Massive C^*-algebras

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I'll be posting lecture slides and recordings at
https://ifarah.mathstats.yorku.ca/teachinig/
Last time: We defined the multiplier algebra of a C^*-algebra A.
Let's revisit the construction.

Weak topology induced by a family of seminorms

Suppose that X is a topological vector space, $\mathcal{N}$ is a family of seminorms on X, and $\mathbb{F}$ is a filter on X.

Def

1. $\mathbb{F}$ *converges to* $x \in X$, $\mathbb{F} \to x$, *if for all* $\rho \in \mathcal{N}$ *and all* $\varepsilon > 0$ *we have* $\{y \in X | \rho(x-y) < \varepsilon\} \in \mathbb{F}$.
2. $\mathbb{F}$ *is Cauchy if for all* $\rho \in \mathcal{N}$ *and all* $\varepsilon > 0$ *we have* $Y \in \mathbb{F}$ *such that* $|\rho(x-y)| < \varepsilon$ *for all* x *and* y *in* Y. *Thus*

$$\tilde{\rho}(\mathbb{F}) := \lim_{x \to \mathbb{F}} \rho(x)$$

$\mathbb{F} \to x$

$\rho : X \to \mathbb{C}$

is well-defined for all ρ.

3. *X is complete (with respect to the topology induced by $\mathcal{N}$) if every Cauchy filter on X converges.*

$\rho(x)$

$X \hookrightarrow \mathbb{C}^{\mathcal{N}}$

The completion $\tilde{X}$ of X with respect to $\mathcal{N}$ is defined in a natural way—see e.g., Gabriel Nagy's lecture notes (https://www.math.ksu.edu/ nagy/func-an-F07-S08.html, lecture TVS IV.). This is not a time or a place to go over the details of the construction, but I ought to say a few things.

The completion of an algebra X with respect to $\mathcal{N}$

$\mathbb{CF}(X)$: The space of all Cauchy filters on X
$(F+G := \{x+y | x \in F, y \in G\}$, etc.)

$$\mathbb{F}+\mathbb{G} := \{F+G | F \in \mathbb{F}, G \in \mathbb{G}\}$$
$$\mathbb{F}\mathbb{G} := \{FG | F \in \mathbb{F}, G \in \mathbb{G}\}$$
$$\lambda\mathbb{F} := \{\lambda F | F \in \mathbb{F}\}$$
$$\mathbb{F}^* := \{F^* | F \in \mathbb{F}\}$$
$$\tilde{\rho}(\mathbb{F}) := \lim_{x\to\mathbb{F}} \rho(x) \qquad \rho \in \mathcal{N}$$
$$\mathbb{F} \approx \mathbb{G} \Leftrightarrow \mathbb{F} + (-1)\mathbb{G} \to 0.$$
$$X \mapsto \mathbb{CF}(X)/\approx : x \mapsto \{Y \subseteq X | x \in Y\}.$$

Claim $\tilde{X} = \mathbb{CF}(X)/\approx$ *is an algebra complete w.r.t.*
$\tilde{\mathcal{N}} := \{\tilde{\rho} | \rho \in \mathcal{N}\}$.

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Claim $\tilde{X} = \mathbb{CF}(X)/\approx$ *is an algebra complete w.r.t.* $\tilde{\mathcal{N}} := \{\tilde{\rho} | \rho \in \mathcal{N}\}$.

$\rho(\mathbb{F}) = \lambda \Leftrightarrow U_{\rho,\lambda,\varepsilon} \in \mathbb{F} \ \forall \varepsilon > 0$

Proof: Use the sets $U_{\rho,\lambda,\varepsilon} := \{x \in X | |\rho(x) - \lambda| < \varepsilon\}$.

