After having done the preparations, today we will state and prove Theorem 17.8.2.

Coherent families of unitaries

We will need the notation from the proof that CH implies $\mathcal{Q}(H)$ has an outer automorphism.

For i and j in $\mathbb{N}$, x and y in $\mathbb{T}^{\mathbb{N}}$, and $F \subseteq \mathbb{N}$ let

$$\Delta_{\{i,j\}}(x,y) := |x(i)\overline{x(j)} - y(i)\overline{y(j)}|, \text{ and} \tag{1}$$

$$\Delta_F(x,y) := \sup_{i,j \in F} \Delta_{\{i,j\}}(x,y). \tag{2}$$

For i and j in $\mathbb{N}$, x and y in $\mathbb{T}^{\mathbb{N}}$, and $F \subseteq \mathbb{N}$ let

$$\Delta_{\{i,j\}}(x,y) := |x(i)\overline{x(j)} - y(i)\overline{y(j)}|, \text{ and} \tag{1}$$

$$\Delta_F(x,y) := \sup_{i,j \in F} \Delta_{\{i,j\}}(x,y). \tag{2}$$

Lemma 17.1.5 *If $F \subseteq \mathbb{N}$ is nonempty, i,j are in $\mathbb{N}$, and x,y,z are in $\mathbb{T}^{\mathbb{N}}$ then the following hold.*

1. $\Delta_{\{i,j\}}(x,y) = |x(i)\overline{y(i)} - x(j)\overline{y(j)}|$.
2. $\Delta_F(x,1) = \operatorname{diam}(\{x(i) : i \in F\})$.
3. $\Delta_{\{i,k\}}(x,y) \leq \Delta_{\{i,j\}}(x,y) + \Delta_{\{j,k\}}(x,y)$, *hence* $\Delta_{\{\cdot,\cdot\}}(x,y)$ *is a pseudometric on* $\mathbb{N}$.
4. $\Delta_F(x,z) \leq \Delta_F(x,y) + \Delta_F(y,z)$, *hence* Δ_F *is a pseudometric on* $\mathbb{T}^{\mathbb{N}}$.
5. $\Delta_F(x,y) = \Delta_F(xz,yz)$.

For i and j in $\mathbb{N}$, x and y in $\mathbb{T}^{\mathbb{N}}$, and $F \subseteq \mathbb{N}$ let

$$\Delta_{\{i,j\}}(x,y) := |x(i)\overline{x(j)} - y(i)\overline{y(j)}|, \text{ and} \tag{1}$$

$$\Delta_F(x,y) := \sup_{i,j\in F} \Delta_{\{i,j\}}(x,y). \tag{2}$$

Lemma 17.1.5 *If $F \subseteq \mathbb{N}$ is nonempty, i,j are in $\mathbb{N}$, and x,y,z are in $\mathbb{T}^{\mathbb{N}}$ then the following hold.*

1. $\Delta_{\{i,j\}}(x,y) = |x(i)\overline{y(i)} - x(j)\overline{y(j)}|$.
2. $\Delta_F(x,1) = \operatorname{diam}(\{x(i) : i \in F\})$.
3. $\Delta_{\{i,k\}}(x,y) \leq \Delta_{\{i,j\}}(x,y) + \Delta_{\{j,k\}}(x,y)$, *hence $\Delta_{\{\cdot,\cdot\}}(x,y)$ is a pseudometric on $\mathbb{N}$.*
4. $\Delta_F(x,z) \leq \Delta_F(x,y) + \Delta_F(y,z)$, *hence Δ_F is a pseudometric on $\mathbb{T}^{\mathbb{N}}$.*
5. $\Delta_F(x,y) = \Delta_F(xz,yz)$.
6. $\min_{\lambda\in\mathbb{T}} \sup_{i\in F} |x(i) - \lambda y(i)| \leq \Delta_F(x,y) \leq 2\min_{\lambda\in\mathbb{T}} \sup_{i\in F} |x(i) - \lambda y(i)|$ *(a proof is on the following page).*

$$\Delta_{\{i,j\}}(x,y) := |x(i)\overline{x(j)} - y(i)\overline{y(j)}|, \text{ and} \tag{3}$$

$$\Delta_F(x,y) := \sup_{i,j \in F} \Delta_{\{i,j\}}(x,y). \tag{4}$$

Lemma *If $u \in \mathcal{U}(\mathcal{B}(H))$, then* $\mathbb{T} \cdot 1 = Z(U(B(H)))$

$$\underline{\operatorname{dist}(u, \mathbb{T}\cdot 1)} \leq \underline{\operatorname{diam}(\operatorname{sp}(u))} \leq \underline{2\operatorname{dist}(u, \mathbb{T}\cdot 1)}$$

$$\operatorname{dist}(u, \mathbb{T}\cdot 1) \leq \sup_{\|a\|\leq 1} \|uau^* - a\| \leq 2\operatorname{dist}(u, \mathbb{T}\cdot 1).$$

