xi_j(x_{n(j)})$. This is a lifting of Φ, but not necessarily of a product type.~~

$\Xi^\circ(a) = \xi_x \hat{\Xi}(a) \xi_x$

Ξ° is a lifting but not of a product type.

Let $\varepsilon_j = \max \| [\xi_x, \tilde{\hat{\Xi}}_j(b)] \|$

$b \in D_{n(j)}$

Fact $\varepsilon_j \to 0,\ j \to \infty.$

Pf otherwise, fix $\varepsilon > 0$ and $Y \subseteq X$, infinite, and $b_j \in D_{n(j)}$, for $j \in Y$,

$$\|[\Sigma_X, \widehat{\Xi}_j(b_j)]\| \geq \varepsilon_j. \quad (*)$$

Let $b = \sum_j b_j.$

Then $\dot{p}_X \dot{b} = \dot{b}\dot{p}_X = \dot{b}$ $(b \in D_X)$,

also $\dot{p}_X(\dot{b} + \dot{s}) = (\dot{b} + \dot{s})\dot{p}_X = \dot{b}$

So $\Sigma_X \Theta(b+s) - \Theta(b+s)\Sigma_X \in K(H)$

and $[\Sigma_X, \widehat{\Xi}(b+s)] \in K(H)$

contradiction with $(*)$.

So $\varepsilon_j = \max_{b \in D_{n(j)}} \|[\Sigma_X, \Xi_j(b)]H\|$

satisfies $\varepsilon_j \to 0$.

choose $Y \subseteq X$, s.t. that

$$\sum_{i\in Y} \varepsilon_i < \infty$$

Then, on D_Y, let

$$\Xi^\circ(a) = \sum_{i\in Y} \xi_X \hat{\Xi}_i(a_{n(i)})\, \xi_X .$$

Then $\Xi^\circ(a)$ is of product type. Also, for $a \in D_Y$,

equal mod $K(H)$

$$\Xi^\circ(a) =^K \xi_X \sum_{i\in Y} \hat{\Xi}_i(a_{n(i)})\, \xi_X$$

$$=^K \xi_X\, \Theta(a+S)\, \xi_X =^K \Theta(a).$$

Before moving on, let's take a look at history.

An automorphism of ℓ_∞/c_0 is trivial if it has a lifting that is a $*$-homomorphism from ℓ_∞ into ℓ_∞.

Before moving on, let's take a look at history.

An automorphism of ℓ_∞/c_0 is trivial if it has a lifting that is a *-homomorphism from ℓ_∞ into ℓ_∞.

Thm (Shelah, 1979) *The assertion 'all automorphisms of ℓ_∞/c_0 are trivial' is relatively consistent with ZFC.*

Before moving on, let's take a look at history.

An automorphism of ℓ_∞/c_0 is trivial if it has a lifting that is a *-homomorphism from ℓ_∞ into ℓ_∞.

Thm (Shelah, 1979) *The assertion 'all automorphisms of ℓ_∞/c_0 are trivial' is relatively consistent with ZFC.*

Thm (Veličković, 1989) OCA_T *and Martin's Axiom together imply that all automorphisms of ℓ_∞/c_0 are trivial.*

Each of the proofs proceeds in three stages.

Fix $\phi \in \mathrm{Aut}(\ell_\infty/c_0)$

1 — There are many $X \subseteq \mathbb{N}$, X ∞,

such that $\phi \restriction \ell_\infty(X)$

OCA_T

$C_0(X)$

is trivial

2 MA (+ OCA$_T$)

3 OCA$_T$

The isometry trick

The following will not be used explicitly in the proof.

Lemma *Suppose that Φ is an automorphism of $\mathcal{Q}(H)$ and $p \in \mathcal{Q}(H)$ is a projection* $p \neq 0$ *such that the restriction of Φ to $p\mathcal{Q}(H)p$ is implemented by a unitary. Then Φ is implemented by a unitary.*

Pf Fix $v \in Q(H)$ $v^*v = 1_{Q(H)}$, $vv^* = p$.

Fix w, s.t. $\forall a \in pQ(H)p$

$w a w^* = \phi(a)$.

For $a \in Q(H)$,

$a = v^* v a v^* v$. , $vav^* \in pQ(H)p$

$pvav^* = vv^*vav^* = vav^*$

$\phi(a) = \phi(v^*)\, \phi(vav^*)\, \phi(v)$

$$= \underbrace{\phi(v^*) w v}\, a\, \underbrace{v^* w^* \phi(v)}$$

Let $u = \phi(v^*) w v$.

Then $\phi(a) = u a u^*$

Also, $u^* u = 1 = u u^*$.

The isometry trick

The following will not be used explicitly in the proof.