$(\mathbb{F}_\mu)^- \qquad \varrho \in W \qquad \varrho(\mathbb{F}_\mu) \to \lambda$

Strict topology

Def 13.1.1 *Suppose $A \leq M$. To every $h \in A$ we associate two seminorms on M, $\lambda_h(b) := \|hb\|$ and $\rho_h(b) := \|bh\|$. The weak topology induced by these seminorms is called the A-strict topology, or just the strict topology if A is clear from the context.*

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Lemma 13.1.5 *The completion $\mathcal{M}(A)$ of A in the strict topology is equipped with a unital C^*-algebra structure such that A is an essential ideal in $\mathcal{M}(A)$.*

$A \trianglelefteq M$ essential ideal

$\forall b \in M$ $\boxed{\text{(if } ab = 0,\ \forall a \in A \Rightarrow b = 0.}$

$A^{\perp} = \{0\}.$

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Proof: In addition to taking the completion as before, we need to define the norm on $\mathcal{M}(A)$. Fix an approximate unit $\mathcal{E}$ for A.

$\underline{\underline{\mathcal{E}}} \subseteq A_{+,1}$ $\lim_{e \to \mathcal{E}} \|e\underline{a} - \underline{\underline{a}}\| = 0$

$a = a^*$

Strict topology

$C_0(X)$ $\quad C(\beta X)$

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Claim *If $\mathbb{F}$ is a Cauchy filter on A, then $\underline{\sup_{e\in\mathcal{E}} \lim_{x\to\mathbb{F}} \lambda_e(x) < \infty}$. (I.e., $\mathbb{F}$ is bounded.)*

Otherwise: Fix $e_n \in \mathcal{E}$ $\quad \tilde{\lambda}_{e_n}(\mathbb{F}) > 2^{2n}$

$e = \sum 2^{-n} e_n \qquad \|e\| \leq 1$

$\|ea\| \geq 2^{-n} \|e_n a\|, \ \forall a$

$e \geq 2^{-n} e_n$

$\tilde{\lambda}_e(\mathbb{F}) \leq K < \infty$

$2^n > K + \varepsilon$

$F \in \mathbb{F} \qquad \tilde{\lambda}_e(\mathbb{F}) < K$

$\tilde{\lambda}_{e_n}(\mathbb{F}) > 2^{2n} + \varepsilon$

$\forall x \in F \qquad \|ex\| < K$

$\|e_n x\| > 2^{2n} + \varepsilon$

$K < K$

$A \hookrightarrow A^{**}$ 2^{c}

Strict topology

$M(A)$ A

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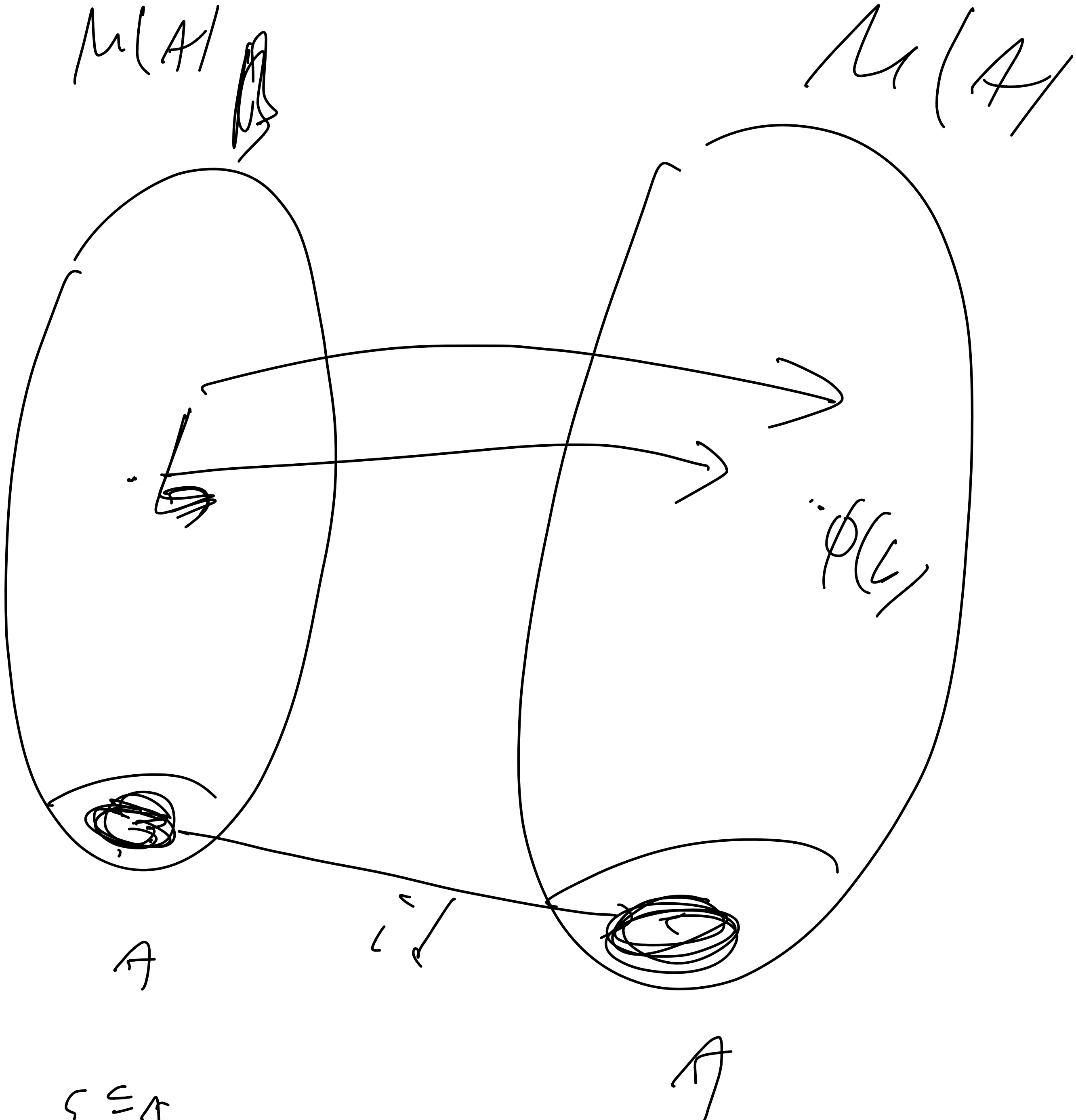
If $\mathbb{F}$ is bounded, let $\|\mathbb{F}\| := \sup_{e\in\mathcal{E}} \tilde{\lambda}_e(\mathbb{F})$.

