We are still proving that $\mathrm{OCA_T}$ implies all automorphisms of $\mathcal{Q}(H)$ are inner (believe it or not!).

$\Phi \in \mathrm{Aut}(\mathcal{Q})$ $\quad$ $\mathrm{OCA_T}$

(1) $\forall E \in \mathrm{Part}_{\mathbb{N}}$ "$\Phi \restriction \pi[\mathcal{F}[E]]$ is implemented by a unitary u_E"

(2) $\mathrm{OCA_T} \Rightarrow (E, u_E)$ can be "uniformized" by a single unitary.

Today:

Analyzing $D[E]$

Last time: Meager Subsets of Product Spaces.

Suppose D_n, for $n \in \mathbb{N}$, are finite sets. Then for $X \subseteq \mathbb{N}$

$$D_X := \prod_{n \in X} D_n$$

is compact with respect to $d(a,b) = 1/(\min\{n : a_n \neq b_n\} + 1)$.

The basic open subsets of $D_{\mathbb{N}}$ have the form

$[l, r] := \{a : a \upharpoonright l = r\}$ for some $l \in \mathbb{N}$ and $r \in D_l$.

$D_{\mathbb{N}} \to D_X : a \to a \upharpoonright X$

We'll also think of D_X as a subspace of $D_{\mathbb{N}}$: Assume $0 \in D_n$. Identify $a \in D_X$ with $\tilde{a}$, $\tilde{a}(n) = a(n)$, $n \in X$, $\tilde{a}(n) = 0$, $n \notin X$.

Last time: Meager Subsets of Product Spaces.
Suppose D_n, for $n \in \mathbb{N}$, are finite sets. Then for $\mathrm{X} \subseteq \mathbb{N}$

$$\mathrm{D}_\mathrm{X} := \prod_{n \in \mathrm{X}} \mathrm{D}_n$$

is compact with respect to $d(a, b) = 1/(\min\{n : a_n \neq b_n\} + 1)$.
The basic open subsets of $\mathrm{D}_\mathbb{N}$ have the form
$[I, r] := \{a : a \restriction I = r\}$ for some $I \Subset \mathbb{N}$ and $r \in \mathrm{D}_I$.

Lemma

1. *$I \cap J = \emptyset$ implies $[I, r] \cap [J, s] = [I \cup J, rs]$ where $(rs)(i) = r(i)$ if $i \in I$ and $(rs)(i) = s(i)$ if $i \in J$.*
2. *$I \cap J = \emptyset$ implies $[I, r] \cap [J, s] \neq \emptyset$.*
3. *$[I, r] \supseteq [J, s]$ if and only if $I \subseteq J$ and $s \cap I = r$.*

Thm 9.9.1 *Some* $\mathcal{A} \subseteq \mathsf{D}_{\mathbb{N}}$ *is relatively comeager in* $\mathsf{D}_{\mathbb{N}}$ *if and only if there are disjoint* $I(n) \Subset \mathbb{N}$, *for* $n \in \mathbb{N}$, *and* $s(n) \in \mathsf{D}_{I(n)}$ *such that* $\bigcap_m \bigcup_{n \geq m} [I(n), s(n)] \subseteq \mathcal{A}$.

We will need a classical result from descriptive set theory:

Thm (Jankov, von Neumann), B.2.13 *If* X *and* Y *are Polish spaces then every analytic* $A \subseteq X \times Y$ *can be uniformized by a C-measurable function.*

The image of a Borel subset of a Polish space by a Borel-measurable function.

the graph of $f \subseteq A$

$(\sigma(\Sigma^1_1)$-measurable)

Fact ① Analytic sets are Baire-measurable (i.e., equal to an open set mod meager).

② C-measurable $\Rightarrow$ Baire-measurable

Fact If X, Y are Polish, $f : X \to Y$ is Baire-measurable, then $\exists\, G \subseteq X$, dense G_δ, $f \upharpoonright G$ is continuous.

