Massive C^*-algebras, Winter 2021

Ilijas Farah. Lecture 6, January 30. Recall:

$\mathfrak{F}_A$: the algebra of of formulas over A with a seminorm,

$$\|\varphi(\bar{x})\| = \sup_{B,\bar{b}} |\varphi^B(\bar{b})|.$$

$\mathfrak{W}_A$: the (real) Banach algebra $(\mathfrak{F}_A/\|\cdot\|)$.

Thm 16.2.8, Łoś's Theorem *If $\varphi(\bar{x}) \in \mathfrak{F}_A$ then*

$$\varphi^{\prod_{\mathcal{U}} A_j}(\bar{a}) = \lim_{j\to\mathcal{U}} \varphi^{A_j}(\bar{a}_j)$$

for every $\bar{a} = (\bar{a}_j)_{j\in\mathbb{J}}$ in $\prod_{\mathcal{U}} A_j$ of the appropriate sort.

Def 16.1.4 *If $A \leq C$ and $\bar{b} \in C^{\mathbb{N}}$, the type of $\bar{b}$, $\mathrm{type}_C(\bar{b}/A)$, is the character $\mathfrak{F}_A^{\bar{x}} \mapsto \mathbb{R}$: $\varphi(\bar{x}) \mapsto \varphi^C(\bar{b})$. Alternatively, type t is identified with the set of conditions $\varphi(\bar{x}) = r$, with $\varphi(\bar{x}) - r \in \ker(\mathrm{t})$.*

C is countably saturated if every countable, approximately satisfiable type over C is realized in it.

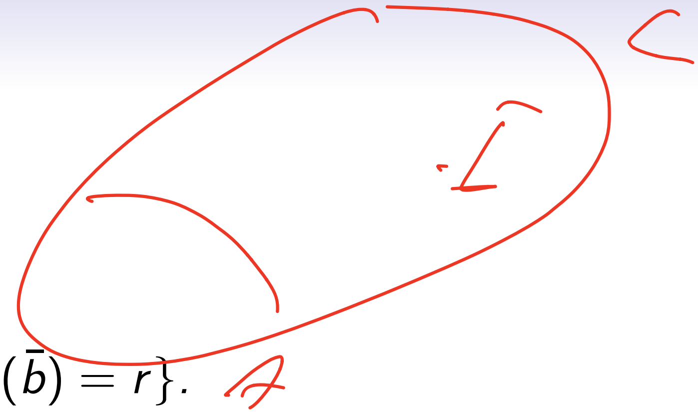
Thm 16.4.1 *Suppose that $\mathcal{U}$ is a countably incomplete ultrafilter on $\mathbb{J}$ and that A_j, for $j \in \mathbb{J}$, are C^*-algebras. Then the ultraproduct $C := \prod_{\mathcal{U}} A_j$ is countably saturated.*

Last time:

Lemma

If $A \leq C$, $\bar{b} \in C^n$, *then*
$\ker(\mathrm{type}_C(\bar{b}/A)) = \{\varphi(\bar{x}) - r : \varphi^C(\bar{b}) = r\}$.

Last time:



Lemma

If $A \leq C$, $b \in C^n$, then
$\ker(\mathrm{type}_C(\bar{b}/A)) = \{\varphi(\bar{x}) - r : \varphi^C(\bar{b}) = r\}$.

Thm *C is countably saturated if and only if for every $\underline{n \geq 1}$ and every separable $A \leq C$, the set*

$$\{\mathrm{type}_C(\bar{b}/A) | \bar{b} \in C^n\}$$

is weak-closed.*

Let's see why this is true (a vague discussion of the above result with no concrete applications follows).

The w^*-closure of $\{\mathrm{type}_C(\bar{b}/A) | \bar{b} \in C^n\}$ is the set of approx. satisfiable types $/A$.

A type $\mathrm{t}(\bar{x})$ over A can be identified with a functional $\tilde{\mathrm{t}} \colon \mathfrak{F}_A^{\bar{x}} \to \mathbb{R}$ such that

$$\tilde{\mathrm{t}}(\varphi) = r$$

if the condition $\varphi(\bar{x}) = r$ belongs to t. Suppose $\bar{x} = (x_0, \ldots, x_{n-1})$.