$\boxed{A \lhd B}$

$\cap$

$M(A)$



$E \subseteq A$

$eb,\ e \in E$

$eb \to b$

X

$\boxed{C_b(X)} = \{f : X \to \mathbb{C},\ \|f\|_\infty < \infty\}$

$A = C_0(X)$

$B = \{\varphi \in A^* \mid \varphi \geq 0 \ \ \|\varphi\| = \varphi(1) = 1\}$

$X = \partial B$

(Continuing the sketch of the proof.) One can prove that this is a norm on $\tilde{A}$, that $\tilde{A}$ is a Banach algebra, and that the C^*-equality holds. This is the sort of a proof that should not be presented in public; I'll post the details.

$$\eta_1 = \|\cdot\| = \rho_1$$

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Def 13.1.6 *$\mathcal{M}(A)$ is the multiplier algebra of A.*

Example 13.2.4

1. *If X is a locally compact Hausdorff space then* $\mathcal{M}(C_0(X)) \cong C(\beta X)$.
2. $\mathcal{M}(\mathcal{K}(H)) \cong \mathcal{B}(H)$.
3. *If B_n, for $n \in \mathbb{N}$, are unital* C^**-algebras, then* $\mathcal{M}(\bigoplus_n B_n) \cong \prod_n B_n$.

Coronas

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2. If X is a locally compact Hausdorff space, then $\mathcal{Q}(C_0(X)) \cong C(\beta X)/C_0(X) \cong C(\beta X \setminus X)$.
3. If $X = \mathbb{N}$ (with discrete topology) then $C_0(\mathbb{N}) \cong c_0$, $C(\beta\mathbb{N}) \cong \ell_\infty$, and $C(\beta\mathbb{N} \setminus \mathbb{N}) \cong \ell_\infty/c_0$.
4. If $\mathbb{J} \subseteq \mathbb{N}$ is infinite, the corona of $\bigoplus_{n\in\mathbb{J}} M_n(\mathbb{C})$ is isomorphic to $\prod_{n\in\mathbb{J}} M_n(\mathbb{C}) / \bigoplus_{n\in\mathbb{J}} M_n(\mathbb{C})$.