Proof: Only one of the inequalities is non-elementary.

- Fix $\underline{\lambda \in Sp(u)}$
  $\forall z \in Sp(u) \quad |z - \lambda| \leq \operatorname{diam}(Sp(u))$
  $\operatorname{dist}(u, \mathbb{T}\cdot 1) \leq \operatorname{dist}(u, \lambda \cdot 1)$
- triangle $\quad \|u - \lambda\cdot 1\| \leq \operatorname{dist}(u, \mathbb{T}\cdot 1)$
- Fix $\varepsilon > 0$, fix λ

Fix a, $\|a\| \le 1$

$$\|uau^* - a\| = \|ua - au\|$$
$$\le \|ua - \lambda a\| + \|\lambda a - au\|$$
$$\le \operatorname{dist}(u, \mathbb{T}\cdot 1) + \varepsilon$$

- Let μ be the Haar measure on $U(\mathbb{B}(H))$.

Let $\varepsilon = \sup_{\|a\| \le 1} \|uau^* - a\|$

$$v = \int wuw^* \, d\mu(w) \in U(\mathbb{B}(H))$$

$$\|v - u\| \le \int \underbrace{\|wuw^* - u\|}_{\|wu - uw\| \le \varepsilon} d\mu(w)$$

$$\le \varepsilon$$

$v \cdot w = wv$, $\forall w \in U(\mathbb{B}(H))$

so $v \in Z(U(\mathbb{B}(H)))$

so $v \in \mathbb{T}\cdot 1$.

$$\Delta_{\{i,j\}}(x,y) := |x(i)\overline{x(j)} - y(i)\overline{y(j)}|, \text{ and} \tag{3}$$

$$\Delta_F(x,y) := \sup_{i,j\in F} \Delta_{\{i,j\}}(x,y). \tag{4}$$

Lemma *If $u \in \mathcal{U}(\mathcal{B}(H))$, then*

$$\operatorname{dist}(u, \mathbb{T}\cdot 1) \leq \operatorname{diam}(\operatorname{sp}(u)) \leq 2\operatorname{dist}(u, \mathbb{T}\cdot 1)$$

$$\operatorname{dist}(u, \mathbb{T}\cdot 1) \leq \sup_{\|a\|\leq 1} \|uau^* - a\| \leq 2\operatorname{dist}(u, \mathbb{T}\cdot 1).$$

Proof: Only one of the inequalities is non-elementary.

Corollary

For w and v in $\mathcal{U}(\mathcal{B}(H))$,

$$\tfrac{1}{2}\sup_{\|a\|\leq 1} \|(\operatorname{Ad} w)(a) - (\operatorname{Ad} v)(a)\|$$

$$\leq \operatorname{diam}(\operatorname{sp}(v^* w)) \leq 2 \sup_{\|a\|\leq 1} \|(\operatorname{Ad} w)(a) - (\operatorname{Ad} v)(a).$$

Let $u = v^* w$, use Lemma.

Recall: Part $_{\mathbb{N}}$

1. $u \sim_{\mathrm{E}} v$ if and only if $\operatorname{Ad} u$ and $\operatorname{Ad} v$ agree on $\mathcal{F}[\mathrm{E}]$,
2. we identify $\mathbb{T}^{\mathbb{N}}$ with $\mathcal{U}(\ell_\infty) \subseteq \mathcal{U}(\mathcal{B}(H))$,
3. $x \sim_{\mathrm{E}} y$ if and only if $\limsup_i \Delta_{E_i \cup E_{i+1}}(x, y) = 0$.

