Massive C^*-algebras, Winter 2021, I. Farah, Lecture 16

Recall

Def 17.3.1 *A lifting of a *-homomorphism $\Phi\colon \mathcal{Q}(A) \to \mathcal{Q}(B)$ is a function $\Phi_*\colon \mathcal{M}(A) \to \mathcal{M}(B)$ such that the following diagram commutes (π_A and π_B denote the quotient maps).*

$$\begin{tikzcd}
\mathcal{M}(A) \arrow[r, "\Phi_*"] \arrow[d, "\pi_A"'] & \mathcal{M}(B)) \arrow[d, "\pi_B"'] \\
\mathcal{Q}(A) \arrow[r, "\Phi"] & \mathcal{Q}(B)
\end{tikzcd}$$

If this diagram commutes on some $\mathcal{X} \subseteq \mathcal{M}(A)$, then Φ_* is called a lifting of Φ on $\mathcal{X}$. When convenient, instead we say that Φ ~~is~~ has a lifting on $\pi[\mathcal{X}]$.

In order to prove the following lemma, we will need to go back to basic theory of C^*-algebras.

Lemma ≈17.3.2 *Every* **-homomorphism* $\Phi\colon \mathcal{Q}(A) \to \mathcal{Q}(B)$ *has a lifting* Φ_* *such that the following holds*

1. $\Phi_*(a)$ *is self-adjoint if* a *is self-adjoint.*
2. $\|\Phi_*(a)\| \leq \|a\|$ *for all* a.
3. *If* $\mathcal{M}(B) = \mathcal{B}(H)$ *then we can assure that* $\Phi_*(p)$ *is a projection for every projection* p.

In order to prove the following lemma, we will need to go back to basic theory of C^*-algebras.

Lemma ≈17.3.2 *Every *-homomorphism $\Phi\colon \mathcal{Q}(A) \to \mathcal{Q}(B)$ has a lifting Φ_* such that the following holds*

1. *$\Phi_*(a)$ is self-adjoint if a is self-adjoint.*
2. *$\|\Phi_*(a)\| \leq \|a\|$ for all a.*
3. *If $\mathcal{M}(B) = \mathcal{B}(H)$ then we can assure that $\Phi_*(p)$ is a projection for every projection p.*

It suffices to prove the following:

Lemma *Suppose J is an ideal in A and $\pi\colon A \to A/J$ is the quotient map.*

1. *If $a \in A/J$ is self-adjoint, then there exists a self-adjoint $a_0 \in A$ such that $\pi(a_0) = a$.*
2. *If $a \in A/J$ then there exists $a_0 \in A$ such that $\pi(a_0) = a$ and $\|a_0\| \leq \|a\|$.*
3. *If $A = \mathcal{B}(H)$, $J = \mathcal{K}(H)$, and $p \in \mathcal{B}(H)/\mathcal{K}(H)$ is a projection, then there exists a projection $p_0 \in \mathcal{B}(H)$ such that $\pi(p_0) = p$.*

① $\Rightarrow$ ①: $a = a^* \Rightarrow \phi(a) = \phi(a^*)$

If $a_0 \in B(H)$, $\pi(a_0) = \phi(a)$

and $a_0 = a_0^*$, then let $\phi_*(a) = a_0$

② $\|\phi(a)\| \leq \|a\|$

---

pf (the second lemma)

① If $b \in A$, $\pi(b) = a$

Let $a_0 = \frac{1}{2}(b + b^*)$. Then $a_0 = a_0^*$

and $\pi(a_0) = \frac{1}{2}(a + a^*) = a$

[② - need more lemmas, they are coming up.]

③ Fix $p \in Q(H)$ projection.

By ①, fix $a_0 \in B(H)$, $a_0 = a_0^*$, and $\pi(a_0) = p$. Then

$b = a_0^2 - a_0 \in K(H)$ (b/c $p^2 = p$

By the spectral thm for each self-adj. operator,

$b = \sum_i r_i q_i$ where

ε_j is a projection of finite rank

$r_j \in \mathbb{R}$, $r_j \to 0$ r_j distinct

$\varepsilon_j \varepsilon_k = 0$ if $j \neq k$

$\sum r_j \varepsilon_j$ is SOT-convergent.

$[\varepsilon_j, b] = 0$, $\forall j$

$\varepsilon = \sum \varepsilon_j$. (Then $(1-\varepsilon) a_0$ is a projection.

