Massive C*-algebras, Winter 2021, I. Farah, Lecture 10

Recall from the last time: we started the proof of

Thm (Phillips–Weaver, 2008) *CH implies that $\mathcal{Q}(H)$ has $2^{\aleph_1}$ outer automorphisms.*

$\aleph_1 = 2^{\aleph_0}$

$\shortparallel$

$|\mathcal{Q}(H)|$

$\mathrm{Part}_{\mathbb{N}}$: the set of all partitions $\mathsf{E} = \langle E_j : j \in \mathbb{N}\rangle$ of $\mathbb{N}$, where $E_j = [n(j), n(j+1))$ and $n(0) < n(1) < n(2) < \dots$.

Def 9.7.2 *On* $\mathrm{Part}_{\mathbb{N}}$ *define*
$\mathsf{E} \leq^* \mathsf{F}$ *if* $(\forall^\infty m)(\exists n) E_n \subseteq F_m$ *(equivalently, if* $(\forall^\infty i)(\exists j) E_i \cup E_{i+1} \subseteq F_j \cup F_{j+1}$*)*

Def 9.7.5 *Consider* H *with an orthonormal basis* (ξ_n)*. For* $\mathsf{E} \in \mathrm{Part}_{\mathbb{N}}$ *and* $\mathsf{X} \subseteq \mathbb{N}$ *let* $p^{\mathsf{E}}_{\mathsf{X}} := \mathrm{proj}_{\overline{\mathrm{span}}\{\xi_i : i \in \bigcup_{n\in \mathsf{X}} E_n\}}$*, and let*

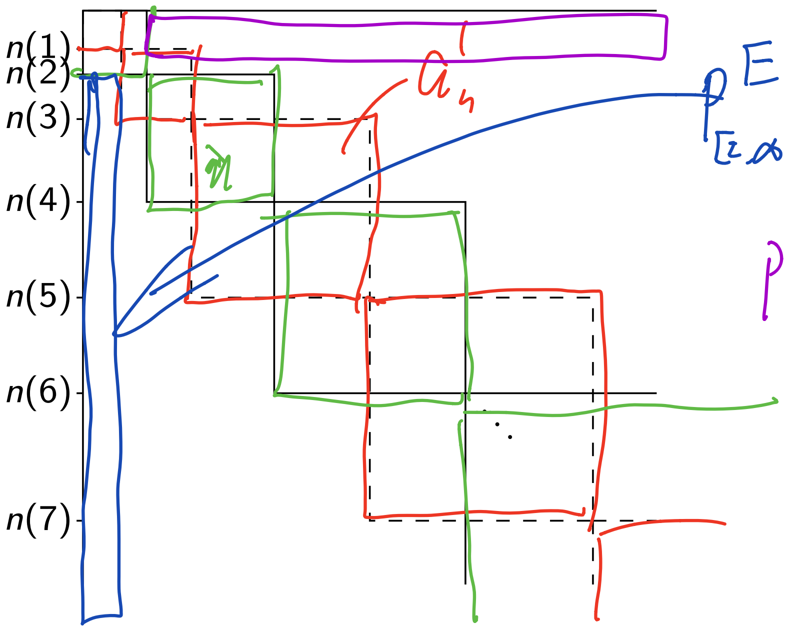
$$\mathcal{D}[\mathsf{E}] := \{a \in \mathcal{B}(H) : (\forall m)(\forall n)((a\xi_m|\xi_n) \neq 0 \text{ implies } (\exists j)\{m,n\} \subseteq E_j)\},$$

$$\mathcal{A}[\mathsf{E}] := \{\textstyle\sum_n \lambda_n p^{\mathsf{E}}_{\{n\}} | (\lambda_n) \in \ell_\infty\} \qquad (= \mathrm{W}^*\{p^{\mathsf{E}}_{\mathsf{X}} : \mathsf{X} \subseteq \mathbb{N}\}).$$

Lemma $\mathcal{D}[\mathsf{E}] \cong \prod_n M_{k(n)}(\mathbb{C})$ *and* $\mathcal{A}[\mathsf{E}] = Z(\mathcal{D}[\mathsf{E}])$.

For E $\in \mathrm{Part}_{\mathbb{N}}$ define two coarser partitions, E^{even} and E^{odd}, by (with $E_{-1} := \emptyset$)

$$E_n^{\mathrm{even}} := E_{2n} \cup E_{2n+1},$$
$$E_n^{\mathrm{odd}} := E_{2n-1} \cup E_{2n}.$$



$p^E_{[2,\infty)} \, a \, p^E_1$

$p^E_1 \, a \, p^E_{[2,\infty)}$

I owe you a proof of:

Lemma 9.7.6 *Let H be a Hilbert space with an orthonormal basis ξ_n, for $n \in \mathbb{N}$. For a sequence a_n, for $n \in \mathbb{N}$ in $\mathcal{B}(H)$ there are $\mathsf{E} \in \mathrm{Part}_{\mathbb{N}}$, $a_n^0 \in \mathcal{D}[\mathsf{E}^{\mathrm{even}}]$ and $a_n^1 \in \mathcal{D}[\mathsf{E}^{\mathrm{odd}}]$ such that $a_n - a_n^0 - a_n^1$ is compact for each n.*

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Proof: It suffices to show this for a single a. Choose $0 = n(0) < n(1) < \dots$ so that if $E_j := [n(j), n(j+1))$ then (recall that $p^{\mathsf{E}}_{\mathsf{X}} := \mathrm{proj}_{\overline{\mathrm{span}}\{\xi_i : i \in \bigcup_{n \in \mathsf{X}} E_n\}}$)

$$\|p^{\mathsf{E}}_{[n+1,\infty)} a p^{\mathsf{E}}_n\| < 2^{-n},$$

$$\|p^{E}_{n} a p^{E}_{[n+1,\infty)}\| = \|p^{\mathsf{E}}_{[n+1,\infty)} a^* p^{\mathsf{E}}_n\| < 2^{-n}.$$

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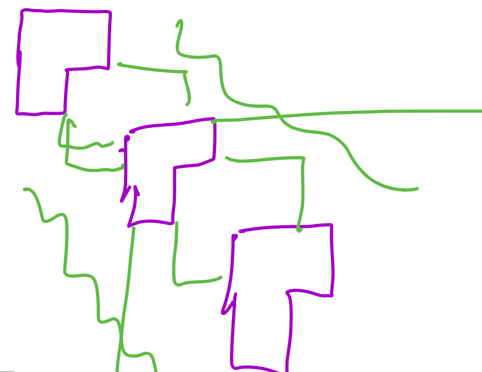


$$\|p_{[n+1,\infty)}^{\mathsf{E}} a p_n^{\mathsf{E}}\| < 2^{-n},$$
$$\|p_{[n+1,\infty)}^{\mathsf{E}} a^* p_n^{\mathsf{E}}\| < 2^{-n}.$$

Then let

$$a^0 := \sum_{n=0}^{\infty} \underline{p_{\{2n,2n+1\}}^{\mathsf{E}}} a \underline{p_{\{2n,2n+1\}}^{\mathsf{E}}} - \underline{p_{\{2n+1\}}^{\mathsf{E}} a p_{\{2n+1\}}^{\mathsf{E}}}$$
$$a^1 := \sum_{n=0}^{\infty} p_{\{2n+1,2n+2\}}^{\mathsf{E}} a p_{\{2n+1,2n+2\}}^{\mathsf{E}} - p_{\{2n+2\}}^{\mathsf{E}} a p_{\{2n+2\}}^{\mathsf{E}}$$

2^{-k}

$a - a^0 - a^1$

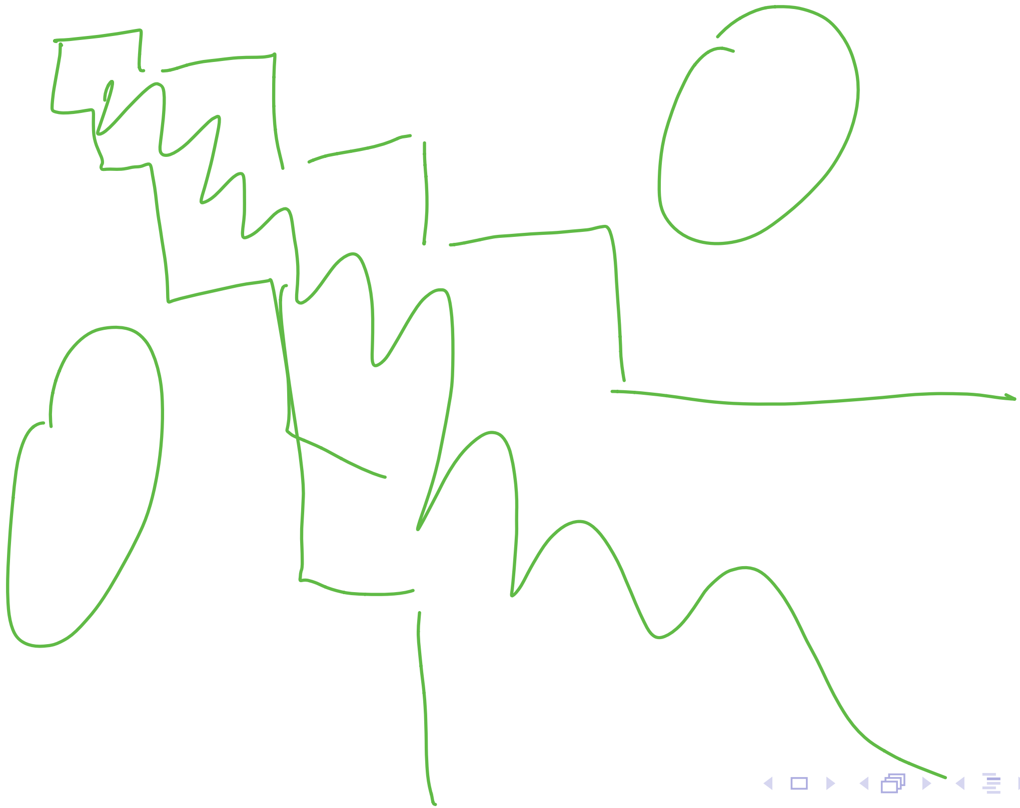
Exercise. Suppose that A is a σ-unital, non-unital C^*-algebra that has an approximate unit (r_n) consisting of projections. Formulate and prove an analog of Lemma 9.7.6 for a countable subset of $\mathcal{M}(A)$ (with the appropriately defined $\mathcal{D}[\mathsf{E}\,\mathrm{even}]$ and $\mathcal{D}[\mathsf{E}^{\mathrm{odd}}]$).