Recall:

1. $u \sim_{\mathrm{E}} v$ if and only if $\mathrm{Ad}\, u$ and $\mathrm{Ad}\, v$ agree on $\mathcal{F}[\mathrm{E}]$,
2. we identify $\mathbb{T}^{\mathbb{N}}$ with $\mathcal{U}(\ell_\infty) \subseteq \mathcal{U}(\mathcal{B}(H))$,
3. $x \sim_{\mathrm{E}} y$ if and only if $\limsup_i \Delta_{E_i \cup E_{i+1}}(x, y) = 0$.

$u = x$

$v = y$

Def 17.8.1 *A family $\mathbb{F}$ of pairs (E, x) for $\mathrm{E} \in \mathrm{Part}_{\mathbb{N}}$ and $x \in \mathbb{T}^{\mathbb{N}}$ is a coherent family of unitaries if $\{\mathrm{E} : (\mathrm{E}, x) \in \mathbb{F}$ for some $x\}$ is $\leq^*$-cofinal in $\mathrm{Part}_{\mathbb{N}}$ and $u \sim_{\mathrm{E}} v$ whenever (E, u) and (F, v) belong to $\mathbb{F}$ and $\mathrm{E} \leq^* \mathrm{F}$.*

$G \leq^* E \qquad (E, x) \in \mathbb{F}$

$\mathbb{F} \cup \{(G, x)\}$

Recall:

1. $u \sim_{\mathsf{E}} v$ if and only if $\mathrm{Ad}\, u$ and $\mathrm{Ad}\, v$ agree on $\mathcal{F}[\mathsf{E}]$,
2. we identify $\mathbb{T}^{\mathbb{N}}$ with $\mathcal{U}(\ell_\infty) \subseteq \mathcal{U}(\mathcal{B}(H))$,
3. $x \sim_{\mathsf{E}} y$ if and only if $\limsup_i \Delta_{E_i \cup E_{i+1}}(x, y) = 0$.

Def 17.8.1 *A family* $\mathbb{F}$ *of pairs* (E, x) *for* $\mathsf{E} \in \mathrm{Part}_{\mathbb{N}}$ *and* $x \in \mathbb{T}^{\mathbb{N}}$ *is a coherent family of unitaries if* $\{\mathsf{E} : (\mathsf{E}, x) \in \mathbb{F}$ *for some* $x\}$ *is* $\leq^*$*-cofinal in* $\mathrm{Part}_{\mathbb{N}}$ *and* $u \sim_{\mathsf{E}} v$ *whenever* (E, u) *and* (F, v) *belong to* $\mathbb{F}$ *and* $\mathsf{E} \leq^* \mathsf{F}$*.*

Recall:

1. $u \sim_{\mathsf{E}} v$ if and only if $\operatorname{Ad} u$ and $\operatorname{Ad} v$ agree on $\mathcal{F}[\mathsf{E}]$,
2. we identify $\mathbb{T}^{\mathbb{N}}$ with $\mathcal{U}(\ell_\infty) \subseteq \mathcal{U}(\mathcal{B}(H))$,
3. $x \sim_{\mathsf{E}} y$ if and only if $\limsup_i \Delta_{E_i \cup E_{i+1}}(x, y) = 0$.

Def 17.8.1 *A family $\mathbb{F}$ of pairs (E, x) for $\mathsf{E} \in \mathrm{Part}_{\mathbb{N}}$ and $x \in \mathbb{T}^{\mathbb{N}}$ is a coherent family of unitaries if $\{\mathsf{E} : (\mathsf{E}, x) \in \mathbb{F}$ for some $x\}$ is $\leq^*$-cofinal in $\mathrm{Part}_{\mathbb{N}}$ and $u \sim_{\mathsf{E}} v$ whenever (E, u) and (F, v) belong to $\mathbb{F}$ and $\mathsf{E} \leq^* \mathsf{F}$.*

The following was proved a couple of classes ago:

Lemma 17.1.4 *Every coherent family of unitaries $\mathbb{F}$ defines a unique automorphism $\Phi = \Phi_{\mathbb{F}}$ of $\mathcal{Q}(H)$ such that the restriction of Φ to $\mathcal{F}[\mathsf{E}]$ agrees with $\operatorname{Ad} u$ for every pair $(\mathsf{E}, u) \in \mathbb{F}$.*

Recall:

1. $u \sim_{\mathrm{E}} v$ if and only if $\mathrm{Ad}\, u$ and $\mathrm{Ad}\, v$ agree on $\mathcal{F}[\mathrm{E}]$,
2. we identify $\mathbb{T}^{\mathbb{N}}$ with $\mathcal{U}(\ell_\infty) \subseteq \mathcal{U}(\mathcal{B}(H))$,
3. $x \sim_{\mathrm{E}} y$ if and only if $\limsup_i \Delta_{E_i \cup E_{i+1}}(x, y) = 0$.

