We are continuing the proof of (a version of) Keisler's theorem: all ultrapowers of a fixed separable C^*-algebra associated with nonprincipal ultrafilters on $\mathbb{N}$ are isomorphic, assuming:
(CH) Every set of cardinality $\mathfrak{c}$ has a well-ordering such that every proper initial segment is countable (i.e., a well-ordering of *type* $\aleph_1$).
Recall:

Lemma *If C is a C^*-algebra of density character $\aleph_1$, then $C = \bigcup_{\alpha<\aleph_1} C_\alpha$ for a continuous $\aleph_1$-chain of separable elementary submodels C_α, for $\alpha < \aleph_1$.*

(*Continuous* means that $C_\beta = \lim_{\alpha<\beta} C_\alpha$ for every limit ordinal β.)

Lemma *Suppose that A and B have density character $\aleph_1$ and Φ is an isomorphism from A onto B. Then A and B can be represented as increasing unions of countable chains of separable elementary substructures, $A = \bigcup_\alpha A_\alpha$, $B = \bigcup_\alpha B_\alpha$, so that $\Phi[A_\alpha] = B_\alpha$ for all α.*

A correction to the statement made last time (thanks to Ben for pointing this out):

Fact

There are unital, separable, AF algebras A and B such that $A \equiv B$ and $A \not\cong B$, but no explicit example of a pair of such simple algebras is known.
There is a family of $\aleph_1$ abelian examples (Eagle–Vignati): $C(\alpha + 1)$, for α a countable indecomposable ordinal.

$\alpha = \beta + \gamma$

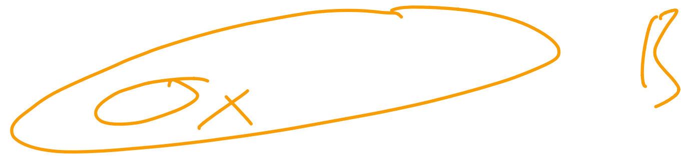
$\Rightarrow \alpha = \beta$ or $\alpha = \gamma$

$\omega + \omega$

ω^2

ω^3

Two facts about types and elementary equivalence



We can define a type over $X \subseteq A$, for X which is not a subalgebra.

Lemma *If* t *is a type over* $X \subseteq A$ *and* $\Phi\colon A \preceq C$, *then* t *is approximately satisfiable in* A *if and only if the (naturally defined) type* $\Phi(\mathrm{t})$ *(over* $\Phi[X]$*) is approximately satisfiable in* C.

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Exercise.

1. If $A \prec B$ and $B \prec C$, then $A \prec C$.
2. If $A \prec C$, $B \prec C$, and $A \subseteq B$, then $A \prec B$.
3. If $A_n \prec C$, and $A_n \subseteq A_{n+1}$ for all n, then $\overline{\bigcup_n A_n} \prec C$.

A universality property of countably saturated algebras

Thm *Suppose that C is countably saturated.*

1. *If A is separable and $A \equiv C$, then there is $\Phi\colon A \preceq C$.*
2. *If A and B are separable, $\Phi\colon A \preceq C$ and $\Psi\colon A \preceq B$, then there exists $\Theta\colon B \preceq C$ such that $\Theta \circ \Psi = \Phi$.*

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Proof: (1) Fix a sequence a_n, for $n \in \mathbb{N}$, dense in A. $\text{type}_A(a_0/\emptyset)$ is approximately satisfiable in C.

Fix $c_0 \in C$ $\quad \text{type}_A(a_0/\emptyset) = \text{type}_C(c_0/\emptyset)$