$\mathbb{N}$

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Exercise. How many nonisomorphic algebras as in (4) can you find?

The Calkin algebra, $\mathcal{B}(H)/\mathcal{K}(H)$

'The' Calkin algebra is associated with the separable, infinite-dimensional H.

Lemma

$\mathcal{B}(H)$ has exactly one nontrivial ideal, $\mathcal{K}(H)$.

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$\mathcal{Q}(H)$ is simple.

Exercise. Suppose that κ is an infinite cardinal. Describe all (two-sided, norm-closed, proper, nontrivial) ideals of $\mathcal{B}(H)$. $= \ell_2(\kappa)$
(Hint: If $\kappa = \aleph_n$, the n-th infinite cardinal, then there are $n+1$ such ideals. Counting starts at 0, i.e., $\aleph_0$ is the smallest infinite cardinal.)

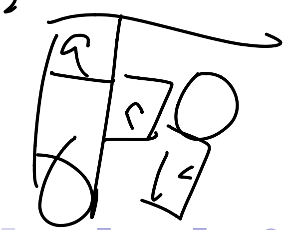
$$\|(a_n)\| = \left(\sum \|a_n\|^2\right)^{1/2}$$

Lemma 12.1.3 *$\mathcal{B}(H)$ is isomorphic to a C^*-subalgebra of $\mathcal{Q}(H)$. Therefore every separable C^*-algebra is isomorphic to a C^*-subalgebra of $\mathcal{Q}(H)$.*

$$H \cong \bigoplus_{\ell_2} H \qquad H \otimes \ell_2(\mathbb{N})$$

$$B(H) \hookrightarrow \prod_{\aleph_0} B(H) \hookrightarrow B\Big(\bigoplus_{\ell_2} H\Big)$$

$$a \to (a, a, \ldots, a)$$



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A bit of rambling:

$\mathcal{B}(H)$: quantization of each one of ℓ_∞, $\mathcal{P}(\mathbb{N})$, and $\beta\mathbb{N}$.
$\mathcal{Q}(H)$: quantization of each one of $\underline{\ell_\infty/c_0}$, $\underline{\mathcal{P}(\mathbb{N})/\operatorname{Fin}}$, and $\underline{\beta\mathbb{N}\setminus\mathbb{N}}$.

Lemma

There is a family X_r, $r \in \mathbb{R} \setminus \mathbb{Q}$, of infinite subsets of $\mathbb{N}$ such that $X_r \cap X_s$ is finite for all $r \neq s$.

$\mathcal{P}(\mathbb{N})/\mathrm{Fin}$

$r \in \mathbb{R} \setminus \mathbb{Q}$

$X_r \subseteq \mathbb{Q}$

$X_r \to r$

Lemma

There is a family X_r, $r \in \mathbb{R}$, *of infinite subsets of* $\mathbb{N}$ *such that* $X_r \cap X_s$ *is finite for all* $r \neq s$.

Lemma

There is a family of $\mathfrak{c} := 2^{\aleph_0}$ *orthogonal projections in* ℓ_∞ / c_0.

$X \subseteq \mathbb{N}$ $\qquad \chi_X \quad 1_X$

$$\ell_\infty / c_0 \hookrightarrow B(H)$$

The *density character* $\chi(X)$ of a topological space X is the minimal cardinality of a dense subset.

Example

$\chi(X) \leq \aleph_0$ if and only if X is separable.
$\chi(\mathcal{B}(H)) = \mathfrak{c}$ ($\mathfrak{c} := 2^{\aleph_0}$, the cardinality of $\mathbb{C}$.)

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Prop 12.1.4 *The Calkin algebra $\mathcal{Q}(H)$ has density character $\mathfrak{c}$. It has a representation on a Hilbert space K if and only if the density character of K is at least $\mathfrak{c}$.*

Projections in coronas

Def $\mathrm{Proj}(A)$ *is the poset of projections in A.*

$$= \{ p \in A \mid \underbrace{p = p^*,\ p = p^2} \}$$

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Fact. If $a \in \mathcal{B}(H)$, then p is a projection if and only if there is a closed subspace K of H such that p is the orthogonal projection to K.