Def 17.8.1 *A family $\mathbb{F}$ of pairs (E, x) for $\mathrm{E} \in \mathrm{Part}_{\mathbb{N}}$ and $x \in \mathbb{T}^{\mathbb{N}}$ is a coherent family of unitaries if $\{\mathrm{E} : (\mathrm{E}, x) \in \mathbb{F}$ for some $x\}$ is $\leq^*$-cofinal in $\mathrm{Part}_{\mathbb{N}}$ and $u \sim_{\mathrm{E}} v$ whenever (E, u) and (F, v) belong to $\mathbb{F}$ and $\mathrm{E} \leq^* \mathrm{F}$.*

The following was proved a couple of classes ago:

Lemma 17.1.4 *Every coherent family of unitaries $\mathbb{F}$ defines a unique automorphism $\Phi = \Phi_{\mathbb{F}}$ of $\mathcal{Q}(H)$ such that the restriction of Φ to $\mathcal{F}[\mathrm{E}]$ agrees with $\mathrm{Ad}\, u$ for every pair $(\mathrm{E}, u) \in \mathbb{F}$.*

We will now prove:

Thm 17.8.2 *If $\mathrm{OCA_T}$ holds then the automorphism $\Phi_{\mathbb{F}}$ is inner for every coherent family of unitaries $\mathbb{F}$.*

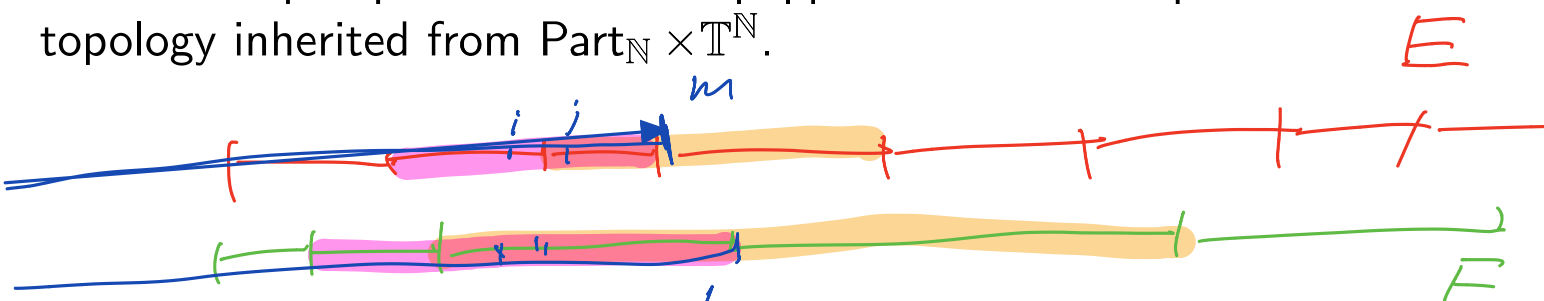
We say that such $\mathbb{F}$ is *trivial*.

Proof that $\mathrm{OCA_T}$ implies every coherent family of unitaries $\mathbb{F}$ is trivial

Fix $\mathbb{F}$. Fix $d \geq 1$ and define a partition $[\mathbb{F}]^2 = L_0^d \cup L_1^d$ by $\{(E, u), (F, v)\} \in L_0^d$ if

(L_0^d) For some m and n, the interval
$I := (E_m \cup E_{m+1}) \cap (F_n \cup F_{n+1})$ satisfies $\Delta_I(u, v) > 2^{-d}$.

This is an open partition if $\mathbb{F}$ is equipped with the subspace topology inherited from $\mathrm{Part}_{\mathbb{N}} \times \mathbb{T}^{\mathbb{N}}$.