A solution to this exercise, together with the upcoming proof, will show that CH implies $\mathcal{M}(A)/A$ has $2^{\aleph_1}$ automorphisms when $A \cong B \otimes \mathcal{K}(H)$ for a unital C^*-algebra B.

Let

$$\mathcal{F}[\mathsf{E}] := \{a_0 + a_1 : a_0 \in \mathcal{D}[\mathsf{E}^{\mathrm{even}}], a_1 \in \mathcal{D}[\mathsf{E}^{\mathrm{odd}}]\}.$$

This is a Banach subspace, but not a subalgebra, of $\mathcal{B}(H)$.



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Lemma 17.1.1 *For* $\mathsf{E} \in \mathrm{Part}_{\mathbb{N}}$ *we have*

$$\mathcal{F}[\mathsf{E}] = \{a \in \mathcal{B}(H) : (\forall m \in \mathbb{N})(\forall n \in \mathbb{N}) \\ (a\xi_m|\xi_n) \neq 0 \textit{ implies } (\exists j)\{m, n\} \subseteq E_j \cup E_{j+1}\}.$$

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Prop 17.1.2 *For every separable subalgebra* A *of* $\mathcal{B}(H)$ *there is* $\mathsf{E} \in \mathrm{Part}_{\mathbb{N}}$ *such that* $\pi[A] \subseteq \pi[\mathcal{F}[\mathsf{E}]]$.

$\pi : \mathcal{B}(H) \to \mathcal{Q}(H)$

$E \leq^* F \Rightarrow \pi[\mathcal{F}[E]] \subseteq \pi[\mathcal{F}[F]]$

$\ell_\infty(\mathbb{N})$

Write $\mathcal{U}(A)$ for the unitary group of A. Identify $\mathcal{U}(\ell_\infty)$ with $\mathbb{T}^{\mathbb{N}}$, and identify ℓ_∞ with the algebra of diagonal operators (with respect to a fixed basis) in $\mathcal{B}(H)$.

$\mathbb{T} = \{ z \in \mathbb{C}$
$|z| = 1 \}$

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Def 17.1.3 *For* $\mathsf{E} \in \mathrm{Part}_{\mathbb{N}}$ *and* u *and* v *in* $\mathbb{T}^{\mathbb{N}}$ *we write* $u \sim_{\mathsf{E}} v$ *if* $uau^* - vav^* \in \mathcal{K}(H)$ *for all* $a \in \mathcal{F}[\mathsf{E}]$.

$(\mathrm{Ad}\, u)(a)$

$\Updownarrow$

$\mathrm{Ad}\, \dot{u}\,[\pi[\mathcal{F}[E]]]$

$= \mathrm{Ad}\, \dot{v}\,[\pi[\mathcal{F}[E]]]$

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Lemma 17.1.4 *Suppose* $\mathcal{E}$ *is a* $\leq^*$*-cofinal subset of* $\mathrm{Part}_{\mathbb{N}}$ *and* $u_{\mathsf{E}} \in \mathbb{T}^{\mathbb{N}}$, *for* $\mathsf{E} \in \mathcal{E}$, *satisfy* $u_{\mathsf{E}} \sim_{\mathsf{E}} u_{\mathsf{F}}$ *whenever* $\mathsf{E} \leq^* \mathsf{F}$ *for* E *and* F *in* $\mathcal{E}$. *Then there exists a unique automorphism of* $\mathcal{Q}(H)$ *which agrees with* $\mathrm{Ad}\,\pi(u_{\mathsf{E}})$ *on* $\pi[\mathcal{F}[\mathsf{E}]]$ *for all* $\mathsf{E} \in \mathcal{E}$.