Lemma

A type $\mathrm{t}(\bar{x})$ *over* $A \leq B$ *is approximately satisfiable in* C *if and only if* $\tilde{\mathrm{t}}$ *is in the weak*-closure of* $\mathrm{type}_C(\bar{b}/A)$, *for* $\bar{b}$ *in* C^n

A formula φ is *quantifier-free* if it does not involve quantifiers sup or inf; that is, $\varphi = f(\psi_0, \ldots, \psi_{n-1})$ for atomic formulas ψ_j, $j < n$. Quantifier-free formulas over A form an algebra.

A formula φ is *quantifier-free* if it does not involve quantifiers sup or inf; that is, $\varphi = f(\psi_0, \ldots, \psi_{n-1})$ for atomic formulas ψ_j, $j < n$.
Quantifier-free formulas over A form an algebra.
Quantifier-free conditions, quantifier-free types, and quantifier-free saturation are defined in a natural way.
For $n \geq 2$, degree-n conditions, degree-n types, and degree-n saturation are defined in a natural way.

deg-2 : $\left.\begin{array}{l} \|X^*X - 1\| = 0 \\ \|XX^* - 1\| = 0 \end{array}\right)$ X is a unitary

A formula φ is *quantifier-free* if it does not involve quantifiers sup or inf; that is, $\varphi = f(\psi_0, \ldots, \psi_{n-1})$ for atomic formulas ψ_j, $j < n$.
Quantifier-free formulas over A form an algebra.
Quantifier-free conditions, quantifier-free types, and quantifier-free saturation are defined in a natural way.
For $n \geq 2$, degree-n conditions, degree-n types, and degree-n saturation are defined in a natural way.

Fact

Saturation $\Rightarrow$ *quantifier-free saturation* $\Rightarrow \ldots \Rightarrow$ *degree-*$n+1$ *saturation* $\Rightarrow$ *degree-n saturation* $\Rightarrow \ldots \Rightarrow$ *degree-2 saturation* $\Rightarrow$ *degree-1 saturation*

Q: Which, if any, of these arrows are reversible?

Fact: deg-2-sat. $\Rightarrow$ deg.-3 sat.

$XY \rightsquigarrow \|XY - z\| = 0$

$XY^2 = z$?

A formula φ is *quantifier-free* if it does not involve quantifiers sup or inf; that is, $\varphi = f(\psi_0, \ldots, \psi_{n-1})$ for atomic formulas ψ_j, $j < n$.
Quantifier-free formulas over A form an algebra.
Quantifier-free conditions, quantifier-free types, and quantifier-free saturation are defined in a natural way.
For $n \geq 2$, degree-n conditions, degree-n types, and degree-n saturation are defined in a natural way.

Fact

Saturation $\Rightarrow$ quantifier-free saturation $\Rightarrow \ldots \Rightarrow$ degree-$n+1$ saturation $\Rightarrow$ degree-n saturation $\Rightarrow$ $\ldots \Rightarrow$ degree-2 saturation $\Rightarrow$ degree-1 saturation

Q: Which, if any, of these arrows are reversible?

Prop *If C is countably saturated and $A \leq C$ is separable, then $A' \cap C$ is countably quantifier-free saturated but not necessarily countably saturated.*

sup
$\|A\|s$,

$A' \cap C$

(Proof available upon request.)

Notably, the proofs of Łoś's Theorem and countable saturation of ultraproducts have nothing to do with C^*-algebras. They are general theorems of model theory, applicable to arbitrary (appropriately defined) metric structures.
Let's take a look at a relevant example.

Tracial ultraproducts

Def *A state* τ *on a* C^**-algebra is a positive functional of norm 1.*
It is tracial if $\tau(ab) = \tau(ba)$ *for all a and b in A.*
$T(A) := \{\tau | \tau$ *is a tracial state on* $A\}$.

$\tau[A_+] \subseteq \mathbb{R}_+$

$M_n(\mathbb{C})$ $\frac{1}{n}$trace

$\bigotimes_{\mathbb{N}} M_2(\mathbb{C})$

$T(A) \subseteq (A^*)_1$

Tracial ultraproducts

Def *A state* τ *on a* C^**-algebra is a positive functional of norm 1. It is tracial if* $\tau(ab) = \tau(ba)$ *for all* a *and* b *in* A*.*
$T(A) := \{\tau | \tau \text{ is a tracial state on } A\}$.