$\{(E,u),(F,v)\} \in L_0^d$

Part

$\cancel{\Delta}\ \Gamma_n$ $\sigma \Lambda_n$

Proof that $\mathrm{OCA_T}$ implies every coherent family of unitaries $\mathbb{F}$ is trivial

Fix $\mathbb{F}$. Fix $d \geq 1$ and define a partition $[\mathbb{F}]^2 = L_0^d \cup L_1^d$ by $\{(\mathrm{E}, u), (\mathrm{F}, v)\} \in L_0^d$ if

(L_0^d) For some m and n, the interval $I := (E_m \cup E_{m+1}) \cap (F_n \cup F_{n+1})$ satisfies $\Delta_I(u, v) > 2^{-d}$.

This is an open partition if $\mathbb{F}$ is equipped with the subspace topology inherited from $\mathrm{Part}_{\mathbb{N}} \times \mathbb{T}^{\mathbb{N}}$.

Claim. *All L_0^d-homogeneous subsets of $\mathbb{F}$ are countable.*

Pf Assume otherwise, fix $d \geq 1$, $\underline{X} \subseteq \mathbb{F}$, $|X| = \aleph_1$, $[X]^2 \subseteq L_0^d$. wlog

$\underline{X}_0 = \{E \mid (\exists x)\ (E,x) \in \underline{X}\}$, $\underline{X}_0 \subseteq Part_{\mathbb{N}}$ $\boxed{|\underline{X}_0| \leq \aleph_1}$

(Assume $\forall E\ \exists_{\leq 1} x\ (E,x) \in \mathbb{F}$)

By the last claim, $\exists F \in Part$ such that $|\{E \in \underline{X}_0 \mid E \leq F\}| = \aleph_1$

w.l.o.g., $\exists \underline{x}$, $(F,x) \in \mathbb{F}$.

$(E_n \cup E_{n+1})$

$\cap (F_n \cup F_{n+1})$

$\subseteq E_n \cup E_{n+1}$

Fix (E,y), $E \in \underline{X}_0$

$(E,y), (F,x)$ $\quad x \sim_E y$

$\Delta_{(E_n \cup E_{n+1}) \cap (F_n \cup F_{n+1})}(x,y)$

$\lim_{n \to \infty} \Delta_{E_n \cup E_{n+1}}(x,y) \neq 0$.

Fix $k = k(E,y)$ $\forall n \geq k$

$\Delta_{E_n \cup E_{n+1}}(x,y) < 2^{-d-1}$

w.l.o.g., $\exists k = k(E,y)$, $\forall E \in \underline{X}_0$.

and $\ell = \min E_k$ is the same $\forall E \in \underline{X}_0$

ℓ

wlog, $\forall\ (E, \mathbf{y}), (E', \mathbf{y}')$ in $\mathcal{X}$

$d(\mathbf{y}\restriction \ell, \mathbf{y}'\restriction \ell) < 2^{-d-1}$

so, for $(E, \mathbf{y}), (E', \mathbf{y}')$

$$\Delta_{(E_n \cup E_{n+1} \mid A(E'_n \cup E'_{n+1}))}(\mathbf{y}, \mathbf{y}') < 2^{-d}$$

(F, x)

We have $\{(\mathrm{E}, u), (\mathrm{F}, v)\} \in L_1^d$ if

(L_1^d) For all m, n, the interval $I := (E_m \cup E_{m+1}) \cap (F_n \cup F_{n+1})$ satisfies $\Delta_I(u, v) \leq 2^{-d}$.

We have $\{(\mathrm{E}, u), (\mathrm{F}, v)\} \in L_1^d$ if

(L_1^d) For all m, n, the interval $I := (E_m \cup E_{m+1}) \cap (F_n \cup F_{n+1})$ satisfies $\Delta_I(u, v) \leq 2^{-d}$.

For $\mathbb{X} \subseteq \mathbb{F}$ write $\mathbb{X}_0 := \{\mathrm{E} : (\mathrm{E}, u) \in \mathbb{X}\}$.

Claim. *If $\mathbb{X}_0$ is $\leq^*$-cofinal in $\mathrm{Part}_{\mathbb{N}}$ and $\mathbb{X}$ is partitioned into countably many pieces, then for one of these pieces, $\mathbb{Y}$, the set $\mathbb{Y}_0$ is $\leq^*$-cofinal in $\mathrm{Part}_{\mathbb{N}}$.*

$OCA_T \Longrightarrow \forall d, \ \mathbb{X} = \bigcup_{n \in \mathbb{N}} \mathbb{Y}_n^d$

$[\mathbb{Y}_n^d]^2 \subseteq L_1^d$

We have $\{(\mathsf{E}, u), (\mathsf{F}, v)\} \in L_1^d$ if

(L_1^d) For all m, n, the interval $I := (E_m \cup E_{m+1}) \cap (F_n \cup F_{n+1})$ satisfies $\Delta_I(u, v) \leq 2^{-d}$.