$\mathcal{F}[F]$

$\cap$

$\mathcal{F}[E]$

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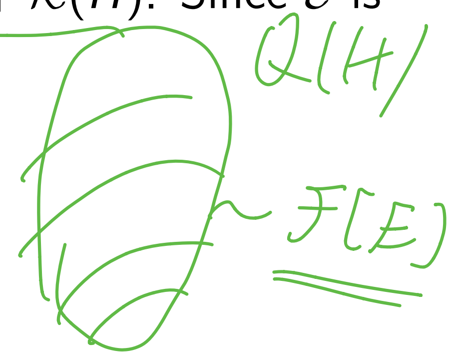
Proof: Note that $\mathsf{E} \leq^* \mathsf{F}$ implies $\mathcal{F}[\mathsf{E}] \subseteq \mathcal{F}[\mathsf{F}] + \mathcal{K}(H)$. Since $\mathcal{E}$ is cofinal, $\mathcal{Q}(H) = \bigcup_{\mathsf{E}\in\mathcal{E}} \pi[\mathcal{F}[\mathsf{E}]]$.



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$a \in \pi[\mathcal{F}[E]]$

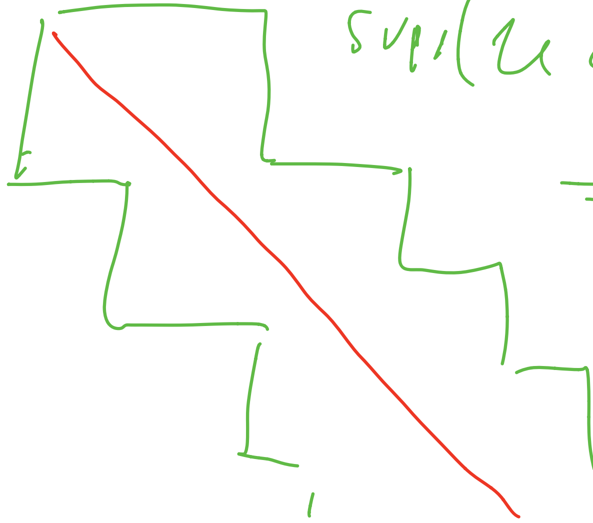
For $a \in \mathcal{Q}(H)$ let

$$\Phi(a) := \mathrm{Ad}(\pi(u_{\mathsf{E}}))(a), \text{ for } \mathsf{E} \in \mathcal{E} \text{ such that } a \in \pi[\mathcal{F}[\mathsf{E}]].$$

$\mathrm{Ad}\,u[\mathcal{F}[E]] = \mathcal{F}[E]$

$a, b \in \mathcal{F}[E]$ $\quad \phi(ab) = \phi(a)\phi(b)$ $\quad u \in \ell_\infty$

$a \in \mathcal{F}(E)$



$\sup(u a u^*)$

$= \sup(a)$

$$\phi[Q(H)] = Q(H)$$

$$\phi[\pi(\mathcal{F}(E))] = \pi(\mathcal{F}(E)),$$

$$\forall E.$$

Constructing an outer automorphism of $\mathcal{Q}(H)$ (using CH)

The plan:

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3. Then $(\mathsf{E}_\alpha, u_\alpha)$, for $\alpha < \aleph_1$, defines an automorphism of $\mathcal{Q}(H)$ (Lemma 17.1.4).
4. We can also enumerate $\mathcal{U}(\mathcal{Q}(H))$ as w_α, for $\alpha < \aleph_1$ and assure that $w_\alpha \not\sim_{\mathsf{E}_\alpha} u_\alpha$, and therefore $\Phi \neq \mathrm{Ad}\, w_\alpha$, for all α.

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5. Even better, we can recursively (along $\underline{\{0,1\}^{<\aleph_1}}$) construct $2^{\aleph_1}$ distinct automorphisms (and $2^{\aleph_1} > \aleph_1 = 2^{\aleph_0}$).

Recall that $u \sim_{\mathrm{E}} v \Leftrightarrow uau^* - vav^* \in \mathcal{K}(H)$ for all $a \in \mathcal{F}[\mathrm{E}]$; we need a working reformulation of this relation (Lemma 17.1.9 below).

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Two illuminating remarks:

Recall that $u \sim_E v \Leftrightarrow uau^* - vav^* \in \mathcal{K}(H)$ for all $a \in \mathcal{F}[E]$; we need a working reformulation of this relation (Lemma 17.1.9 below).
Two illuminating remarks:
(1) The following lemma will not be used explicitly:

Lemma *Suppose $A \leq C$, and u, v are in $\mathcal{U}(C)$. TFAE:*

1. $\underline{\operatorname{Ad} u}(a) = \underline{\operatorname{Ad} v}(a)$ *for all* $a \in A$
2. $\underline{uv^*} \in \underline{C \cap A'}$.

$a \in A$

$uau^* = vav^*$

$\underline{v^*ua} = av^*u$ then

$v^*u \in C \cap A'$

$\Leftarrow$ $uv^* \in C \cap A'$

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$D[E^e]$

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$A[E^e]$

1. $\operatorname{Ad} u(a) = \operatorname{Ad} v(a)$ *for all* $a \in A$
2. $uv^* \in C \cap A'$.