Also, $[\varepsilon_j, a_0] = 0$, ($C^*(a_0)$ is abelian, b, $\varepsilon_j \in C^*(a_0)$)

$\varepsilon_j \in C^*(b)$ $\quad f(r_j) = 1$, $f(r_k) = 0$ $k \neq j$

r_0 $\cdots$ r_j $\cdots$ 0 $\quad f(b) = \varepsilon_j$

$g : \mathbb{R} \to \{0, 1\}$

$g(t) = 0$, $\quad t < \frac{1}{2}$

$g(t) = 1$, $\quad t \geq \frac{1}{2}$

$sp(a_0)$

$sp(b) = \{t - t^2 \mid t \in sp(a_0)\} = \{r_j \mid j \in \mathbb{N} \cup \{0\}\}$.

$g(a_0 \varepsilon_j)$ is

a projection, $\forall j$.

Let $p_0 = \sum_j g(a_0 \xi_j) + (1-\varepsilon) a_0$

Then $\pi(p_0) = p$.

(Proving ②, that wlog $\|\phi_*(a)\| \leq \|a\|$, $\forall a$.)

The easy case: If $a = a^*$.

choose b, $b = b^*$, $\phi(a) = \pi(b)$

Let $f : \mathbb{R} \to \mathbb{R}$ be:

$$f(t) = \begin{cases} -\|a\|, & t \leq -\|a\| \\ t, & -\|a\| < t < \|a\| \\ \|a\|, & t \geq \|a\|. \end{cases}$$

Let $\phi_*(a) = f(b)$. Then

$f(b) = f(b)^*$, $\|b\| \leq \|a\|$ $\quad \|\cdot\| \leq \|a\|$

$\pi(f(b)) = f(\pi(b)) = f(\phi(\pi(a)) = \phi(\pi(a)))$

$$\overline{U}(L^{q}) = \left(\overline{U(L)}\right)^{q}$$

$\mathbb{C} \subseteq A$

$\lambda \sim \lambda \cdot 1_A$

Coro 1.6.12 *Suppose a and b_1 belong to a C^*-algebra A. If $0 \le a$ and $a \le b_1^* b_1$ then $a \in \overline{b_1^* A b_1}$.*

Pf Let $b = b_1^* b_1$

Let, for $n \ge 1$,

$x_n = a\left(b + \frac{1}{n}\right)^{-1} b$

① claim $x_n \to a$.

$a - x_n = a - a\left(b + \frac{1}{n}\right)^{-1} b$

$= a\left(1 - \left(b + \frac{1}{n}\right)^{-1} b\right)$

$$= a\left(b+\tfrac{1}{n}\right)^{-1}\left(b+\tfrac{1}{n}-b\right)$$

$$= a(nb+1)^{-1}$$

$$\|a-x_n\|^2 = \|a(nb+1)^{-1}\|^2$$

$$= \|(nb+1)^{-1}a^2(nb+1)^{-1}\|$$

$a^2 \le b^2$ $$\le \|(nb+1)^{-1}b^2(nb+1)^{-1}\|$$

$$= \|b(nb+1)^{-1}\| \quad (b \geq 0)$$

$$= \|f_n(b)\|$$

Let $f_n(t) = \dfrac{t}{nt+1}$ $\quad t \geq 0, \; n \geq 1$

$f_n(t) \to 0, \quad n \to 0$

$\left(f_n(t) \le \frac{1}{n}\right)$

so $\|f_n(b)\| \to 0$.

So, $x_n \to a$ (in $\|\cdot\|$).

$0 \le a \le b \Rightarrow xax^* \le xbx^*$)

$b-a \geq 0$ $\qquad x(b-a)x^* \geq 0$

$x_n = a\left(b+\frac{1}{n}\right)^{-1}b$

$x_n^* x_n = b\ldots \quad b \in \overline{bAb} \subseteq \overline{b_1^* A b_1}$

so $x_n^* x_n \to a^2$, and $a^2 \in \overline{b_1^* A b_1}$,

since $a \geq 0$, $a = (a^2)^{1/2}$, and

$$a \in C^*(a^2) \subseteq \overline{b_1^* A b_1}.$$

Content advisory: The following proposition is very general and parsing its statement may take a while.

Content advisory: The following proposition is very general and parsing its statement may take a while.

Prop 1.6.9 *Suppose* b, c, d *belong to a* C^**-algebra* A, $f, g \in C([0, \|b\|])_+$, *and* $h(t) := f(t)g(t)t^{-1}$ *continuously extends to* $[0, \|b\|]$. *If* $0 \leq b$, $c^*c \leq f(b)^2$, *and* $dd^* \leq g(b)^2$ *then the sequence* $a_n := c(b + 1/n)^{-1}d$ *norm converges to a limit* a *with* $\|a\| \leq \|h(b)\|$.

Pf: $\|a_m - a_n\|^2$

Content advisory: The following proposition is very general and parsing its statement may take a while.