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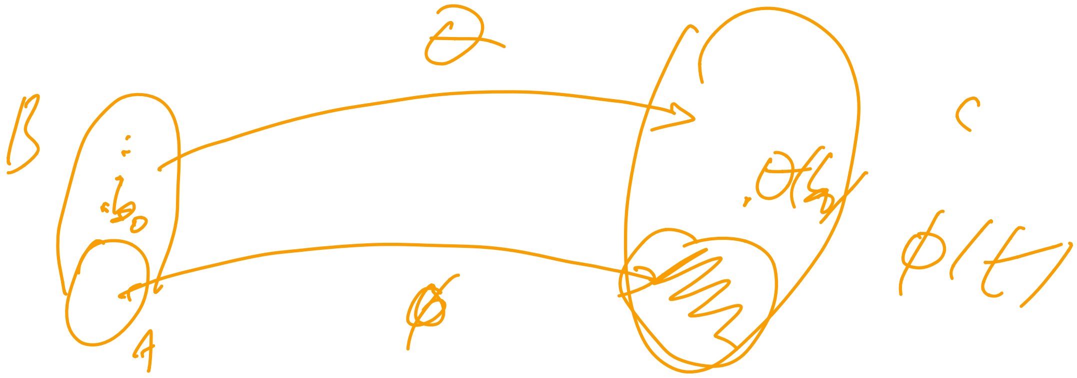
Proof: Fix a sequence a_n, for $n \in \mathbb{N}$, dense in A. $\mathrm{type}_A(a_0/\emptyset)$ is approximately satisfiable in C.

We'll define $\Phi(a_n)$ by recursion, so that for all n:

$$\Phi(\mathrm{type}_A(a_n/\{a_j | j < n\})) = \mathrm{type}_C(\Phi(a_n)/\{\Phi(a_j) | j < n\}).$$

$t = t_{\psi, e_B}(b_0/A)$



Thm (Corollary 16.6.5, Keisler) *Suppose that C and D are countably saturated, elementarily equivalent, and of density character $\aleph_1$. Then $C \cong D$.*

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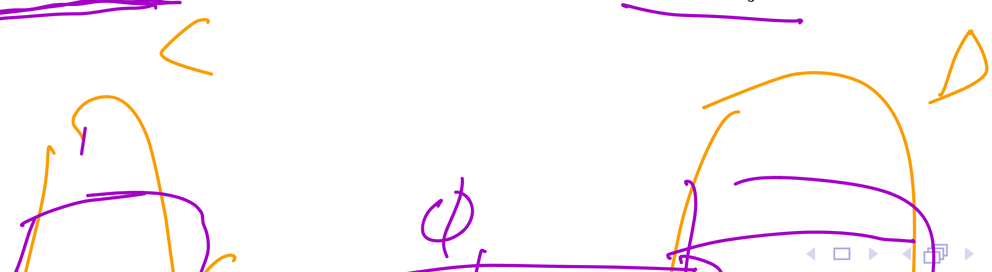
separable

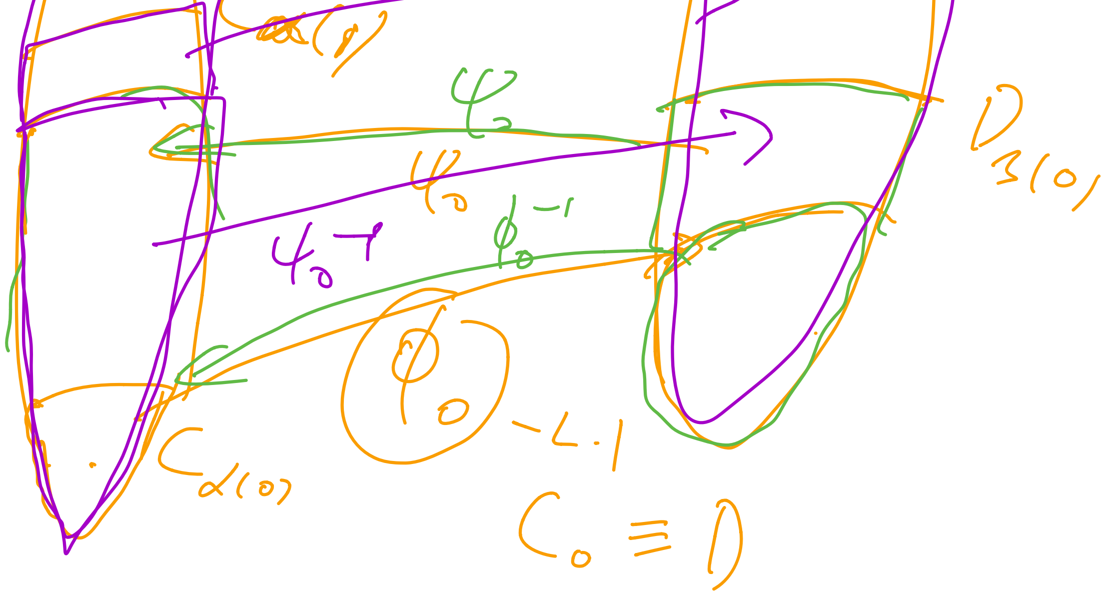
Thm (Corollary 16.6.5, Keisler) *Suppose that C and D are countably saturated, elementarily equivalent, and of density character $\aleph_1$. Then $C \cong D$.*