(2) The following strengthening of Theorem 12.3.2 can be used to shorten the discussion:

Thm (Popa, J. Func. Anal 71, 393–408 (1987)) *If M is a von Neumann subalgebra of $\mathcal{B}(H)$, then $\mathcal{Q}(H) \cap \pi[M]' = \pi[M']$.*

$\supseteq$

For i and j in $\mathbb{N}$, x and y in $\mathbb{T}^{\mathbb{N}}$, and $F \subseteq \mathbb{N}$ let

$$\Delta_{\{i,j\}}(x,y) := |x(i)\overline{x(j)} - y(i)\overline{y(j)}|, \text{ and} \tag{1}$$

$$\Delta_F(x,y) := \sup_{i,j \in F} \Delta_{\{i,j\}}(x,y). \tag{2}$$

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Lemma 17.1.5 *If $F \subseteq \mathbb{N}$ is nonempty, i, j are in $\mathbb{N}$, and x, y, z are in $\mathbb{T}^{\mathbb{N}}$ then the following hold.*

1. $\Delta_{\{i,j\}}(x,y) = |x(i)\overline{y(i)} - x(j)\overline{y(j)}|$.

$$\overline{x(j)}\,x(j) = 1$$

$$|x(i)\overline{x(j)} - y(i)\overline{y(j)}|$$
$$= |x(i) - y(i)\overline{y(j)}\,x(j)|$$
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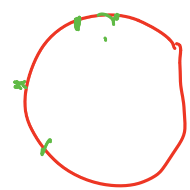
1. $\Delta_{\{i,j\}}(x,y) = |x(i)\overline{y(i)} - x(j)\overline{y(j)}|$.
2. $\Delta_F(x,1) = \operatorname{diam}(\{x(i) : i \in F\})$.

$x(i)$



$y = 1$

$\Delta_{i,j}(x,1) = |x(i) - x(j)|$

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2. $\Delta_F(x,1) = \operatorname{diam}(\{x(i) : i \in F\})$.
3. $\Delta_{\{i,k\}}(x,y) \le \Delta_{\{i,j\}}(x,y) + \Delta_{\{j,k\}}(x,y)$, *hence* $\Delta_{\{\cdot,\cdot\}}(x,y)$ *is a pseudometric on* $\mathbb{N}$.

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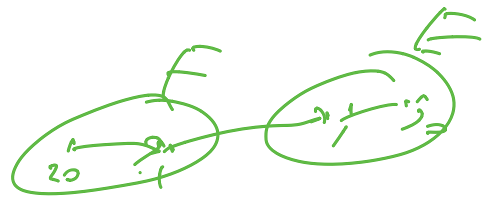
4. $\Delta_F(x,z) \leq \Delta_F(x,y) + \Delta_F(y,z)$, *hence* Δ_F *is a pseudometric on* $\mathbb{T}^{\mathbb{N}}$.

For i and j in $\mathbb{N}$, x and y in $\mathbb{T}^{\mathbb{N}}$, and $F \subseteq \mathbb{N}$ let

$$\Delta_{\{i,j\}}(x,y) := |x(i)\overline{x(j)} - y(i)\overline{y(j)}|, \text{ and} \tag{1}$$

$$\Delta_F(x,y) := \sup_{i,j \in F} \Delta_{\{i,j\}}(x,y). \tag{2}$$

Lemma 17.1.5 *If $F \subseteq \mathbb{N}$ is nonempty, i,j are in $\mathbb{N}$, and x,y,z are in $\mathbb{T}^{\mathbb{N}}$ then the following hold.*



1. $\Delta_{\{i,j\}}(x,y) = |x(i)\overline{y(i)} - x(j)\overline{y(j)}|$.
2. $\Delta_F(x,1) = \mathrm{diam}(\{x(i) : i \in F\})$.
3. $\Delta_{\{i,k\}}(x,y) \leq \Delta_{\{i,j\}}(x,y) + \Delta_{\{j,k\}}(x,y)$, *hence $\Delta_{\{\cdot,\cdot\}}(x,y)$ is a pseudometric on $\mathbb{N}$.*
4. $\Delta_F(x,z) \leq \Delta_F(x,y) + \Delta_F(y,z)$, *hence Δ_F is a pseudometric on $\mathbb{T}^{\mathbb{N}}$.*
5. $\Delta_F(x,y) = \Delta_F(xz, yz)$.

$x(i)z(i)\overline{x(i)z(i)} = x(i)\overline{x(i)}$

Lemma 17.1.6 *Let F and E be finite subsets of $\mathbb{N}$. Then for all $i \in F$, $j \in E$, and all x and y we have*

$$\Delta_{F \cup E}(x, y) \leq \Delta_F(x, y) + \Delta_E(x, y) + \Delta_{\{i,j\}}(x, y).$$

Lemma 17.1.6 *Let F and E be finite subsets of $\mathbb{N}$. Then for all $i \in F$, $j \in E$, and all x and y we have $\Delta_{F \cup E}(x, y) \leq \Delta_F(x, y) + \Delta_E(x, y) + \Delta_{\{i,j\}}(x, y)$.*

Lemma *For* $\mathrm{E} \in \mathrm{Part}_{\mathbb{N}}$,

$$\Delta_{\mathrm{E}}(x, y) := \limsup_{j \to \infty} \Delta_{E_j \cup E_{j+1}}(x, y)$$

defines a pseudometric on $\mathbb{T}^{\mathbb{N}}$.