Coro 9.9.2 *If Y is a second countable space and $f_n \colon \mathsf{D}_{\mathbb{N}} \to Y$, for $n \in \mathbb{N}$, are Baire-measurable, then there are infinite $\mathsf{X} \subseteq \mathbb{N}$ and $b \in \mathsf{D}_{\mathbb{N} \setminus \mathsf{X}}$ such that the function $g_n \colon \mathsf{D}_{\mathsf{X}} \to Y$ defined by $g_n(a) := f_n(a + b)$ is continuous for all $n \in \mathbb{N}$.*

$$(a+b)(n) = \begin{cases} a(n), & n \in X \\ b(n), & n \notin X \end{cases}$$

If $\exists$ dense G_δ, $A \subseteq D_{\mathbb{N}}$, such that $f_n \upharpoonright A$ is cts, for all n. Fix

$\ldots [I(n), S(n)],$

so that

$$A \supseteq \bigcap_m \bigcup_{n \geq m} [I(n), S(n)].$$

Go to a subsequence, so that

$X := \mathbb{N} \setminus \bigcup_n I(n)$ is ∞.

Let $b \in D_{\mathbb{N} \setminus X} =$ "$\sum_n$" $S(n)$

Then $D_X \to Y : a \to f_n(a+b)$

is cts, hom, because $a+b \in A$.

Back to proving that $\mathrm{OCA_T}$ implies every $\Phi \in \mathrm{Aut}(\mathcal{Q}(H))$ is inner.

$$H = \bigoplus_{i=0}^{\infty} \operatorname{span}\{\xi_i \mid i \in E_n\} \quad (\ell_2\text{-sum})$$

Back to proving that $\mathrm{OCA_T}$ implies every $\Phi \in \mathrm{Aut}(\mathcal{Q}(H))$ is inner. Fix a separable Hilbert space H with an orthonormal basis (ξ_n), $\Phi \in \mathrm{Aut}(\mathcal{Q}(H))$, and a lifting Φ_* such that $\Phi_*(p)$ is a projection if p is a projection and $\|\Phi_*(a)\| \leq \|a\|$ for all a.

Def 17.4.1 *If* $\mathsf{E} \in \mathrm{Part}_{\mathbb{N}}$ *and* $\mathsf{X} \subseteq \mathbb{N}$ *then let* $p_{\mathsf{X}}^{\mathsf{E}} := \mathrm{proj}_{\overline{\mathrm{span}}\{\xi_i : i \in \bigcup_{n \in \mathsf{X}} E_n\}}$ *and* $q_{\mathsf{X}}^{\mathsf{E}} := \Phi_*(p_{\mathsf{X}}^{\mathsf{E}})$. *Also let*

$$\mathcal{D}_{\mathsf{X}}[\mathsf{E}] := p_{\mathsf{X}}^{\mathsf{E}} \mathcal{D}[\mathsf{E}] p_{\mathsf{X}}^{\mathsf{E}}.$$

Def 17.4.2 *Let* $A(n) := \mathcal{D}_{\{n\}}[\mathsf{E}]$.

Def 17.4.2 *Let* $A(n) := \mathcal{D}_{\{n\}}[\mathrm{E}]$. *Then* $A(n) \cong M_m(\mathbb{C})$ *with* $m = |E_n|$. *Let* $\mathrm{D}(n)$ *be a finite,* 2^{-n}*-dense, subset of the unit ball of* $A(n)$ *such that* $\{0, 1\} \subseteq \mathrm{D}(n)$ *and* $\mathrm{D}(n) \cap \mathrm{U}(A(n))$ *is* 2^{-n}*-dense in* $\mathrm{U}(A(n))$.

Def 17.4.2 *Let* $A(n) := \mathcal{D}_{\{n\}}[\mathsf{E}]$. *Then* $A(n) \cong M_m(\mathbb{C})$ *with* $m = |E_n|$. *Let* $\mathsf{D}(n)$ *be a finite,* 2^{-n}*-dense, subset of the unit ball of* $A(n)$ *such that* $\{0,1\} \subseteq \mathsf{D}(n)$ *and* $\mathsf{D}(n) \cap \mathsf{U}(A(n))$ *is* 2^{-n}*-dense in* $\mathsf{U}(A(n))$.