Proof: Write $C = \bigcup_{\alpha<\aleph_1} C_\alpha$ and $D = \bigcup_{\alpha<\aleph_1} D_\alpha$, continuous chains of elementary submodels. We will find continuous increasing families $\alpha(\xi)$, $\beta(\xi)$, $\xi < \aleph_1$, and elementary embeddings $\Phi_\xi \colon C_{\alpha(\xi)} \to D$ and $\Psi_\xi \colon D_{\beta(\xi)} \to C$ such that for all $\xi < \eta$

1. $\Phi_\xi[C_{\alpha(\xi)}] \subseteq D_{\beta(\xi)}$. $\Psi_\xi[D_{\beta(\xi)}] \subseteq C_{\alpha(\xi+1)}$.
2. $\Psi_\xi \circ \Phi_\xi(a) = a$, for $a \in C_{\alpha(\xi)}$, $\Phi_{\xi+1} \circ \Psi_\xi(b) = b$, for $b \in D_{\alpha(\xi)}$,
3. Φ_η extends Φ_ξ, Ψ_η extends Ψ_ξ.

Then $\Phi := \bigcup_\xi \Phi_\xi$ is an isomorphism, and $\Psi := \bigcup_\xi \Psi_\xi$ is its inverse.




ψ_2
ψ_0
ϕ_0^{-1}
ψ_0^{-1}
$D_{3}(0)$
$C_{\alpha}(0)$
$\phi_0 - L.1$
$C_0 = D$

The proof gives a more precise statement:

Thm (Corollary 16.6.5) *Suppose that C and D are countably saturated, elementarily equivalent, and of density character $\aleph_1$. Then $C \cong D$, and the isomorphism can be chosen so that it extends any fixed isomorphism $\Phi_0 \colon C_0 \to D_0$ between separable elementary submodels of C and D.*

Corollary

If the Continuum Hypothesis (CH) holds, A is a separable C^*-algebra, and $\mathcal{U}$ and $\mathcal{V}$ are nonprincipal ultrafilters on $\mathbb{N}$, then there is an isomorphism $\Phi\colon A_{\mathcal{U}} \to A_{\mathcal{V}}$ that commutes with the diagonal embedding of A.

$A \prec A_{\mathcal{U}}$

$|$

$A \prec A_{\mathcal{V}}$

$A_{\mathcal{U}} \cap A' \to A_{\mathcal{V}} \cap A'$

[1] All ultrafilters are nonprincipal and over $\mathbb{N}$.

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If A is a separable C^*-algebra, then CH implies that all ultrapowers of A on $\mathbb{N}$ are isomorphic, and all relative commutants of A in its ultrapowers are isomorphic.[1]

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Corollary

There are nonisomorphic separable, simple, AF algebras with isomorphic ultrapowers.

2004 $Th(A)$ $Th(A_{\mathcal{U}} \cap A') = ?$

Prop (Kirchberg) $A \equiv B$ *does not imply* $A_{\mathcal{U}} \cap A' \equiv B_{\mathcal{U}} \cap B'$*, even for separable, simple, A and B.*

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$$R^{\mathcal{U}} := \ell_\infty(R)/\{(a_n) \mid \lim_{n\to\mathcal{U}} \|a_n\|_2 = 0\}.$$

$A^{\mathcal{U}} \cap A'$

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Thus $R^{\mathcal{U}} \cong (M_{2^\infty})^{\mathcal{U}}$ and $R^{\mathcal{U}} \cap R' \cong (M_{2^\infty})^{\mathcal{U}} \cap (M_{2^\infty})'$.

Corollary

If $\mathcal{U}$ and $\mathcal{V}$ are nonprincipal ultrafilters on $\mathbb{N}$, then $R^{\mathcal{U}} \cong R^{\mathcal{V}}$ and $R^{\mathcal{U}} \cap R' \cong R^{\mathcal{V}} \cap R'$.