Def *A unital* C^**-algebra* A *is finite if there is no* $v \in A$ *such that* $v^*v = 1_A$ *and* $vv^* < 1_A$*.*
It is stably finite if $M_n(A)$ *is finite for all* n*.*

Tracial ultraproducts

Def *A state* τ *on a* C^**-algebra is a positive functional of norm 1. It is tracial if* $\tau(ab) = \tau(ba)$ *for all* a *and* b *in* A*.*
$T(A) := \{\tau | \tau \text{ is a tracial state on } A\}$.

Def *A unital* C^**-algebra* A *is finite if there is no* $v \in A$ *such that* $v^*v = 1_A$ *and* $vv^* < 1_A$*.*
It is stably finite if $M_n(A)$ *is finite for all* n*.*

Fact
if $T(A) \neq \emptyset$ and A is simple *then* A *is stably finite.*

Tracial ultraproducts

Def *A state τ on a C^*-algebra is a positive functional of norm 1.*
It is tracial if $\tau(ab) = \tau(ba)$ for all a and b in A.
$T(A) := \{\tau | \tau$ *is a tracial state on* $A\}$.

Def *A unital C^*-algebra A is finite if there is no $v \in A$ such that*
$v^*v = 1_A$ *and* $vv^* < 1_A$.
It is stably finite if $M_n(A)$ is finite for all n.

Fact

+ A small

if $T(A) \neq \emptyset$ then A is stably finite.

The converse is an open problem (deep partial results by Haagerup, Kirchberg, Haagrup–Thornbjørsen.)
(Note that '*A* is not finite' is equivalent to $\psi^A = 0$, with ψ defined as

$$\psi: \inf_{\|x\| \leq 1} \|1 - x^*x\| + |1 - \|1 - xx^*\||.$$

Lemma *If $\tau \in T(A)$ then*

$$\|a\|_{2,\tau} := \tau(a^* a)^{1/2}$$

is a seminorm on A and $J_\tau := \{a \,|\, \|a\|_{2,\tau} = 0\}$ is an ideal of A.

$\|\tau\| = 1$

$\|a\|_{2,\tau} \leq \|a\|$

Lemma *If $\tau \in T(A)$ then*

$$\|a\|_{2,\tau} := \tau(a^* a)^{1/2}$$

is a seminorm on A and $J_\tau := \{a \| \|a\|_{2,\tau} = 0\}$ is an ideal of A.
If $T(A) \neq \emptyset$, then

$$\|a\|_{2,u} := \sup_{\tau \in T(A)} \|a\|_{2,\tau}$$

is a seminorm on A and $J := \{a \| \|a\|_{2,u} = 0\}$ is an ideal of A.

Lemma *If* $\tau \in T(A)$ *then*

$$\|a\|_{2,\tau} := \tau(a^*a)^{1/2}$$

is a seminorm on A *and* $J_\tau := \{a \mid \|a\|_{2,\tau} = 0\}$ *is an ideal of* A.
If $T(A) \neq \emptyset$, *then*

$$\|a\|_{2,u} := \sup_{\tau \in T(A)} \|a\|_{2,\tau}$$

is a seminorm on A *and* $J := \{a \mid \|a\|_{2,u} = 0\}$ *is an ideal of* A.

Exercise. If A is abelian, then $\|\cdot\|$ and $\|\cdot\|_{2,u}$ agree on A.

Caveat: $\|\cdot\|_{2,u}$ is uniformly continuous with respect to $\|\cdot\|$, but not vice versa, except in very specific situations.

Tracial ultraproduct

$A, T(A)$

$(A, \| \|_{2,u})$

Def D.2.14, C.7.1 *Suppose $\mathcal{U}$ is an ultrafilter on an index set $\mathbb{J}$, A_j, for $j \in \mathbb{J}$, are unital C^*-algebras with $T(A_j) \neq \emptyset$. Then*

$$J_{\mathcal{U}} := \{a \in \prod_j A_j : \lim_{j \to \mathcal{U}} \|a_j\|_{2,u} = 0\}$$

is a two-sided, self-adjoint, norm-closed ideal of $\prod_j A_j$.