Fix an infinite $\mathsf{X} \subseteq \mathbb{N}$ *and let* $\mathsf{D}[\mathsf{E}] := \prod_n \mathsf{D}(n)$ *and* $\mathsf{D}_\mathsf{X}[\mathsf{E}] := \prod_{m \in \mathsf{X}} \mathsf{D}(m)$.

$D_X[E] \subseteq D[E]$

Def 17.4.2 *Let* $A(n) := \mathcal{D}_{\{n\}}[\mathsf{E}]$. *Then* $A(n) \cong M_m(\mathbb{C})$ *with* $m = |E_n|$. *Let* $\mathsf{D}(n)$ *be a finite,* 2^{-n}*-dense, subset of the unit ball of* $A(n)$ *such that* $\{0,1\} \subseteq \mathsf{D}(n)$ *and* $\mathsf{D}(n) \cap \mathsf{U}(A(n))$ *is* 2^{-n}*-dense in* $\mathsf{U}(A(n))$.
Fix an infinite $\mathsf{X} \subseteq \mathbb{N}$ *and let* $\mathsf{D}[\mathsf{E}] := \prod_n \mathsf{D}(n)$ *and* $\mathsf{D}_\mathsf{X}[\mathsf{E}] := \prod_{m \in \mathsf{X}} \mathsf{D}(m)$.
Then $\mathsf{D}[\mathsf{E}]$ *is a discretization of* $\mathcal{D}[\mathsf{E}]$ *and* $\mathsf{D}_\mathsf{X}[\mathsf{E}]$ *is a discretization of* $\mathcal{D}_\mathsf{X}[\mathsf{E}]$. *For* $a \in \mathsf{D}$ *let* $\operatorname{supp}(a) := \{n : a(n) \neq 0\}$ *and identify* $\mathsf{D}_\mathsf{X}[\mathsf{E}]$ *with* $\{a \in \mathsf{D}[\mathsf{E}] : \operatorname{supp}(a) \subseteq \mathsf{X}\}$.

D is $D[E]$
D_X is $D_X[E]$

Recall: WOT on $\mathcal{B}(H)_1$ is cpct, metrizable.

$D[E] \subseteq \mathcal{B}(H)_1$ is a subspace (its topology agrees with WOT).

Lemma 17.4.3 *The relation* $\approx_\varepsilon^{\mathcal{K}}$ *on* $\mathcal{B}(H)_{\leq 1}$ *defined by* $x \approx_\varepsilon^{\mathcal{K}} y$ *if* $\|\pi(x-y)\| \leq \varepsilon$ *is Borel in the weak operator topology for* $\varepsilon \geq 0$.

Pf Fix $(r_n)_{n\in\mathbb{N}}$, approx. unit of $\mathcal{K}(H)$ consisting of projections.

Then $\|\pi(x)\| = \lim_{n\to\infty} \|(1-r_n)x\|$

$= \inf_n \|(1-r_n)x\|$

Fact $\{x \mid \|x\| > r\}$ is WOT-open

$\{x \mid \|x\| \le r\}$ is not τ-closed.

$\|\pi(x-y)\| < \varepsilon \iff \exists m \quad \|(1-r_m)(x-y)\| < \varepsilon$

Bond

$\|\pi(x-y)\| \le \varepsilon \iff \forall \delta > 0$

$\|\pi(x-y)\| < \varepsilon + \delta$

Lemma 17.4.3 *The relation $\approx_\varepsilon^{\mathcal{K}}$ on $\mathcal{B}(H)_{\leq 1}$ defined by $x \approx_\varepsilon^{\mathcal{K}} y$ if $\|\pi(x-y)\| \leq \varepsilon$ is Borel in the weak operator topology for $\varepsilon \geq 0$.*