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$\forall^{\infty} j \ \exists i \ E_j \cup E_{j+1} \subseteq F_i \cup F_{i+1}$

Lemma 17.1.7 *If* $\mathsf{E} \le^* \mathsf{F}$ *and* x, y, z *are in* $\mathbb{T}^{\mathbb{N}}$ *then* $\Delta_{\mathsf{E}}(x,z) \le \Delta_{\mathsf{E}}(x,y) + \Delta_{\mathsf{E}}(y,z)$ *and* $\Delta_{\mathsf{E}}(x,y) \le \Delta_{\mathsf{F}}(x,y)$.

$$\lim_{j\to\infty} \Delta_{E_j \cup E_{j+1}}(x,1) = 0$$

$\Updownarrow$

$$\lim_{j\to\infty} \operatorname{dim}(\operatorname{Sp}(xP^{E}_{\{j,j+1\}}))$$

Def 17.1.8 *Let* $\mathsf{F}_\mathsf{E} := \{x \in \mathbb{T}^\mathbb{N} : \Delta_\mathsf{E}(x,1) = 0\}$, *and* $\mathsf{G}_\mathsf{E} := \mathbb{T}^\mathbb{N} / \mathsf{F}_\mathsf{E}$, *for* $\mathsf{E} \in \mathsf{Part}_\mathbb{N}$.

$$F_E \triangleleft \mathbb{T}^N$$

Def 17.1.8 *Let* $\mathsf{F_E} := \{x \in \mathbb{T}^{\mathbb{N}} : \Delta_{\mathsf{E}}(x,1) = 0\}$, *and* $\mathsf{G_E} := \mathbb{T}^{\mathbb{N}} / \mathsf{F_E}$, *for* $\mathsf{E} \in \mathsf{Part}_{\mathbb{N}}$.

$F_F \trianglelefteq F_E$

Then $\mathsf{F_E}$ is a subgroup of $\mathbb{T}^{\mathbb{N}}$ and $\mathsf{E} \leq^* \mathsf{F}$ implies $\mathsf{F_E} \supseteq \mathsf{F_F}$ and therefore $\mathsf{G_F} = \mathsf{G_E} / (\mathsf{F_F} / \mathsf{F_E})$. Also,

$$
\begin{tikzcd}
0 \arrow[r] & \mathsf{F_F} \arrow[r] \arrow[d, "\Delta\ \iota_{\mathsf{EF}}"'] & \mathbb{T}^{\mathbb{N}} \arrow[r] \arrow[d, "\mathrm{id}"'] & \mathsf{G_F} \arrow[r] \arrow[d, "\pi_{\mathsf{EF}}"'] & 0 \\
0 \arrow[r] & \mathsf{F_E} \arrow[r] & \mathbb{T}^{\mathbb{N}} \arrow[r] & \mathsf{G_E} \arrow[r] & 0
\end{tikzcd}
$$

Higson Corona

Def 17.1.8 *Let* $\mathsf{F}_\mathsf{E} := \{x \in \mathbb{T}^\mathbb{N} : \Delta_\mathsf{E}(x,1) = 0\}$, *and* $\mathsf{G}_\mathsf{E} := \mathbb{T}^\mathbb{N} / \mathsf{F}_\mathsf{E}$, *for* $\mathsf{E} \in \mathrm{Part}_\mathbb{N}$.

Then F_E is a subgroup of $\mathbb{T}^\mathbb{N}$ and $\mathsf{E} \leq^* \mathsf{F}$ implies $\mathsf{F}_\mathsf{E} \supseteq \mathsf{F}_\mathsf{F}$ and therefore $\mathsf{G}_\mathsf{F} = \mathsf{G}_\mathsf{E} / (\mathsf{F}_\mathsf{F} / \mathsf{F}_\mathsf{E})$. Also,

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{F}_\mathsf{F} \arrow[r] \arrow[d, "\iota_{\mathsf{E}\,\mathsf{F}}"'] & \mathbb{T}^\mathbb{N} \arrow[r] \arrow[d, "\mathrm{id}"'] & \mathsf{G}_\mathsf{F} \arrow[r] \arrow[d, "\pi_{\mathsf{E}\,\mathsf{F}}"'] & 0 \\
0 \arrow[r] & \mathsf{F}_\mathsf{E} \arrow[r] & \mathbb{T}^\mathbb{N} \arrow[r] & \mathsf{G}_\mathsf{E} \arrow[r] & 0
\end{tikzcd}$$

Lemma 17.1.9 *Suppose* $\mathsf{E} \in \mathrm{Part}_\mathbb{N}$ *and* u *and* v *belong to* $\mathbb{T}^\mathbb{N}$. *Then* $u \sim_\mathsf{E} v$ *if and only if* $\underline{uv^* \in \mathsf{F}_\mathsf{E}}$.