Fact. $\operatorname{Proj}(\mathcal{B}(H))$ is a lattice.

$p \wedge \varepsilon \qquad p \vee \varepsilon$

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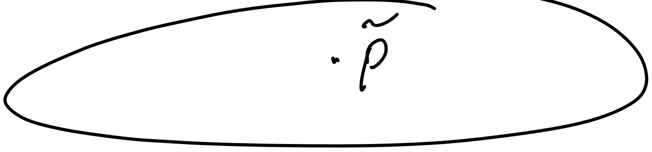
Prop (Weaver) *The poset* $\mathrm{Proj}(\mathcal{Q}(H))$ *is not a lattice.*

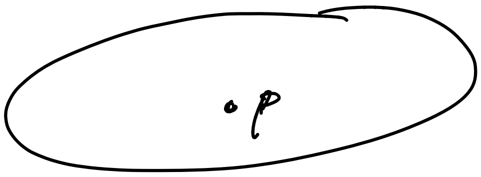
(For a proof see Proposition 13.3.3.)

Lemma 12.2.5 *Assume p_n, for $n \in \mathbb{N}$, is a decreasing sequence of projections in $\mathcal{Q}(H)$. Then there is a nonzero projection p in $\mathcal{Q}(H)$ such that $p \leq p_n$ for all n. Therefore there exists a transfinite, uncountable, decreasing 'sequence' of projections in $\mathcal{Q}(H)$.*

$p_n p_{n+1} = p_{n+1}$



$B(H)$



$Q(H)$

$\pi : B(H) \to Q(H)$ - quotient map

Lemma $p \in Q(H)$, projection, then $\exists\, \tilde p \in B(H)$, projection, $\pi(\tilde p) = p$.