For $\mathbb{X} \subseteq \mathbb{F}$ write $\mathbb{X}_0 := \{\mathsf{E} : (\mathsf{E}, u) \in \mathbb{X}\}$.

Claim. *If $\mathbb{X}_0$ is $\leq^*$-cofinal in $\mathrm{Part}_{\mathbb{N}}$ and $\mathbb{X}$ is partitioned into countably many pieces, then for one of these pieces, $\mathbb{Y}$, the set $\mathbb{Y}_0$ is $\leq^*$-cofinal in $\mathrm{Part}_{\mathbb{N}}$.*

By $\mathrm{OCA_T}$, $\mathbb{F}$ can be covered by countably many L_1^d-homogeneous sets. Recursively choose $\mathbb{F}(d) \subseteq \mathbb{F}$ for $d \geq 1$ so that for all d:

1. $[\mathbb{F}(d)]^2 \subseteq L_1^d$,
2. $\mathbb{F}(d) \supseteq \mathbb{F}(d+1)$, and
3. $\mathbb{F}(d)_0$ is $\leq^*$-cofinal in $\mathrm{Part}_{\mathbb{N}}$

$\mathbb{F}(1) \subseteq \mathbb{F}$

$\mathbb{F}(2) \subseteq \mathbb{F}(1)$

$L_1^d \supseteq L_1^{d+1}$

We have $\{(\mathsf{E}, u), (\mathsf{F}, v)\} \in L_1^d$ if

(L_1^d) For all m, n, the interval $I := (E_m \cup E_{m+1}) \cap (F_n \cup F_{n+1})$ satisfies $\Delta_I(u, v) \leq 2^{-d}$.

For $\mathbb{X} \subseteq \mathbb{F}$ write $\mathbb{X}_0 := \{\mathsf{E} : (\mathsf{E}, u) \in \mathbb{X}\}$.

Claim. *If $\mathbb{X}_0$ is $\leq^*$-cofinal in $\mathrm{Part}_{\mathbb{N}}$ and $\mathbb{X}$ is partitioned into countably many pieces, then for one of these pieces, $\mathbb{Y}$, the set $\mathbb{Y}_0$ is $\leq^*$-cofinal in $\mathrm{Part}_{\mathbb{N}}$.*

By $\mathrm{OCA_T}$, $\mathbb{F}$ can be covered by countably many L_1^d-homogeneous sets. Recursively choose $\mathbb{F}(d) \subseteq \mathbb{F}$ for $d \geq 1$ so that for all d:

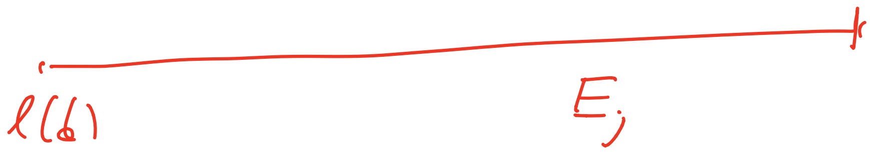
1. $[\mathbb{F}(d)]^2 \subseteq L_1^d$,
2. $\mathbb{F}(d) \supseteq \mathbb{F}(d+1)$, and
3. $\mathbb{F}(d)_0$ is $\leq^*$-cofinal in $\mathrm{Part}_{\mathbb{N}}$

By Lemma 9.7.9, for every d there are infinitely many k such that there is $\mathsf{f} = \langle F_0, \dots, F_{k-1} \rangle$ and for every m there is $\mathsf{E} \in [\mathsf{f}] \cap \mathcal{E}$ for which $\max(E_k) > m$.

Find $\ell(d)$, $d \in \mathbb{N}$,

such that ① $\bigcup \{ E_j \mid \exists x\ (E, x) \in \mathbb{F}(d),\ \min E_j = \ell(d) \} = [\ell(d), \infty)$

② $\ell(d) < \ell(d+1)$, $\forall d$



Fix $(E^{d,n}, x^{d,n}) \in \mathbb{F}(d)$, $n \in \mathbb{N}$,

so that ① $\underset{X^d}{\underline{\lim_{n \to \infty} x^{d,n} \in \mathbb{T}^{\mathbb{N}}}}$ exists.