Tracial ultraproduct

$(A, \| \|_{2,u})$

Def D.2.14, C.7.1 *Suppose $\mathcal{U}$ is an ultrafilter on an index set $\mathbb{J}$, A_j, for $j \in \mathbb{J}$, are unital C^*-algebras with $T(A_j) \neq \emptyset$. Then*

$$J_{\mathcal{U}} := \{a \in \textstyle\prod_j A_j : \lim_{j \to \mathcal{U}} \|a_j\|_{2,u} = 0\}$$

is a two-sided, self-adjoint, norm-closed ideal of $\prod_j A_j$. The quotient

$$\textstyle\prod^{\mathcal{U}} A_j := \prod_j A_j / J_{\mathcal{U}}$$

is the (tracial) ultraproduct associated to $\mathcal{U}$. If all A_j are equal to some A, the tracial ultraproduct is denoted $A^{\mathcal{U}}$ and called tracial ultrapower.

(See e.g., C. Schafhauser *A new proof of the Tikuisis–White–Winter theorem*, Crelle, 2020 or Castillejos et. al., *Nuclear dimension of simple* C^**-algbras*, Inv. Math. 2020)

Formulas, revisited

Recall that $A[\bar{x}]$ is the algebra of *-polynomials in $\bar{x}$ with coefficients in A, called *terms*. Suppose $T(A) \neq \emptyset$ and A is unital.

Def D.2.2 *Formulas over A are defined recursively:*

1. *The atomic formulas are expressions of the form $\|P(\bar{x})\|_2$ for $P(\bar{x})$ a term over A.*
2. *If $n \geq 1$, $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function, and $\varphi_0, \dots, \varphi_{n-1}$ are formulas then $f(\varphi_0, \dots, \varphi_{n-1})$ is a formula.*
3. *If φ is a formula, x is a variable, and $k < \infty$, then both $\inf_{\|x\| \leq k} \varphi$ and $\sup_{\|x\| \leq k} \varphi$ are formulas.*

The space $\mathfrak{F}_{A,t}$ of formulas over A has an algebra structure.

Formulas, revisited

Recall that $A[\bar{x}]$ is the algebra of *-polynomials in $\bar{x}$ with coefficients in A, called *terms*. Suppose $T(A) \neq \emptyset$ and A is unital.

Def D.2.2 *Formulas over A are defined recursively:*

1. *The atomic formulas are expressions of the form $\|P(\bar{x})\|_2$ for $P(\bar{x})$ a term over A.*
2. *If $n \geq 1$, $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function, and $\varphi_0, \ldots, \varphi_{n-1}$ are formulas then $f(\varphi_0, \ldots, \varphi_{n-1})$ is a formula.*
3. *If φ is a formula, x is a variable, and $k < \infty$, then both $\inf_{\|x\| \leq k} \varphi$ and $\sup_{\|x\| \leq k} \varphi$ are formulas.*

The space $\mathfrak{F}_{A,t}$ of formulas over A has an algebra structure.

Def *If* $\varphi(\bar{x})$ *is in* $\mathfrak{F}_{A,t}$, $A \leq B$, $T(B) \neq \emptyset$, $\bar{b}$ *in* B *of the same 'sort' as* $\bar{x}$, *define the interpretation* $\varphi^B(\bar{b})$ *by recursion on complexity of* φ.

Łoś's Theorem and countable saturation hold for tracial ultraproducts

Łoś's Theorem and countable saturation hold for tracial ultraproducts

Thm 16.2.8, Łoś's Theorem *If* $A \leq A_j$ *are unital,* $T(A_j) \neq \emptyset$ *for all* $j \in \mathbb{J}$*,* $\mathcal{U}$ *is an ultrafilter on* $\mathbb{J}$*,* $\varphi(\bar{x}) \in \mathfrak{F}_{A,t}$*, and* $C := \prod^{\mathcal{U}} A_j$*, then* $\varphi^C(\bar{a}) = \lim_{j \to \mathcal{U}} \varphi^{A_j}(\bar{a}_j)$ *for all* $\bar{a}$ *in* $\prod^{\mathcal{U}} A_j$ *of the appropriate sort.*

Thm 16.4.1 *Suppose that* $\mathcal{U}$ *is a countably incomplete ultrafilter on* $\mathbb{J}$ *and that* A_j*, for* $j \in \mathbb{J}$*, are unital* C^**-algebras with* $T(A_j) \neq \emptyset$*. Then the ultraproduct* $C := \prod^{\mathcal{U}} A_j$ *is countably saturated (with respect to the tracial language* $\mathfrak{F}_{C,t}$*).*