Lemma *Suppose that Φ is an automorphism of $\mathcal{Q}(H)$ and $p \in \mathcal{Q}(H)$ is a projection such that the restriction of Φ to $p\mathcal{Q}(H)p$ is implemented by a unitary. Then Φ is implemented by a unitary.*

von Neumann algebra

Lemma *Suppose that Φ is an automorphism of $\mathcal{Q}(H)$ such that the restriction of Φ to $\mathcal{D}_{\mathsf{X}}[\mathsf{E}]$ is implemented by a unitary for some $\mathsf{E} \in \mathsf{Part}_{\mathbb{N}}$ such that $|E_n| \to \infty$ as $n \to \infty$. Then the restriction of Φ to $\mathcal{D}[\mathsf{F}]$ is implemented by a unitary for every $\mathsf{F} \in \mathsf{Part}_{\mathbb{N}}$.*

The isometry trick

The following will not be used explicitly in the proof.

Lemma *Suppose that Φ is an automorphism of $\mathcal{Q}(H)$ and $p \in \mathcal{Q}(H)$ is a projection such that the restriction of Φ to $p\mathcal{Q}(H)p$ is implemented by a unitary. Then Φ is implemented by a unitary.*

Lemma *Suppose that Φ is an automorphism of $\mathcal{Q}(H)$ such that the restriction of Φ to $\mathcal{D}_{\mathrm{X}}[\mathrm{E}]$ is implemented by a unitary for some $\mathrm{E} \in \mathrm{Part}_{\mathbb{N}}$ such that $|E_n| \to \infty$ as $n \to \infty$. Then the restriction of Φ to $\mathcal{D}[\mathrm{F}]$ is implemented by a unitary for every $\mathrm{F} \in \mathrm{Part}_{\mathbb{N}}$.*

Lemma 17.5.2 *Suppose $\Phi\colon \mathcal{Q}(A) \to \mathcal{Q}(B)$ is a *-homomorphism between coronas of nonunital C^*-algebras, $\mathcal{X} \subseteq \mathcal{M}(A)$, v is an isometry in $\mathcal{M}(A)$, and Υ is a lifting of Φ on $v\mathcal{X}v^*$. Then $b \mapsto \Phi_*(v)^*\Upsilon(vbv^*)\Phi_*(v)$ is a lifting of Φ on $\mathcal{X}$.*

Analogous lemmas, with 'implemented by a unitary' replaced by 'has a continuous/C-measurable ε-approximation', 'has a lifting of product type',… have analogous proofs.

We have H with the basis (ξ_i). For an injection $g\colon \mathbb{N} \to \mathbb{N}$,

$$v_g(\xi_i) := \xi_{g(i)}$$

$H \to H$

$\sum \lambda_i \xi_i \to \sum \lambda_i \xi_{g(i)}$

defines an isometry on H. Such v_g is called an *injection isometry* on H.

We have H with the basis (ξ_j). For an injection $g\colon \mathbb{N} \to \mathbb{N}$,

$$v_g(\xi_i) := \xi_{g(i)}$$

defines an isometry on H. Such v_g is called an *injection isometry* on H.

Partd

Lemma 17.5.1 *Suppose* E *and* F *are in* $\mathrm{Part}_{\mathbb{N}}$, X *and* Y *are infinite subsets of* $\mathbb{N}$, *and* $\lim_{n \in \mathrm{X}} |E_n| = \infty$. *Then there exist a permutation isometry* v *such that* $a \mapsto vav^*$ *defines an isomorphism from* $\mathcal{D}_{\mathrm{Y}}[\mathrm{F}]$ *onto* $vv^*\mathcal{D}_{\mathrm{X}}[\mathrm{E}]vv^*$.

Pf: Find $m(n) \in X$ $m(n) > |F_n|$, $m(n) < m(n+1)$ ...thn.

Pick g so that $g[F_n] \subseteq E_{m(n)}$, injectivity, for $g \in G$.

Then $u_g^* u_g = p^F_\gamma$, $u_g u_g^* \in D[E]$,

$a \mapsto u_g a u_g^*$ sends $D_\gamma[F]$

into $u_g u_g^* D_{\cancel{\gamma}}[E] u_g u_g^*$. $\square$

Lemma 17.5.3 *Suppose* E *and* F *are in* $\mathrm{Part}_{\mathbb{N}}$, $X \subseteq \mathbb{N}$, $v \in \mathcal{B}(H)$ *is an injection isometry such that* $a \mapsto vav^*$ *defines an isomorphism from* $\mathcal{D}[F]$ *onto* $vv^*\mathcal{D}_X[E]vv^*$, *and* Φ *is an endomorphism of* $\mathcal{Q}(H)$.

1. *If the restriction of* Φ *to* $\mathcal{D}_X[E]$ *is implemented by* w, *then the restriction of* Φ *to* $\mathcal{D}[F]$ *is implemented by* $\Phi(v^*)wv$.
2. *If* Φ *has a lifting of product type on* $D_X[E]$ *then it has a lifting of product type on* $D[F]$. (because $a \to vav^*$ is of a product type)
3. *If* Θ *is a C-measurable* ε*-approximation of* Φ *on* $\mathcal{D}_X[E]$ *then*

$$a \mapsto \Phi_*(v^*)\Theta(vav^*)\Phi_*(v)$$

is a C-measurable ε*-approximation of* Φ *on* $\mathcal{D}[F]$.

$a \to vav^*$ is C-meas. isometry

Ulam - stability

We can now change the gears.