Def 17.4.4 *A function $\Theta : \mathrm{D_X[E]} \to \mathcal{B}(H)_{\leq 1}$ is an ε-approximation of Φ on $\mathrm{D_X}$ if $\Theta(a) \approx_\varepsilon^{\mathcal{K}} \Phi_*(a)$ for all $a \in \mathrm{D_X}$.*

$(\varepsilon \geq 0)$

0-approximation $\Leftrightarrow$ lifting

$D[E] \to B(H)_{\leq 1}$

Lemma 17.4.5 *If* $\mathsf{E} \in \mathsf{Part}_{\mathbb{N}}$ *and an endomorphism of* $\mathcal{Q}(H)$ *has a C-measurable* ε*-approximation on* $\mathsf{D[E]}$ *for every* $\varepsilon > 0$*, then it has a continuous lifting on* $\mathsf{D_Y[E]}$ *for some infinite* $\mathsf{Y} \subseteq \mathbb{N}$*.*

Pf Let f_n be a C-measurable $\frac{1}{n}$-approx. on $D[E]$, $n \geq 1$..

Find $X \subseteq N$ infinite and

$b \in D_{N \setminus X}$ such that

$$g_n(a) = f_n(a+b), \text{ for } a \in D_X$$

is ctns, $\forall n$.

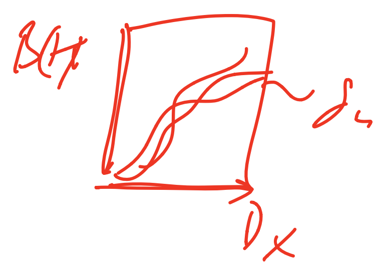
Let $\xi := \xi_X^E \ (= \phi_*(p_X^E))$

Then $\underline{\tilde{g}_n(a) := \xi_X^E \, g_n(a) \, \xi_X^E}$ is a $\frac{1}{n}$-approx. of ϕ on D_X. $\left(\phi(\dot{a}) = \dot{\xi}_X^E \, \phi(\dot{a}+\dot{b}) \, \dot{\xi}_X^E \right.$

$\left. (\dot{a} = \dot{p}_X^E (\dot{a}+\dot{b}) \dot{p}_X^E \right)$

and ctns.

Let $A = \left\{ (a, c) \in D_X \times B(H)_{\leq 1} \;\middle|\; \underline{c \approx^K_{\frac{1}{n}} \tilde{g}_n(a), \ \forall n} \right\}$



Then:

① A is Borel.

② $\forall a \in D_X$, $(a, \phi_*(a)) \in A$

③ $\forall c \in B(H)_{\leq 1}$, $(a, c) \in A$

$\Rightarrow$ $c - \phi_*(a) \in K(H)$

By J-vN, find $h : D_X \to B(H)_{\leq 1}$

c-measurable, and $(a, h(a)) \in A$

$\forall a \in D_X$. Find $Y \subseteq X$, $d \in N^Y$

s.t. $a \longrightarrow h(a+d)$, $a \in D_Y$

is cts. Then

$$a \longrightarrow \Sigma_Y^E h(a+d) \Sigma_Y^E$$

is a cts lifting of ϕ

on D_Y.

(Ξ is the capital ξ.)

Def 17.4.6 *A function $\Xi\colon \mathrm{D} \to \mathcal{B}(H)_{\leq 1}$ is of a product type if there are orthogonal projections $r_n \in \mathcal{B}(H)$ and $\Xi_n\colon \mathrm{D}(n) \to r_n(\mathcal{B}(H)_{\leq 1})r_n$ for $n \in \mathbb{N}$ such that (with the SOT-convergent series) $\Xi(a) = \sum_n \Xi_n(a_n)$ for all $a \in \mathrm{D}$.*

$$D \cong \prod_n D(n)$$

$$D(n) \xrightarrow{\Xi_n} r_n \mathcal{B}(H)_{\leq 1} r_n$$

$$D = \prod D(n) \xrightarrow{\Xi} \prod_n r_n \mathcal{B}(H)_{\leq 1} r_n \subseteq \mathcal{B}(H)_{\leq 1}$$

(Ξ is the capital ξ.)