$\mathfrak{F}[E]$

$\mathrm{Ad}\, i \restriction \pi(\mathfrak{F}[E]) = \mathrm{Ad}\, i \restriction \pi[\mathfrak{F}[E]]$

$M_u(\mathfrak{C})$

Assume $uv^* \in F_E$. Fix $a \in \mathcal{J}[E]$

$uau^* - vav^* \in K(H) \iff$

Note: In $M_n(\mathbb{C})$, $u, v \in U(M_n(\mathbb{C}))$

then $\forall a \in M_n(\mathbb{C}), (\|a\| \leq 1)$

$\|uau^* - vav^*\| \simeq 0$

$\iff \dim(Sp(uv^*)) \simeq \emptyset$

A speedup of the relation $\leq^*$ on $\mathrm{Part}_{\mathbb{N}}$; the $\ll^*$ as defined here is not the same as $\ll^*$ defined earlier

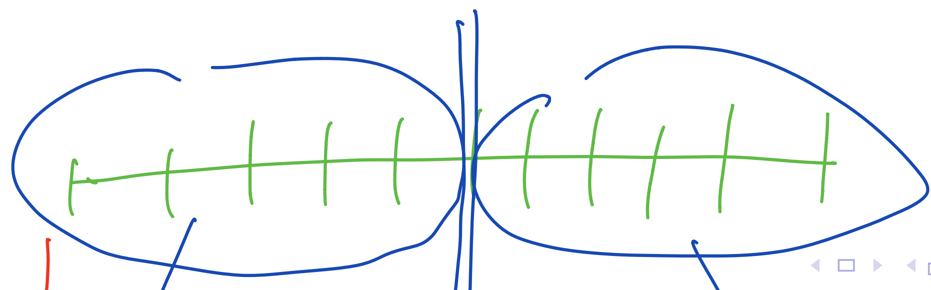
Let E $\ll^*$ F if E $\leq^*$ F and for every m there exist n and k such that $\bigcup_{j=n}^{n+m-1} E_j \subseteq F_k$.

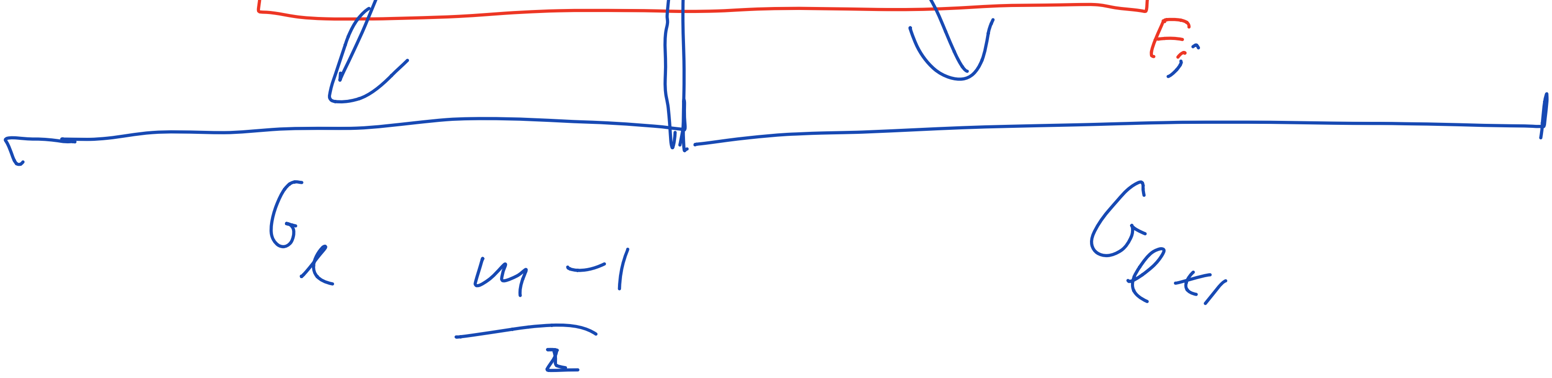
Lemma *$\ll^*$ is a partial ordering on $\mathrm{Part}_{\mathbb{N}}$ and* E $\ll^*$ F *implies* E $\leq^*$ F.

$E \ll^* F$, $F \ll^* G$

m

E





F_i
G_l
$\frac{m-1}{2}$
G_{l+1}

A speedup of the relation $\leq^*$ on $\mathrm{Part}_{\mathbb{N}}$; the $\ll^*$ as defined here is not the same as $\ll^*$ defined earlier

Let $\mathrm{E} \ll^* \mathrm{F}$ if $\mathrm{E} \leq^* \mathrm{F}$ and for every m there exist n and k such that $\bigcup_{j=n}^{n+m-1} E_j \subseteq F_k$.