Łoś's Theorem and countable saturation hold for tracial ultraproducts

Thm 16.2.8, Łoś's Theorem *If $A \leq A_j$ are unital, $T(A_j) \neq \emptyset$ for all $j \in \mathbb{J}$, $\mathcal{U}$ is an ultrafilter on $\mathbb{J}$, $\varphi(\bar{x}) \in \mathfrak{F}_{A,t}$, and $C := \prod^{\mathcal{U}} A_j$, then $\varphi^C(\bar{a}) = \lim_{j \to \mathcal{U}} \varphi^{A_j}(\bar{a}_j)$ for all $\bar{a}$ in $\prod^{\mathcal{U}} A_j$ of the appropriate sort.*

Thm 16.4.1 *Suppose that $\mathcal{U}$ is a countably incomplete ultrafilter on $\mathbb{J}$ and that A_j, for $j \in \mathbb{J}$, are unital C^*-algebras with $T(A_j) \neq \emptyset$. Then the ultraproduct $C := \prod^{\mathcal{U}} A_j$ is countably saturated (with respect to the tracial language $\mathfrak{F}_{C,t}$).*

Coro *If $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$ and A_n, for $n \in \mathbb{N}$, are unital C^*-algebras, $T(A_n) \neq \emptyset$, $C := \prod^{\mathcal{U}} A_n$, then C is countably saturated with respect to $\mathfrak{F}_{C,t}$. It is therefore SAW*, CRISP,...*

Q: If $a \in C$, $0 \leq a \leq 1$ and $0 \in \mathrm{sp}(a)$, is $a^{\perp} \cap C \neq \{0\}$?

types: $\|xa\| = 0$, $\|ax\| = 0$, $\|x\| = 1$

In the traced language:

$\|xa\|_{2,n} = 0$

$\|ax\|_{2,n} = 0$

$\|x\|_{2,n} = 1$?

Łoś's Theorem and countable saturation hold for tracial ultraproducts

Thm 16.2.8, Łoś's Theorem *If* $A \leq A_j$ *are unital,* $T(A_j) \neq \emptyset$ *for all* $j \in \mathbb{J}$*,* $\mathcal{U}$ *is an ultrafilter on* $\mathbb{J}$*,* $\varphi(\bar{x}) \in \mathfrak{F}_{A,t}$*, and* $C := \prod^{\mathcal{U}} A_j$*, then* $\varphi^C(\bar{a}) = \lim_{j \to \mathcal{U}} \varphi^{A_j}(\bar{a}_j)$ *for all* $\bar{a}$ *in* $\prod^{\mathcal{U}} A_j$ *of the appropriate sort.*

Thm 16.4.1 *Suppose that* $\mathcal{U}$ *is a countably incomplete ultrafilter on* $\mathbb{J}$ *and that* A_j*, for* $j \in \mathbb{J}$*, are unital* C^**-algebras with* $T(A_j) \neq \emptyset$*. Then the ultraproduct* $C := \prod^{\mathcal{U}} A_j$ *is countably saturated (with respect to the tracial language* $\mathfrak{F}_{C,t}$*).*

Coro *If* $\mathcal{U}$ *is a nonprincipal ultrafilter on* $\mathbb{N}$ *and* A_n*, for* $n \in \mathbb{N}$*, are unital* C^**-algebras,* $T(A_n) \neq \emptyset$*,* $C := \prod^{\mathcal{U}} A_n$*, then* C *is countably saturated with respect to* $\mathfrak{F}_{C,t}$*. It is therefore SAW*, CRISP,...*

Q: If $a \in C$, $0 \leq a \leq 1$ and $0 \in \mathrm{sp}(a)$, is $a^{\perp} \cap C \neq \{0\}$?
A: Not necessarily! Let's see why.

$|\tau(M_{2^\infty})| = 1$

Example

Let A be the CAR algebra M_{2^∞}. It has a unique tracial state τ. let $C := A^{\mathcal{U}}$. Choose $a \in A_+$ such that $\mathrm{sp}(a) = [0,1]$ and $\tau^{\mathcal{U}} \restriction \mathrm{C}^*(a) \cong C([0,1])$ is the Lebesgue measure. (I.e., $\tau(f(a)) = \int f\, d\lambda$ for all $f \in C([0,1])$.)

$\|xa\|_{2,u} = 0$ $\qquad$ $\|x\|_{2,u} = 1$ $\qquad$ $\tau(x^*x) = 1$

$\tau(a^*x^*xa) \leq \varepsilon$

$\|x\| \leq k$ $\qquad$ $\tau(x^*x) \leq \varepsilon \|x\|^2$

Formulas, re-revisited

Recall that $A[\bar{x}]$ is the algebra of *-polynomials in $\bar{x}$ with coefficients in A, called *terms*.