Prop 1.6.9 *Suppose b, c, d belong to a C^*-algebra A, $f, g \in C([0, \|b\|])_+$, and $h(t) := f(t)g(t)t^{-1}$ continuously extends to $[0, \|b\|]$. If $0 \le b$, $c^*c \le f(b)^2$, and $dd^* \le g(b)^2$ then the sequence $a_n := c(b + 1/n)^{-1}d$ norm converges to a limit a with $\|a\| \le \|h(b)\|$.*

Coro 1.6.13 *Suppose $b = v|b|$ is the polar decomposition of b.*

1. *If $g \in C_0(\mathrm{sp}(|b|))$ then $vg(|b|) \in \mathrm{C}^*(b)$.* $\boxed{g(0)=0}$

   $\setminus\{0\}$
2. *If A is a C^*-algebra such that $b \in A$ and $c \in \overline{b^*Ab}$, then $vc \in A$.*
3. *If $a \le b^*b$ then $va \in \mathrm{C}^*(a, b)$.*

Thm $\forall b \in B(H)$ $\exists v$, partial isometry, $b = v|b|$ ... ($b|b|^{-1} = v$)

(note: $v \notin C^*(b)$, for some b).

Pf ① Let $a_n = b(|b| + 1/n)^{-1} g(|b|)$

Then $b^*b \leq |b|^2$ ... $v|b|$

$g(|b|) \leq g(|b|)$

$t g(t) \cdot t^{-1}$ cts on $[0, \|b\|]$

by Lemma, (a_n) is Cauchy

$\|vg(|b|) - a_n\| \to 0$, $n \to \infty$

(use the same trick).

So $vg(|b|) \in C^*(b)$.

② Like ①, put c in place of $g(|b|)$

③

Back to ϕ_*:

Fix $a \in B(H)$, let $a_0 \in B(H)$

such that $\pi(a_0) = \phi_t(\pi(a))$.

Then $a_0 = v|a_0|$

Let $f : \mathbb{R} \to \mathbb{R}$

$$f(t) = \begin{cases} -\|a_0\|, & t \le -\|a_0\| \\ t, & \cdot \ - \\ \|a_0\|, & t \ge \|a_0\| \end{cases}$$

Let $\phi_*(a) = v f(|a_0|) \in C^*(a_0)$

$$\| v f(|a_0|) \| \le \|v\| \cdot \|a_0\| \le \|a_0\|$$

A side remark: We are within ε of proving Proposition 1.6.14, that is one of the main components in the proof of Kirchberg's Slice Lemma. . . but we'll skip these, since we don't need them in this course.

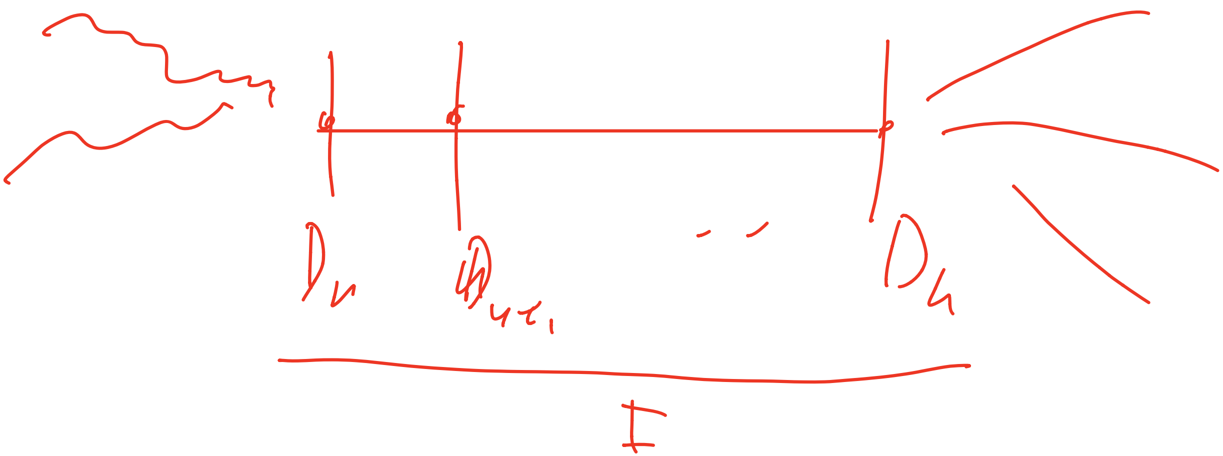
Next, we'll need a standard tricks from set theory.