Lemma *$\ll^*$ is a partial ordering on $\mathrm{Part}_{\mathbb{N}}$ and $\mathrm{E} \ll^* \mathrm{F}$ implies $\mathrm{E} \leq^* \mathrm{F}$.*

Lemma 17.1.10 *If $\mathrm{E} \ll^* \mathrm{F}$, then F_{F} is a proper subgroup of F_{E}.*

$$F_F = \{x \in 2^{\mathbb{N}} \mid \lim_{j\to\infty} \Delta_{F_j \cup F_{j+1}}(x, 1) = 0\}$$

$|4-1|$

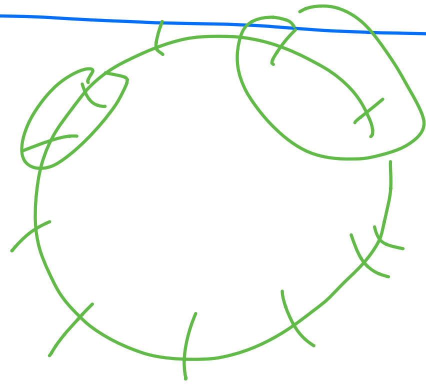
E_j E_{j+1} E_{j+k}

E

$\zeta \quad \zeta^2 \quad \zeta^3 \quad \cdots \quad \zeta^m = 1$

F

$$\zeta = e^{\frac{2\pi i}{m}}$$



2

Prop 17.1.11 *For every $\ll^*$-increasing sequence $\mathsf{E}(\alpha)$, for $\alpha < \aleph_1$, the inverse limit $\varprojlim_\alpha \mathsf{G}_{\mathsf{E}(\alpha)}$ has cardinality $2^{\aleph_1}$.*

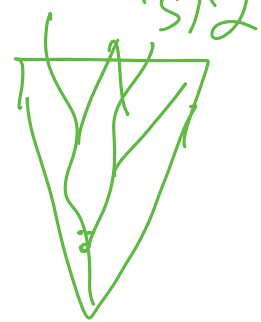
Pf For $s \in \{0,1\}^{<\aleph_1}$ find

$u_s \in G_{E(\alpha)}$ $\qquad \alpha = \mathrm{len}(s)$

s.t. $\alpha < \beta$ $\quad \pi_{\beta\alpha}(u_s) = u_{s\restriction\alpha}$

$\forall s$ $\quad \mathrm{len}(s) = \beta$, o.k.

$u_{s^\frown 0} \neq u_{s^\frown 1}$, $\quad \forall s$.



Prop 17.1.11 *For every* $\ll^*$*-increasing sequence* $\mathsf{E}(\alpha)$, *for* $\alpha < \aleph_1$, *the inverse limit* $\varprojlim_\alpha \mathsf{G}_{\mathsf{E}(\alpha)}$ *has cardinality* $2^{\aleph_1}$.

Proof: Write $\mathsf{G}(\alpha) := \mathsf{G}_{\mathsf{E}(\alpha)}$ and $\mathsf{F}(\alpha) := \mathsf{F}_{\mathsf{E}(\alpha)}$.

$G(\alpha) = \mathbb{T}^{\mathbb{N}} / F(\alpha)$

Prop 17.1.11 *For every* $\ll^*$*-increasing sequence* $\mathsf{E}(\alpha)$*, for* $\alpha < \aleph_1$*, the inverse limit* $\varprojlim_\alpha \mathsf{G}_{\mathsf{E}(\alpha)}$ *has cardinality* $2^{\aleph_1}$*.*

Proof: Write $\mathsf{G}(\alpha) := \mathsf{G}_{\mathsf{E}(\alpha)}$ and $\mathsf{F}(\alpha) := \mathsf{F}_{\mathsf{E}(\alpha)}$.

Claim. *If* α *is a countable limit ordinal then*

$$x \mapsto (\pi_{\mathsf{E}(\beta)\mathsf{E}(\alpha)}(x) : \beta < \alpha)$$

is a surjection from $\mathsf{G}(\alpha)$ *onto* $\varprojlim_{\beta<\alpha} \mathsf{G}(\beta)$*.*

Thm 17.1.12 *CH implies that the Calkin algebra has at least $2^{\aleph_1}$ automorphisms.*

$E(\omega_1) \nearrow \mathrm{Part}_{\mathbb{N}}$

$U(s)$, $s \in \{0,1\}^{<\aleph_1}$

(1) $s \subseteq t$ $\quad \underline{\mathrm{len}(s) = \alpha}$

$U_s \sim_{E(\omega_1)} U_t$

(2) $U_{s^\frown 0} \sim U_{s^\frown 1}$

$\geq 0 \quad E(\alpha+1) \geq 1$

$\phi_f \qquad u_{f \restriction \alpha} \qquad \alpha < \aleph_1$