Def D.2.2 *Formulas in* $\mathfrak{F}_{A,t^+}$ *are defined recursively:*

1. *The atomic formulas are expressions of the form* $\|P(\bar{x})\|$ *or* $\|P(\bar{x})\|_2$ *for* $P(\bar{x})$ *a term over* A.
2. *If* $n \geq 1$, $f : \mathbb{R}^n \to \mathbb{R}$ *is a continuous function, and* $\varphi_0, \ldots, \varphi_{n-1}$ *are formulas then* $f(\varphi_0, \ldots, \varphi_{n-1})$ *is a formula.*
3. *If* φ *is a formula,* x *is a variable, and* $k < \infty$*, then both* $\inf_{\|x\| \leq k} \varphi$ *and* $\sup_{\|x\| \leq k} \varphi$ *are formulas.*

The space $\mathfrak{F}^{\bar{x}}_{A,t^+}$ *of formulas over* A *has an algebra structure.*

Formulas, re-revisited

Recall that $A[\bar{x}]$ is the algebra of *-polynomials in $\bar{x}$ with coefficients in A, called *terms*.

Def D.2.2 *Formulas in $\mathfrak{F}_{A,t^+}$ are defined recursively:*

1. *The atomic formulas are expressions of the form $\|P(\bar{x})\|$ or $\|P(\bar{x})\|_2$ for $P(\bar{x})$ a term over A.*
2. *If $n \geq 1$, $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function, and $\varphi_0, \dots, \varphi_{n-1}$ are formulas then $f(\varphi_0, \dots, \varphi_{n-1})$ is a formula.*
3. *If φ is a formula, x is a variable, and $k < \infty$, then both $\inf_{\|x\| \leq k} \varphi$ and $\sup_{\|x\| \leq k} \varphi$ are formulas.*

The space $\mathfrak{F}^{\bar{x}}_{A,t^+}$ of formulas over A has an algebra structure.

This language describes pairs $(C, C/J)$, where $J = \{a \mid \|a\|_{2,u} = 0\}$ (the quotient map $\pi \colon C \to C/J$ is definable in this language).

Łoś's Theorem and countable saturation hold for tracial ultraproducts

$$\begin{tikzcd} A \arrow[r, "\iota"] \arrow[d, "\mathrm{id}"'] \arrow[dr, "\iota"] & A_{\mathcal{U}} \arrow[d, "\pi"] \\ \overline{A}^{t} \arrow[r, "\pi"'] & A^{\mathcal{U}} \end{tikzcd}$$

$T(A) \neq \emptyset$

the $\|\cdot\|_{2,\mathcal{U}}$-completion of bounded balls of A

unital

Suppose that A is a separable unital C^*-algebra, $T(A) \neq \emptyset$. If $D \leq A_{\mathcal{U}}$ is separable and $\pi(a) \in \pi[D]' \cap A^{\mathcal{U}}$, consider the type with conditions

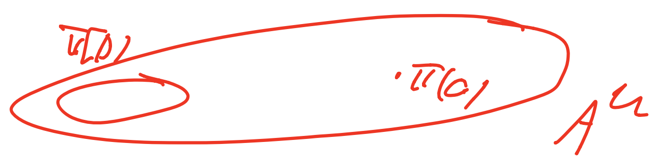
$\pi(a-x) = 0$

$$\|a - x\|_2 = 0, \; \|[d, x]\| = 0, \;\; d \in D.$$

This type is consistent and "countable".

$\pi : A_{\mathcal{U}} \to A^{\mathcal{U}}$

$(A_{\mathcal{U}}, A^{\mathcal{U}}, \pi)$



Fix $F \subset\subset D$, $\varepsilon > 0$

J_u has a quasi-central approx. unit.

Find $e \in J_u$ s.t.