Def 17.2.5 *Given $\varepsilon > 0$ and C^*-algebras A and B, some $\Theta\colon A_1 \to B_1$ is an ε-*-homomorphism if for all x, y in A_1 and $\lambda \in \mathbb{C}$, $|\lambda| \leq 1$, each one of $\Theta(x^*) - \Theta(x)^*$, $\Theta(x+y) - \Theta(x) - \Theta(y)$, $\Theta(xy) - \Theta(x)\Theta(y)$, and $\Theta(\lambda x) - \lambda\Theta(x)$ has norm not greater than ε. It is unital if in addition $\Theta[\mathrm{U}(A)] \subseteq \mathrm{U}(B)$ and $\Theta(1) = 1$.*

Thm (Kanovei - Reeken) $\forall \varepsilon > 0$ There is an ε-homomorphism between metric groups, $f_\varepsilon : G_\varepsilon \to K_\varepsilon$ such that

(① every homo $f: G_\varepsilon \to K_\varepsilon$ is trivial,

$\mathrm{dist}(f_\varepsilon, f) \geq \frac{1}{2}$, $\forall$ hom $f: G_\varepsilon \to H_\varepsilon$

Pf Pick n large enough

$$\mathbb{Z}/n\mathbb{Z} \subseteq \mathbb{T}$$

$$f_\varepsilon : \underline{\mathbb{Z}/n\mathbb{Z}} \to \underline{\mathbb{Z}/(n+1)\mathbb{Z}}$$

$$|f_\varepsilon(x) - x| \leq e^{2\pi i/(n+1)}, \quad \forall x$$

We can now change the gears.

Def 17.2.5 *Given $\varepsilon > 0$ and C^*-algebras A and B, some $\Theta\colon A_1 \to B_1$ is an ε-*-homomorphism if for all x, y in A_1 and $\lambda \in \mathbb{C}$, $|\lambda| \leq 1$, each one of $\Theta(x^*) - \Theta(x)^*$, $\Theta(x+y) - \Theta(x) - \Theta(y)$, $\Theta(xy) - \Theta(x)\Theta(y)$, and $\Theta(\lambda x) - \lambda\Theta(x)$ has norm not greater than ε. It is unital if in addition $\Theta[\mathrm{U}(A)] \subseteq \mathrm{U}(B)$ and $\Theta(1) = 1$.*

Prop 17.5.4 *Suppose that Φ is an endomorphism of the Calkin algebra which has a continuous lifting Θ on $\mathrm{D}[\mathrm{E}]$ for some $\mathrm{E} \in \mathrm{Part}_{\mathbb{N}}$ such that $\lim_n |E_n| = \infty$, Then for every $\mathrm{F} \in \mathrm{Part}_{\mathbb{N}}$, Φ has a lifting (Θ'_n) of product type on $\mathcal{D}[\mathrm{F}]$ such that each Θ'_n is a unital, Borel-measurable, ε_n-*-homomorphism on $\mathcal{D}[\mathrm{F}]$ for some sequence (ε_n) converging to 0.*

$\Theta(a) = \sum \Theta'_n(a_n)$ $\qquad$ $a_n \in \mathcal{D}_{\{n\}}[F]$

Pf First, we have a lifting

$\psi = (\psi_n)$ of product type (Cantor-like space).

on $D[F]$

On $D_{\{n\}}[F]$, let $\alpha_n : \underset{\text{finite}}{\underbrace{(D_{\{n\}}[F])_1}} \to D_n[F]$

be s.t. $\|\alpha_n(x) - x\| < 2^{-n}$ and

α_n is Borel-measurable

Then $\alpha(x) = \sum \alpha_n(x_n)$

satisfies $\alpha(x) - x \in K(H)$

So $\theta = \psi \circ \alpha$ is a lifting of

product type on $D[F]$.

Claim $\forall \varepsilon > 0$ $\forall^\infty n$ θ_n $(= \psi_n \circ \alpha_n)$

is an ε-*-homo.

Pf Assume otherwise.

Assume $\exists \varepsilon > 0$ $\exists^\infty n$ θ_n is not

an ε-*-homo.

Case 1 $\exists^\infty n$ $\exists x_n \in \left(\mathcal{D}_n[F]\right)_1$

$\|\Theta_n(x_n^*) - \Theta_n(x_n)^*\| > \varepsilon$

Then, with $x = \sum_n x_n$

$\Theta(x^*) - \Theta(x)^* \notin K(H)$

– Contradiction.

Case 2–4 – analogous.

In order to prove the following, we will need to introduce a new tool.

Prop 17.5.5 *Suppose that Φ is an endomorphism of the Calkin algebra which has a continuous lifting on* D[E] *for some* $E \in \mathrm{Part}_{\mathbb{N}}$ *such that* $\lim_n |E_n| = \infty$*. Then for every* $F \in \mathrm{Part}_{\mathbb{N}}$, Φ *has a lifting on* $\mathcal{D}[F]$ *which is a* **-homomorphism.*

Next time:

Ulam - Stability.

$\phi : A \to B$

fin. dim.