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(CH)

Corollary

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Before (very briefly) discussing the model in which these conclusions fail, we'll take a look at automorphisms of ultrapowers.

$(L_\infty([0,1], \lambda))$ $\quad \|f\|_{2,\lambda} = \int |f|^2 d\lambda$

Thm *Suppose that C is countably saturated and of density character $\aleph_1$. Then C has $2^{\aleph_1}$ automorphisms.*

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Lemma *Suppose that C is countably saturated, $A \prec C$ is separable, and $b \in C \setminus A$. Then the set*

$$\{c \in C \,|\, \mathrm{type}_C(c/A) = \mathrm{type}_C(b/A)\}$$

is nonseparable.

Let $t = type_C(b/A)$

Let $\varepsilon = dist(b, A)$ $(\varepsilon > 0)$

Let $t_2(X, Y) = t(X) \cup t(Y) \cup \|X - Y\| \geq \varepsilon$

$b_n \quad n \in \mathbb{N}$

$t \ll t_2$

$\|X - b_n\| \geq \varepsilon.$

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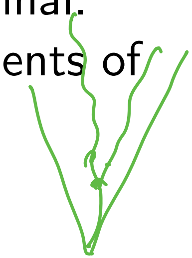
Now for the bookkeeping. Let $\{0,1\}^{<\aleph_1}$ denote the set of all functions $s\colon \alpha \to \{0,1\}$, where α is a countable ordinal. Maximal chains in $\{0,1\}^{<\aleph_1}$ correspond to the elements of $\{0,1\}^{\aleph_1}$.



Proof that if C is countably saturated and of density character $\aleph_1$ then C has $2^{\aleph_1}$ automorphisms.
Write $C = \bigcup_\alpha C_\alpha$, for a continuous chain of separable elementary submodels.
For $s \in \{0,1\}^{<\aleph_1}$ we will find separable $C_s \prec C$ and $D_s \prec C$ and an isomorphism $\Phi_s \colon C_s \to D_s$ (onto) so that for all s we have the following

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   For $s \in \{0,1\}^\alpha$, $C_{s\frown 0} \supseteq C_\alpha$, $C_{s\frown 1} \supseteq C\alpha$, $D_{s\frown 0} \supseteq C_\alpha$ and $D_{s\frown 1} \supseteq C\alpha$.

$\bigcup_\alpha C_{f\restriction\alpha} \supseteq \bigcup C_\alpha = C$

Q: Is there a model of ZFC s.t. all automorphisms of C_u (M_{2^∞}) are "trivial"?

$$\begin{array}{ccc} \ell_\infty(C) & \xrightarrow{\;?\;} & \ell_\infty(C) \\ \downarrow & & \downarrow \\ C_u & \xrightarrow{\;\phi\;} & C_u \end{array}$$

(See Shelah, Vive la différence, Pt. III.)

The dark side

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$|\beta\mathbb{N}| = 2^{2^{\aleph_0}}$

$\kappa \geq \aleph_1$

$\exists\, 2^{\kappa}$ very different

(i.e. orderings of card $= \kappa$.

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Thm (F.–Hart–Sherman (Maharam)) *All tracial ultrapowers of* $L_\infty([0,1],\lambda)$ *are isomorphic (even when the CH fails).*