① $\|e[a,d] - [a,d]\| < \varepsilon/2 \quad \forall d \in F$

② $\|[e,d]\| < \varepsilon/(2\|a\|)$

Let $b = (1-e)a$

$\pi(b) = \pi((1-\underline{e})a) = \pi(a)$

$\underline{\|[b,d]\| < \varepsilon}$

Suppose that A is a separable C^*-algebra, $T(A) \neq \emptyset$. If $D \leq A_{\mathcal{U}}$ is separable and $a \in \pi[D]' \cap A^{\mathcal{U}}$, consider the type with conditions

$$\|a - x\|_2 = 0, \|[d, x]\| = 0, \; d \in D.$$

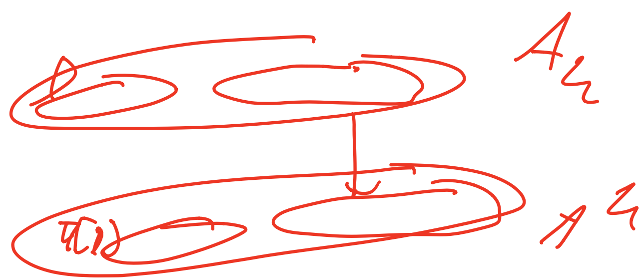
This type is consistent and "countable".
So there is $\tilde{a} \in A_{\mathcal{U}} \cap D'$ such that $\pi(\tilde{a}) = a$.

a unital

Prop (Sato, Kirchberg–Rørdam) *If $T(A) \neq \emptyset$ and $D \leq A_{\mathcal{U}}$ is separable, then $\pi[D' \cap A_{\mathcal{U}}] = \pi[D]' \cap A^{\mathcal{U}}$.*

$\supseteq$

$\subseteq$



Kirchberg's invariant:

$$F(A) = (A_{\mathcal{U}} \cap A')/(A^{\perp} \cap A_{\mathcal{U}}).$$

Sep. unital

$A \hookrightarrow A_{\mathcal{U}}$

Fact

If A is unital, then $F(A) = A_{\mathcal{U}} \cap A'$.

Kirchberg's invariant:
$F(A) = (A_{\mathcal{U}} \cap A')/(A^{\perp} \cap A_{\mathcal{U}})$.

Fact

If A is unital, then $F(A) = A_{\mathcal{U}} \cap A'$.

(Even if not, $F(A)$ still ought to be countably quantifier-free saturated, but nobody verified this yet as far as I know.)

Kirchberg's invariant:
$F(A) = (A_{\mathcal{U}} \cap A')/(A^{\perp} \cap A_{\mathcal{U}})$.

Fact

If A is unital, then $F(A) = A_{\mathcal{U}} \cap A'$.

(Even if not, $F(A)$ still ought to be countably quantifier-free saturated, but nobody verified this yet as far as I know.)
In the following, all ultrafilters are nonprincipal and on $\mathbb{N}$.

Question *(McDuff, 1970) Are all ultrapowers of the hyperfinite II_1 factor R isomorphic?*
($\Leftrightarrow$ are all tracial ultrapowers of M_{2^∞} isomorphic?)

$(M_{2^\infty})^{\mathcal{U}} \cong R^{\mathcal{U}}$

Kirchberg's invariant:
$F(A) = (A_{\mathcal{U}} \cap A')/(A^{\perp} \cap A_{\mathcal{U}})$.

Fact

If A is unital, then $F(A) = A_{\mathcal{U}} \cap A'$.

(Even if not, $F(A)$ still ought to be countably quantifier-free saturated, but nobody verified this yet as far as I know.)

In the following, all ultrafilters are nonprincipal and on $\mathbb{N}$.

Question *(McDuff, 1970) Are all ultrapowers of the hyperfinite II_1 factor isomorphic?*

($\Leftrightarrow$ are all tracial ultrapowers of M_{2^∞} isomorphic?)

(Kirchberg, 2004) If A is a separable C^*-algebra, does $F(A)$ depend on the choice of the ultrafilter?

Kirchberg's invariant:
$F(A) = (A_{\mathcal{U}} \cap A')/(A^{\perp} \cap A_{\mathcal{U}})$.

Fact

If A is unital, then $F(A) = A_{\mathcal{U}} \cap A'$.

(Even if not, $F(A)$ still ought to be countably quantifier-free saturated, but nobody verified this yet as far as I know.)

In the following, all ultrafilters are nonprincipal and on $\mathbb{N}$.

Question *(McDuff, 1970) Are all ultrapowers of the hyperfinite II_1 factor isomorphic?*
($\Leftrightarrow$ are all tracial ultrapowers of M_{2^∞} isomorphic?)
(Kirchberg, 2004) If A is a separable C^*-algebra, does $F(A)$ depend on the choice of the ultrafilter?