Pf Fix $a \in B(H)$, $\pi(a) = p$

Let $a_1 = \frac{a + a^*}{2}$ $\qquad \pi(a_1) = p$

$sp(a_1)$ $\quad 0 \quad 1$ $\qquad \pi(a_1)^2 = \pi(a_1)$

$p_0 \longrightarrow \tilde p_0$ $\qquad B(\tilde p_0 H)$

$\vee$

$p_1 \longrightarrow \tilde p_1$

$p_2 \longrightarrow \tilde p_2$

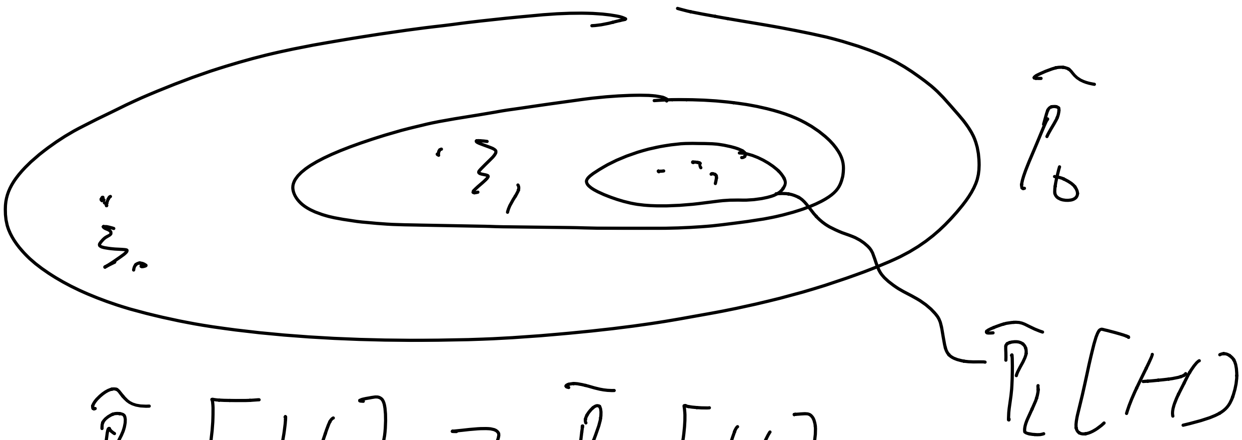
$\vdots$

$\tilde p_n$ in $B(H)$

choose orthonormal

$(\xi_n)_{n=0}^{\infty}$ in H

$\tilde{P}_n \xi_n = \xi_n \quad , \forall n.$



$\hat{P}_0[H] \supseteq \tilde{P}_1[H]$

$K = \overline{\mathrm{span}}\{\xi_n \mid n \in \mathbb{N}\}$

$\tilde{P}_\infty = \mathrm{Proj}_K$

$\tilde{P}_\infty \cdot \tilde{P}_n - \hat{P}_\alpha \in \mathcal{K}(H)$

$p = \pi(\tilde{P}_\infty)$

$A \trianglelefteq B$

Coronas of σ-unital C^*-algebras

$$0 \to \underline{A} \to \underline{B} \to B/A \to 0$$

$\uparrow$ essential $\quad B \to M(A)$

Def 1.6.7 (second part) *A* C^**-algebra is* σ*-unital if it has a countable approximate unit.*

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Coronas of σ-unital C^*-algebras

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Exercise. Every C^*-algebra is isomorphic to a subalgebra of a σ-unital C^*-algebra.

e_n $\qquad$ $\underline{h = \sum 2^{-n} e_n}$

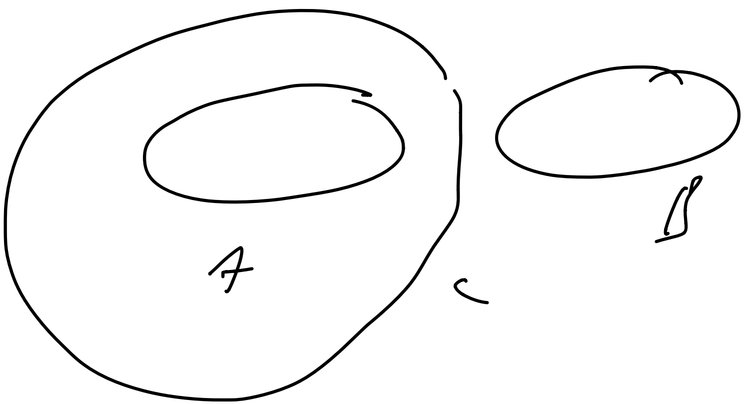
Some unrelated (?) facts

$C = Q(H)$ $\quad \ell_\infty / c_0$

Example

Suppose that $C = \mathcal{M}(A)/A$ is the corona of a σ-unital, non-unital, C^*-algebra A. Then the following holds.

1. If A and B are separable C^*-subalgebras of C and $A \perp B$ (i.e., $ab = 0 = ab^* = a^*b = a^*b^*$ for all $a \in A$ and $b \in B$) then there exists $c \in C$ such that $ac = a$ and $cb = 0$ for all $a \in A$ and $b \in B$.



$C([0,1])$

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$A = K(H)$

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2. For all a_n, b_n, for $n \in \mathbb{N}$, in C_+ satisfying $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$ for all n there exists a positive $c \in C$ such that $a_n \leq c \leq b_n$ for all n.

$a_0 \leq a_1 \leq \dots c \dots \leq b_1 \leq b_0$

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3. For every sequence a_n, $n \in \mathbb{N}$, in $C_{+,1}$ such that $a_n a_{n+1} = a_{n+1}$ for all n there exists $a \in C_{+,1}$ such that $a_n a = a$ for all n.

$a <c$ $\quad \ldots \quad a_{n+1} << a_n$

Some unrelated (?) facts

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4. If $\underline{a \in C_+}$ and $\underline{0 \in \mathrm{sp}(a)}$ then $\underline{a^\perp \cap C \neq \{0\}}$.

" $\{b \mid ab = ba = 0\}$

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Example

Suppose that $C = \mathcal{M}(A)/A$ is the corona of a σ-unital, non-unital, C^*-algebra A. Then the following holds.