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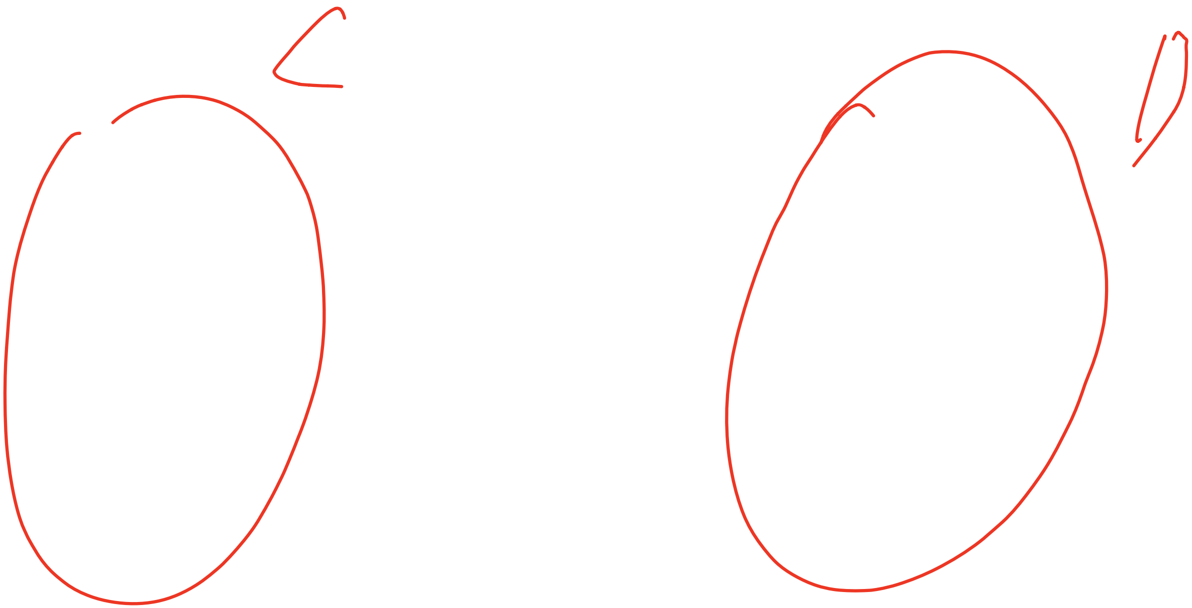
For all practical purposes, all ultrapowers are 'isomorphic'

Def 8.2.8 *Suppose C and D are nonseparable metric structures in the same language. A σ-complete back-and-forth system between C and D is a poset $\mathbb{F}$ with the following properties.*



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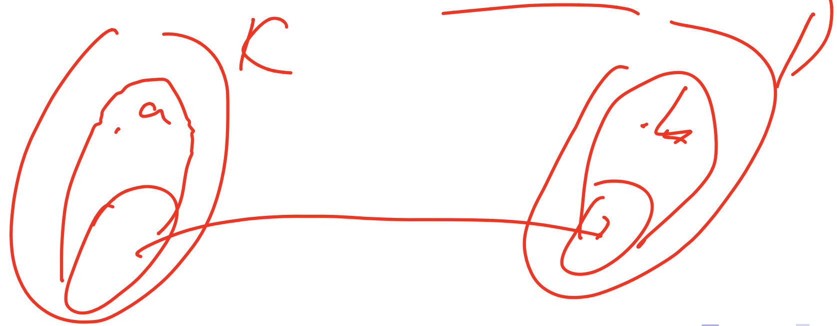
2. *The ordering is defined by $p \leq q$ if $A^p \subseteq A^q$, $B^p \subseteq B^q$, and $\Phi^q \restriction A^p = \Phi^p$.*
3. *For every $p \in \mathbb{F}$ and all $a \in A$ and $b \in B$ there exists $q \geq p$ in $\mathbb{F}$ such that $a \in A^q$ and $b \in B^q$.*
4. *$\mathbb{F}$ is σ-complete: For every increasing sequence $p(n)$, for $n \in \mathbb{N}$, in $\mathbb{F}$ we require that (identifying a function with its graph) $p := (\overline{\bigcup_n A^{p(n)}}, \overline{\bigcup_n B^{p(n)}}, \overline{\bigcup_n \Phi^{p(n)}})$ belongs to $\mathbb{F}$. We write $p = \sup_n p_n$.*