Meager Subsets of Product Spaces

Suppose D_n, for $n \in \mathbb{N}$, are finite sets. Then for $X \subseteq \mathbb{N}$

$$D_X := \prod_{n \in X} D_n$$

is compact with respect to $d(a, b) = 1/(\min\{n : a_n \neq b_n\} + 1)$. The basic open subsets of $D_{\mathbb{N}}$ have the form $[I, r] := \{a : a \upharpoonright I = r\}$ for some $I \in \mathbb{N}$ and $r \in D_I$.



Meager Subsets of Product Spaces

Suppose D_n, for $n \in \mathbb{N}$, are finite sets. Then for $\mathrm{X} \subseteq \mathbb{N}$

$$\mathrm{D}_\mathrm{X} := \prod_{n \in \mathrm{X}} \mathrm{D}_n$$

is compact with respect to $d(a,b) = 1/(\min\{n : a_n \neq b_n\} + 1)$.
The basic open subsets of $\mathrm{D}_\mathbb{N}$ have the form
$[I, r] := \{a : a \restriction I = r\}$ for some $I \Subset \mathbb{N}$ and $r \in \mathrm{D}_I$.

I J

Lemma

1. *$I \cap J = \emptyset$ implies $[I, r] \cap [J, s] = [I \cup J, rs]$ where $(rs)(i) = r(i)$ if $i \in I$ and $(rs)(i) = s(i)$ if $i \in J$.*
2. *$I \cap J = \emptyset$ implies $[I, r] \cap [J, s] \neq \emptyset$.*
3. *$[I, r] \supseteq [J, s]$ if and only if $I \subseteq J$ and $s \cap I = r$.*

Meager $\Leftrightarrow$ 1^{st} category

comeager $\Leftrightarrow$ includes dense G_δ set.

(in a Polish space.)

Thm 9.9.1 *Some $\mathcal{A} \subseteq \mathrm{D}_{\mathbb{N}}$ is relatively comeager in $\mathrm{D}_{\mathbb{N}}$ if and only if there are disjoint $I(n) \Subset \mathbb{N}$, for $n \in \mathbb{N}$, and $s(n) \in \mathrm{D}_{I(n)}$ such that $\underbrace{\bigcap_m \underbrace{\bigcup_{n \ge m} \underbrace{[I(n), s(n)]}_{\text{open}}}_{\text{open}}}_{G_\delta} \subseteq \mathcal{A}$.*

Pf ($\Leftarrow$) Need to check

$\bigcap \bigcup [I(n), s(n)]$ is dense.

$m \qquad n \geq m$

Fix $[J, r]$. Since $I(n)$ are disjoint, $\forall^\infty n$ $I(n) \cap J = \emptyset$.

But then $[J, r] \cap [I(n), S(n)] \neq \emptyset$

The right proof:

$\forall m$ $\bigcup_{n \geq m} [I(n), S(n)]$

is open, dense.

Use Baire Category.

$(\Rightarrow)$ Fix $A \subseteq DN$, comeager. Fix dense open $U_n \subseteq DN$, $n \in N$, so that

$\bigcap_n U_n \subseteq A$.

(WLOG, $U_0 \supseteq U_1 \supseteq U_2 \supseteq \ldots$)

Find $I(n)$, $S(n)$ by recursion.
(we'll assure each $I(n)$ is an interval)

① Fix $I(0)$, $S(0)$, so that

$$[I(0), S(0)] \subseteq U_0$$

② Enumerate $D_{I(0)}$ as v_j, $j < k(0)$.

Find $[I^j(1), S^j(1)]$, $j < k(0)$, as follows.

$$[I(0), v_0] \cap [I^0(1), S^0(1)] \subseteq U_1$$

$$[I(0), v_1] \cap [I^0(1), S^0(1)] \cap [I^1(1), S^1(1)] \subseteq U_1$$

etc.

$$[I(0), v_j] \cap \bigcap_{\ell=0}^{j} [I^\ell(1), S^\ell(1)] \subseteq U_1$$

(note: $I^\ell(1) \cap I^{\ell'}(1) = \emptyset$ $\ell \neq \ell'$

$$I(1) = \bigcup_{\ell=0}^{k(0)-1} I^\ell(1), \quad S(1) = S^0(1) S^1(1) \ldots S^{k(0)-1}(1)$$

$$\ell_{I(0) \cup I(1)}$$

$$\boxed{\forall n, \forall x \quad x \in [I(n), S(n)] \Rightarrow x \in U_n}$$

$$\exists^\infty n \quad x \in [I(n), S(n)]$$

$$\Rightarrow \exists^\infty n \quad x \in U_n$$

$$\Rightarrow x \in \bigcap_n U_n \Rightarrow x \in A.$$