② $\bigcup_j E_j^{d,n} = [\ell(d), \infty)$

Fact $\forall d$ $\forall (E, y) \in \mathbb{F}(d)$, $\forall j$

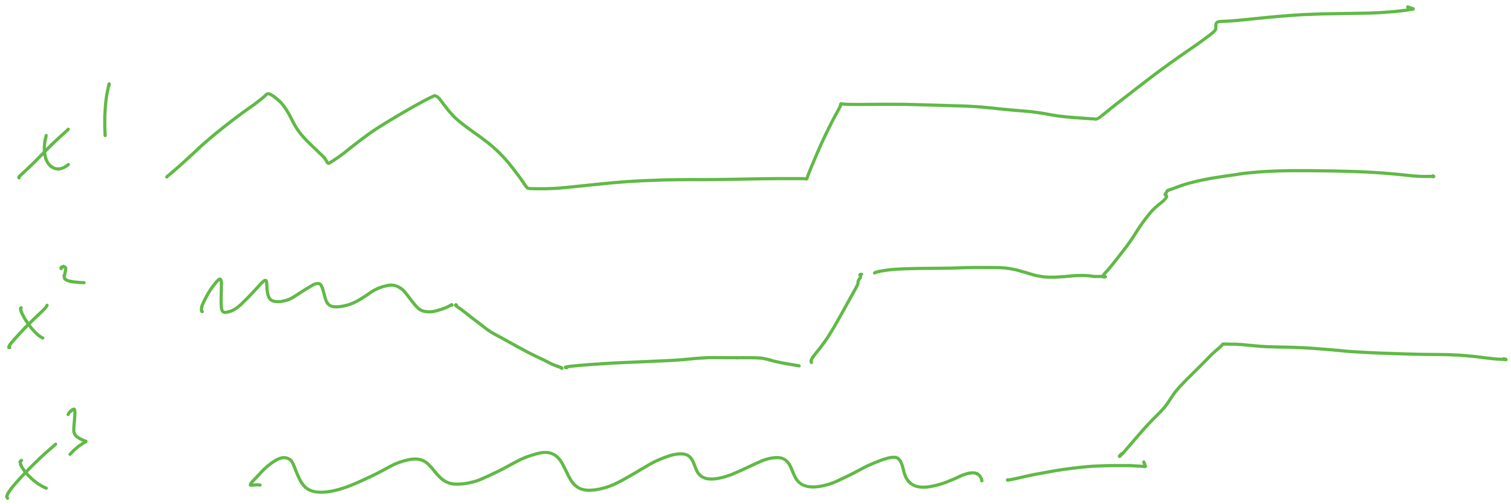
$$\underline{\Delta_{(E_j \cup E_{j+1}) \cap [\ell(d), \infty)} (y, X^d) \le 2^{-d}}$$

Therefore, $\forall d < d'$,

$$\Delta_{[\ell(d'), \infty)} (X^d, X^{d'}) \le 2^{-d}$$

Fact $\forall d$ $\exists \lambda_d \in \mathbb{T}$

$$\sup_{j \ge \ell(d+1)} |X^d(j) - \lambda_d X^{d+1}(j)| \le 2^{-d}$$



Define $z \in \mathbb{T}^{\mathbb{N}}$

$z(i)$, $i < \ell(0)$ — don't care

$i \in [\ell(d), \ell(d+1))$

$$z(i) = \prod_{j<d} \lambda_j \, x^d(i) \in \mathbb{T}^{\mathbb{N}}$$

Then $\forall d < d'$ $\Delta_{[\ell(d), \infty)}(z, x^{d'}) \leq 2^{-d}$

<u>claim</u> z implements $\phi_{\mathbb{F}}$ on $Q(H)$.

<u>Pf</u> Fix $E \in Part_{\mathbb{N}}$, $(E, x) \in \mathbb{F}$.

Fix $\varepsilon > 0$, d, $2^{-d} < \varepsilon$.