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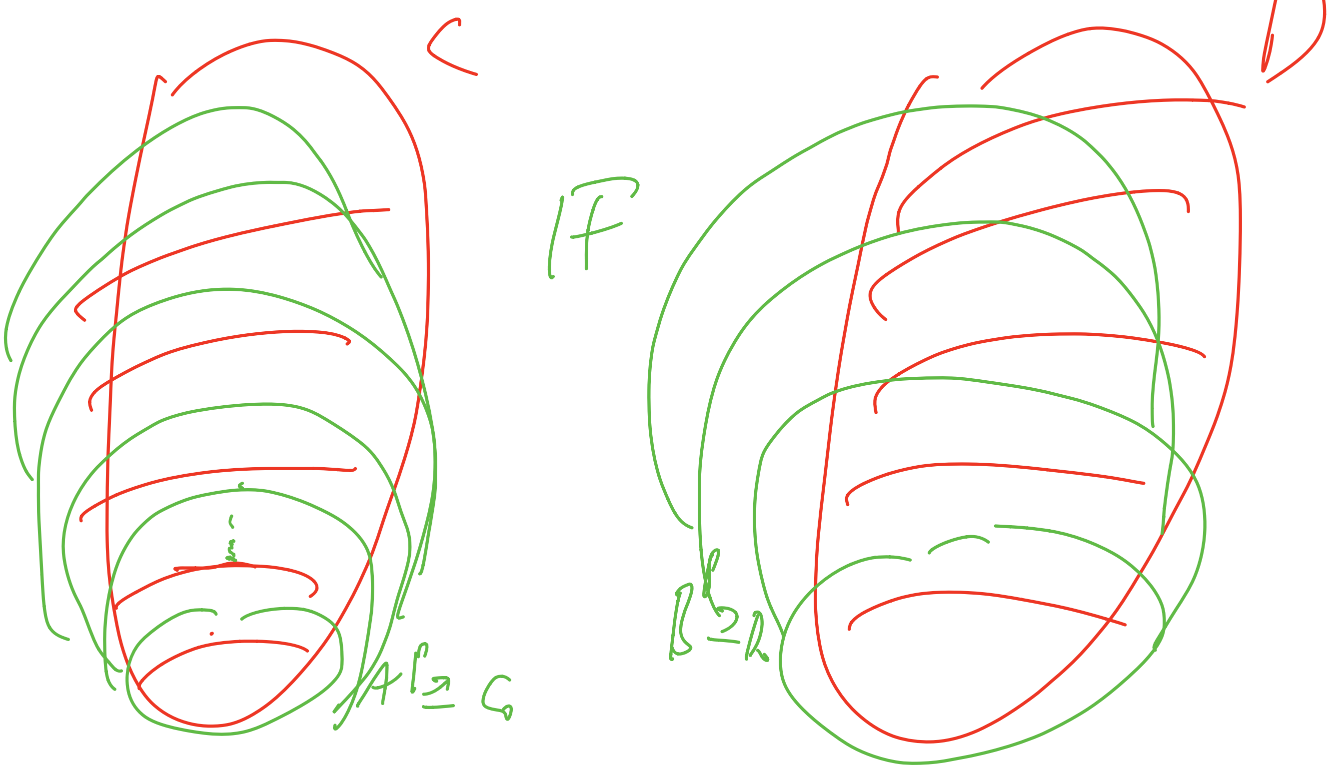
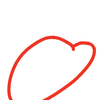
Prop 16.6.1 *Suppose C and D are metric structures of density character $\aleph_1$. The following are equivalent.*

1. *There exists a σ-complete back-and-forth system between C and D.*
2. *The structures C and D are isomorphic.*