1. If A and B are separable C^*-subalgebras of C and $A \perp B$ (i.e., $ab = 0 = ab^* = a^*b = a^*b^*$ for all $a \in A$ and $b \in B$) then there exists $c \in C$ such that $ac = a$ and $cb = 0$ for all $a \in A$ and $b \in B$.
2. For all a_n, b_n, for $n \in \mathbb{N}$, in C_+ satisfying $a_n \le a_{n+1} \le b_{n+1} \le b_n$ for all n there exists a positive $c \in C$ such that $a_n \le c \le b_n$ for all n.
3. For every sequence a_n, $n \in \mathbb{N}$, in $C_{+,1}$ such that $a_n a_{n+1} = a_{n+1}$ for all n there exists $a \in C_{+,1}$ such that $a_n a = a$ for all n.
4. If $a \in C_+$ and $0 \in \mathrm{sp}(a)$ then $a^\perp \cap C \neq \{0\}$.
5. Suppose a_n, b_n, for $n \in \mathbb{N}$, are in C_+ and $a_n \le a_{n+1} \le b_{n+1} \le b_n$ for all n. Furthermore suppose $D \le C$ is separable and $\lim_n \|[a_n, d]\| = 0$ for every $d \in D$. Then there exists $c \in D' \cap C_+$ such that $a_n \le c \le b_n$ for all n.

Def *(commutator)* $[a, b] := ab - ba$.
(relative commutant) If $D \le C$, then $D' \cap C := \{c \in C | [c, d] = 0\}$.

A unified framework for the facts from the previous slide

Taking the syntax seriously will pay off. . . just bear with me.

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$C = Q(H)$

Def 15.1.1 *A degree-1 condition over a* C^**-algebra* C *is an expression of the form*

$$\|a_0 x a_1 + a_2 x^* a_3 + a\| = r \qquad (1)$$

$= P(x)$

with the coefficients in C *and* $r \in \mathbb{R}_+$.
The condition $\|P(x)\| = r$ *is satisfied in* C *by* b *if* $\|P(b)\| = r$.

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Def 15.1.2 *A degree-1 type over* C *is a set of degree-1 conditions over* C. *A type* $\mathrm{t}(x)$ *is realized in* C *if there exists* b *in the unit ball of* C *such that every condition in* $\mathrm{t}(x)$ *is satisfied by* b.

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Def 15.1.2 *A degree-1 type over* C *is a set of degree-1 conditions over* C. *A type* $\mathrm{t}(x)$ *is realized in* C *if there exists* b *in the unit ball of* C *such that every condition in* $\mathrm{t}(x)$ *is satisfied by* b. *A type* $\mathrm{t}(x)$ *is approximately realized in* C *(or satisfiable) if for every finite subset* $\mathrm{t}_0(x)$ *of* $\mathrm{t}(x)$ *and every* $\varepsilon > 0$ *there exists* b *in the unit ball of* C *such that for every condition* $\|P(\bar{x})\| = r$ *in* $\mathrm{t}_0(\bar{x})$ *we have* $|\|P(b)\| - r| < \varepsilon$. *Such* b *is a partial realization of* $\mathrm{t}(x)$.

(All this can be defined for types in n variables for $n \leq \aleph_0$.)

Each of these examples asserts that a certain type is realized

1. If A and B are separable C^*-subalgebras of C and $A \perp B$ (i.e., $ab = 0 = ab^* = a^*b = a^*b^*$ for all $a \in A$ and $b \in B$) then there exists $c \in C$ such that $ac = a$ and $cb = 0$ for all $a \in A$ and $b \in B$.

$t(x)$: $\boxed{\begin{array}{l}\|ax - a\| = 0, \\ \|xb\| = 0,\end{array}}$

$a \in A_0$

$b \in B_0$

$A_0 \subseteq A$ ctbl, dense

$B_0 \subseteq B$

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$a_n \le x$, $\quad x \le b$

Exercise: express "$X \geq 0$" using degree-1 conditions.

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$$\|a_n x - x\| = 0$$

$$\|x\| = 1$$

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$Q(H) \not\hookrightarrow B(H)$ $\quad$ $B(H) \hookrightarrow Q(H)$

$$a \to \begin{pmatrix} a & & & 0 \\ & a & & \\ & & a & \\ 0 & & & \ddots \end{pmatrix}$$

$$b \in B(K)$$

$$\iff \forall \varepsilon > 0 \;\; \exists L \leq H,\ \dim(L)$$

$$\|(1 - P_L) b\| < \varepsilon$$