Thm (F.–Hart–Sherman) *The answer to either question cannot be decided in ZFC.*

We will prove $\frac{1}{2}$ of this theorem (and argue that this is the relevant half).

Kirchberg's invariant:
$F(A) = (A_{\mathcal{U}} \cap A')/(A^{\perp} \cap A_{\mathcal{U}})$.

Fact
If A is unital, then $F(A) = A_{\mathcal{U}} \cap A'$.

(Even if not, $F(A)$ still ought to be countably quantifier-free saturated, but nobody verified this yet as far as I know.)
In the following, all ultrafilters are nonprincipal and on $\mathbb{N}$.
Question *(McDuff, 1970) Are all ultrapowers of the hyperfinite II_1 factor isomorphic?*
($\Leftrightarrow$ are all tracial ultrapowers of M_{2^∞} isomorphic?)
(Kirchberg, 2004) If A is a separable C^*-algebra, does $F(A)$ depend on the choice of the ultrafilter?

Thm (F.–Hart–Sherman) *The answer to either question cannot be decided in ZFC.*

We will prove $\frac{1}{2}$ of this theorem (and argue that this is the relevant half). This will involve a bit of a digression.

A spoiler

Thm (Keisler) *Assume the Continuum Hypothesis. If A is a separable C^*-algebra and $\mathcal{U}$ and $\mathcal{V}$ are nonprincipal ultrafilters on $\mathbb{N}$, then $A_{\mathcal{U}} \cong A_{\mathcal{V}}$.*

(The theorem applies to tracial von Neumann algebras, Banach spaces, countable discrete structures. . .)

Back-and-forth method

A linear ordering is *dense* if $x < y$ implies there is z such that $x < z < y$.

Thm (Cantor) *Every two countable, dense linearly ordered sets without endpoints are isomorphic.*

Pf Fix $(L, <)$, $(M, <)$

$L = \{a_n \mid n \in \mathbb{N}\}$

$M = \{b_n \mid n \in \mathbb{N}\}$

Define $L_n \subset\subset L$, $M_n \subset\subset M$

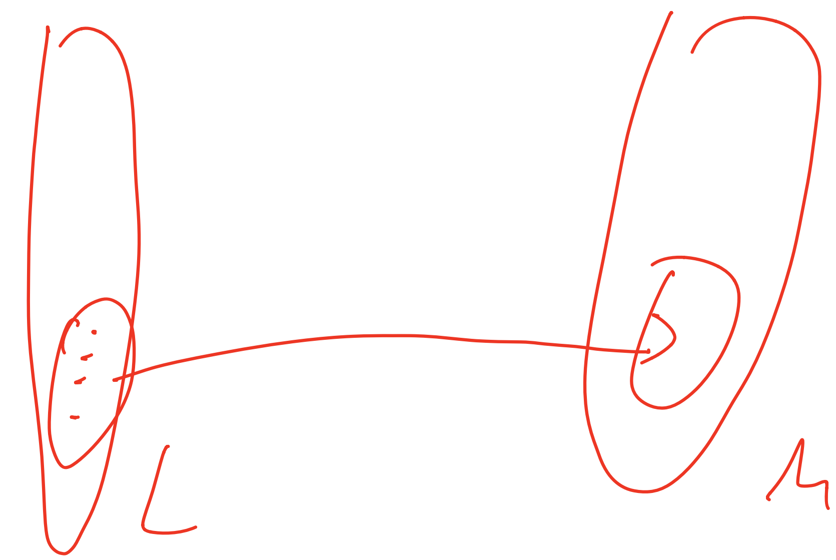
$f_n : L_n \xrightarrow{\cong} M_n$

$L_n \subseteq L_{n+1}$, $M_n \subseteq M_{n+1}$

$f_{n+1} \upharpoonright L_n = f_n$

$\underline{a_n \in L_{n+1}}$, $\underline{b_n \in M_{n+1}}$, $\forall n$

L_n, M_n finite



$a_0 \to b_0$

$M_2(\mathbb{C}) \otimes \mathcal{O}_2$

$\tau(a, b) = \tau(a)$

Back-and-forth method

A linear ordering is *dense* if $x < y$ implies there is z such that $x < z < y$.

dense no endpoints

Thm (Cantor) *Every two countable, linearly ordered sets are isomorphic.*

Exercise. Every two countable atomless Boolean algebras are isomorphic.