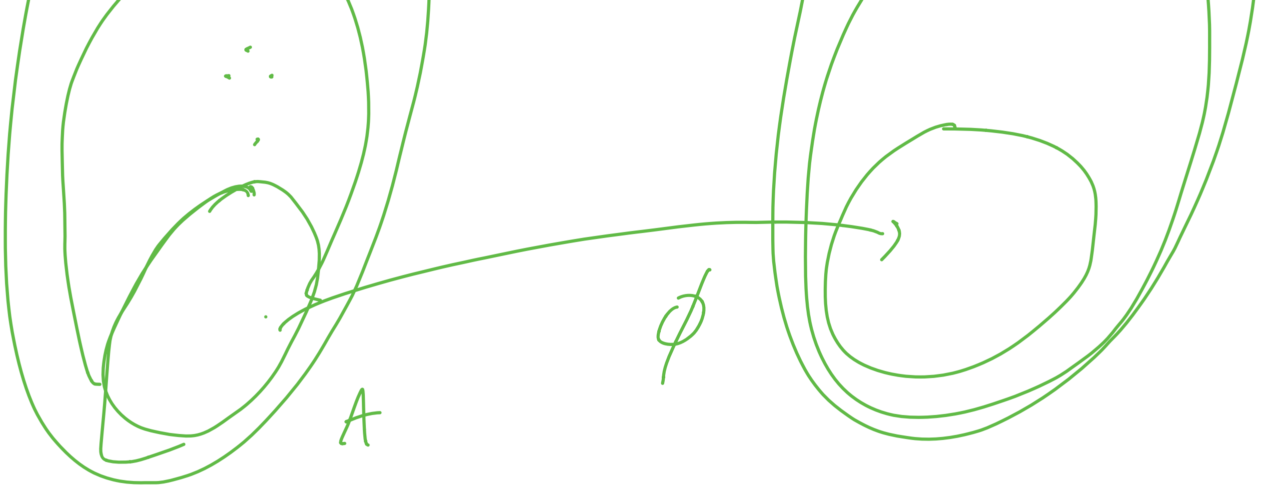

C
D
If
A ⊆ C
B ⊇ D



Thm 16.6.4 *Suppose C and D are countably saturated metric structures. The following are equivalent.*

1. *The metric structures C and D are elementarily equivalent.*
2. *There exists a σ-complete back-and-forth system between C and D.*

Proof: This is what the proof that CH implies $C \cong D$ gives in the absence of CH.



$$\mathbb{F} = \{ (A, B, \phi) \mid A < C,\ B < D$$

set!

$$\phi : A \xrightarrow{\cong} B \}$$

The asymptotic sequence algebra
(a few theorems with proofs omitted; see §16.3, §16.5)

$c_0(B) := \{(a_n) \in \ell_\infty(B) \mid \lim_n \|a_n\| = 0\}$.
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Feferman-Vaught type

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Thm 16.5.1 B_∞ *is countably saturated for every* B.

The asymptotic sequence algebra (a few theorems with proofs omitted; see §16.3, §16.5)

$c_0(B) := \{(a_n) \in \ell_\infty(B) \mid \lim_n \|a_n\| = 0\}$.

The algebra $B_\infty := \ell_\infty(B)/c_0(B)$ is the *asymptotic sequence algebra*.

Łoś's Theorem generally fails for the diagonal embedding of B into B_∞.

Thm 16.3.1 (Ghasemi) *It is possible to compute* $\mathrm{Th}(B_\infty)$ *from* $\mathrm{Th}(B)$.

Thm 16.5.1 B_∞ *is countably saturated for every* B.

Thm (F., 2020) *For any separable B we have*

1. $B \cong B \otimes C(K)$ *(K denotes the Cantor space).*
2. *Any 'diagonal' copy of $B \otimes C(K)$ in B_∞ is an elementary submodel.*
3. *CH implies that $B_\infty \cong (B \otimes C(2^{\mathbb{N}}))_{\mathcal{U}}$, for any nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$.*

Another theorem with its proof omitted

Thm (F.–Hart–Rørdam–Tikuisis, 2017) *CH implies that*[3]

$$(M_{2^\infty})_{\mathcal{U}} \cap (M_{2^\infty})' \cong (M_{2^\infty})_{\mathcal{U}}.$$

(One can replace M_{2^∞} with any strongly self-absorbing C^*-algebra, or with R with respect to a tracial ultrapower.)

$$A_{\mathcal{U}} \cap A' < A_{\mathcal{U}}$$

[3] $\mathcal{U}$ is nonprincipal and over $\mathbb{N}$.

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1. Fin.
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3. $\{X \subseteq \mathbb{N} | \sum_{n \in X} \frac{1}{n+1} < \infty\}$.
4. $\{X \subseteq \mathbb{Q} | X$ is nowhere dense$\}$.
5. For a countable indecomposable ordinal α, $\{X \subseteq \alpha |$ the order type of X is $< \alpha\}$.
6. $\mathcal{Z}_0 := \{X \subseteq \mathbb{N} | \limsup_n |X \cap n|/n = 0\}$

Given a family of C^*-algebras B_j, for $j \in \mathbb{J}$, and an ideal $\mathcal{J}$ on $\mathbb{J}$, we let $\bigoplus_{\mathcal{J}} B_j := \{\bar{b} \in \prod_{j \in \mathbb{J}} B_j : \limsup_{j \to \mathcal{J}} \|b_j\| = 0\}$

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Ghasemi's Feferman–Vaught Theorem still holds, but countable saturation may fail. (There is no known characterization of ideals for which every associated reduced product is countably saturated.)