Fix $\underline{(F, y) \in \mathbb{F}(d)}$, $E \leq^* F$

y implements ϕ_F on $\mathcal{F}[E]$

$(x \sim_E y) \; . \quad \forall m$

$$\Delta_{(E_m \cup E_{m+1}) \cap [\ell(d), \infty)}(z, y) \leq 2^{-d}$$

$$\|(\mathrm{Ad}\, y)(a) - (\mathrm{Ad}\, z)(a)\| \leq 2^{-d+1} \leq 2\varepsilon$$

$$\forall a \in \mathcal{F}[E], \; \|a\| \leq 1.$$

Def 17.3.1 *A lifting of a *-homomorphism $\Phi\colon \mathcal{Q}(A) \to \mathcal{Q}(B)$ is a function $\Phi_*\colon \mathcal{M}(A) \to \mathcal{M}(B)$ such that the following diagram commutes (π_A and π_B denote the quotient maps).*

$[\Phi_*(a)] = \Phi([a])$

$$\begin{tikzcd}
\mathcal{X} \subseteq \mathcal{M}(A) \arrow[r, "\Phi_*"] \arrow[d, "\pi_A"'] & \mathcal{M}(B)) \arrow[d, "\pi_B"] \\
\pi[\mathcal{X}] \subseteq \mathcal{Q}(A) \arrow[r, Rightarrow, "\Phi"] & \mathcal{Q}(B) = \mathcal{M}(B)/B
\end{tikzcd}$$

$= \mathcal{M}(A)/A$

If this diagram commutes on some $\mathcal{X} \subseteq \mathcal{M}(A)$, then Φ_* is called a lifting of Φ on $\mathcal{X}$. When convenient, instead we say that Φ is a lifting on $\pi[\mathcal{X}]$.

$\mathcal{X} = \mathcal{F}[E]$

Liftings (§17.3)

Def 17.3.1 *A lifting of a *-homomorphism $\Phi\colon \mathcal{Q}(A) \to \mathcal{Q}(B)$ is a function $\Phi_*\colon \mathcal{M}(A) \to \mathcal{M}(B)$ such that the following diagram commutes (π_A and π_B denote the quotient maps).*

$$\begin{tikzcd} \mathcal{M}(A) \arrow[r, "\Phi_*"] \arrow[d, "\pi_A"'] & \mathcal{M}(B)) \arrow[d, "\pi_B"'] \\ \mathcal{Q}(A) \arrow[r, "\Phi"] & \mathcal{Q}(B) \end{tikzcd}$$

If this diagram commutes on some $\mathcal{X} \subseteq \mathcal{M}(A)$, then Φ_* is called a lifting of Φ on $\mathcal{X}$. When convenient, instead we say that Φ is a lifting on $\pi[\mathcal{X}]$.

Def 17.3.3 *A *-homomorphism Φ between coronas of separable C^*-algebras is said to be topologically trivial if its restriction to the unit ball has a lifting which is Borel-measurable with respect to the strict topology (this is a Polish topology).*

Shoenfield's Absoluteness

Example 17.3.5 *There is a separable abelian* C^**-algebra* A *such that* $\mathcal{Q}(A)$ *has a topologically trivial automorphism that cannot be lifted by a* **-homomorphism.*

Example 17.3.5 *There is a separable abelian* C^**-algebra* A *such that* $\mathcal{Q}(A)$ *has a topologically trivial automorphism that cannot be lifted by a* **-homomorphism.*

Lemma 17.3.6 *If* A *is a separable, nonunital* C^**-algebra then* $\mathcal{Q}(A)$ *has at most* $\mathfrak{c}$ *topologically trivial automorphisms.*

" $2^{\aleph_0}$

$|\text{Borel}| = 2^{\aleph_0}$

Example 17.3.5 *There is a separable abelian C^*-algebra A such that $\mathcal{Q}(A)$ has a topologically trivial automorphism that cannot be lifted by a *-homomorphism.*

Lemma 17.3.6 *If A is a separable, nonunital C^*-algebra then $\mathcal{Q}(A)$ has at most $\mathfrak{c}$ topologically trivial automorphisms.*

Corollary

$A \otimes \mathcal{K} \cong A$

If the Continuum Hypothesis holds and A is a separable, stable, nonunital C^*-algebra then $\mathcal{Q}(A)$ has topologically nontrivial automorphisms.

$2^{\aleph_1} > 2^{\aleph_0}$ (assuming $\aleph_1 = 2^{\aleph_0}$)

$2^{\aleph_1} > \aleph_1$