Def 17.4.6 *A function $\Xi \colon \mathrm{D} \to \mathcal{B}(H)_{\leq 1}$ is of a* *product type* *if there are orthogonal projections $r_n \in \mathcal{B}(H)$ and $\Xi_n \colon \mathrm{D}(n) \to r_n(\mathcal{B}(H)_{\leq 1})r_n$ for $n \in \mathbb{N}$ such that (with the SOT-convergent series) $\Xi(a) = \sum_n \Xi_n(a_n)$ for all $a \in \mathrm{D}$.*

Lemma 17.4.7 *Suppose $\mathrm{E} \in \mathrm{Part}_{\mathbb{N}}$ and that $\mathrm{D}[\mathrm{E}]$ and $\mathrm{D}'[\mathrm{E}]$ are two discretizations of $\mathcal{D}[\mathrm{E}]$.*

1. *There exists a continuous function of product type $\Theta \colon \mathrm{D}[\mathrm{E}] \to \mathrm{D}'[\mathrm{E}]$ such that $x - \Theta(x) \in \mathcal{K}(H)$ for all x.*
2. *If E ~~$\in \mathrm{Part}_{\mathbb{N}}$ and~~ Φ is an endomorphism of $\mathcal{Q}(H)$, then Φ has a continuous lifting on some discretization of $\mathcal{D}[\mathrm{E}]$ if and only if it has a continuous lifting on every discretization of $\mathcal{D}[\mathrm{E}]$.*

Pf

(1) D: $D[E]$ $\quad D' := D'[E]$

① $D = D[E]$, $D' = D'[E]$

$$D = \prod_n D(n) \qquad D' = \prod_n D'(n)$$

Let $\theta_n : D(n) \to D'(n)$

so that $\boxed{\|\theta_n(x) - x\| < 2^{-n}}$, $\forall x$.

Then $\theta((x_n)) = \sum_n \theta_n(x_n)$

$$x - \theta(x) \in K(H), \quad \forall x \in D.$$

② θ as in ① is ctn.

composition of ctn.

The key lemma

The proof of the following lemma uses the method of stabilizers (Shelah, Just, Veličković, F.).

Lemma 17.4.8 *If* Φ *has a continuous lifting* Θ *on* $\mathsf{D}[\mathsf{E}]$ *for some* $\mathsf{E} \in \mathsf{Part}_{\mathbb{N}}$*, then it has a lifting of product type on* $\mathsf{D}_{\mathsf{X}}[\mathsf{E}]$ *for some infinite* $\mathsf{X} \subseteq \mathbb{N}$*.*

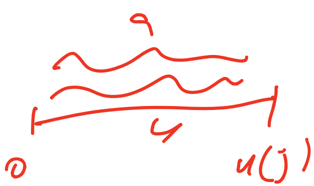
The key lemma

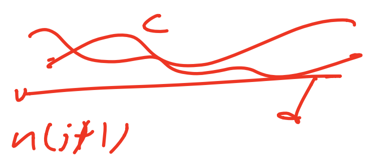
The proof of the following lemma uses the method of stabilizers (Shelah, Just, Veličkoviś, F.).

Lemma 17.4.8 *If Φ has a continuous lifting Θ on $\mathrm{D}[\mathrm{E}]$ for some $\mathrm{E} \in \mathrm{Part}_{\mathbb{N}}$, then it has a lifting of product type on $\mathrm{D}_{\mathrm{X}}[\mathrm{E}]$ for some infinite $\mathrm{X} \subseteq \mathbb{N}$.*

$0 = n(0) < n(1) < n(2) < \dots$

Proof: Recursively find an increasing sequence $(n(j))_j$, $s(j) \in \mathrm{D}_{(n(j),n(j+1))}$ (with $n(0) := 0$), and an increasing sequence of finite-rank projections $(r_j)_j$ so that for all j, all a and b in $\mathrm{D}_{[0,n(j)]}$, and all c and d in $\mathrm{D}_{[n(j+1),\infty)}$:





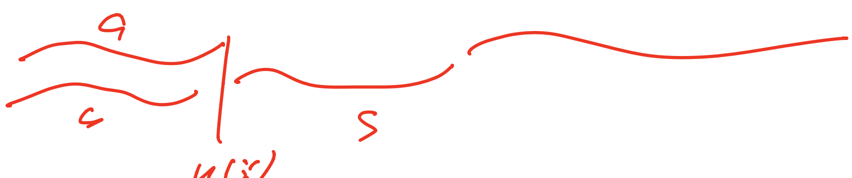
The key lemma

The proof of the following lemma uses the method of stabilizers (Shelah, Just, Veličkoviś, F.).

Lemma 17.4.8 *If* Φ *has a continuous lifting* Θ *on* $\mathrm{D}[\mathrm{E}]$ *for some* $\mathrm{E} \in \mathrm{Part}_{\mathbb{N}}$, *then it has a lifting of product type on* $\mathrm{D}_{\mathrm{X}}[\mathrm{E}]$ *for some infinite* $\mathrm{X} \subseteq \mathbb{N}$.

Proof: Recursively find an increasing sequence $(n(j))_j$, $s(j) \in \mathrm{D}_{(n(j),n(j+1))}$ (with $n(0) := 0$), and an increasing sequence of finite-rank projections $(r_j)_j$ so that for all j, all a and b in $\mathrm{D}_{[0,n(j)]}$, and all c and d in $\mathrm{D}_{[n(j+1),\infty)}$:

1. $\|(\Theta(a + s(j) + c) - \Theta(b + s(j) + c))(1 - \underline{r_j})\| \leq 2^{-j}$,
2. $\|(1 - r_j)(\Theta(a + s(j) + c) - \Theta(b + s(j) + c))\| \leq 2^{-j}$,



$x \approx^{\varepsilon} y$

$\Rightarrow \Theta(x) \approx^{k} \Theta(y)$

The key lemma

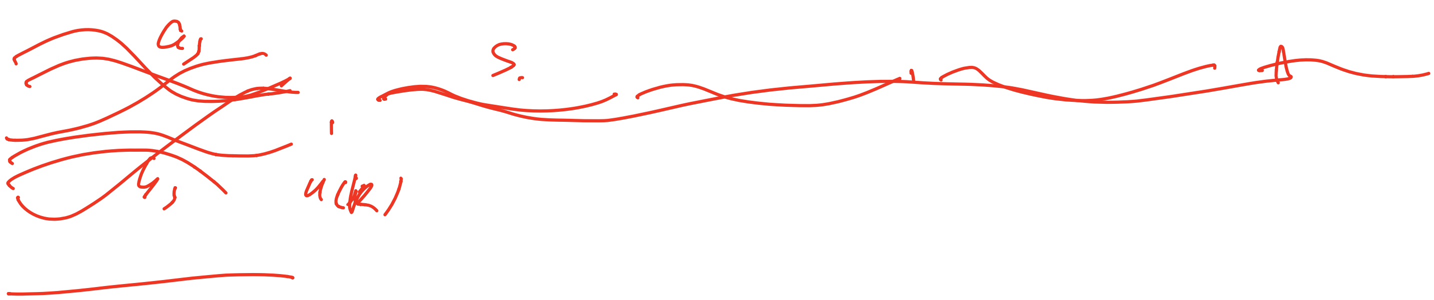
The proof of the following lemma uses the method of stabilizers (Shelah, Just, Veličkoviś, F.).

Lemma 17.4.8 *If Φ has a continuous lifting Θ on $\mathrm{D}[\mathrm{E}]$ for some $\mathrm{E} \in \mathrm{Part}_{\mathbb{N}}$, then it has a lifting of product type on $\mathrm{D}_{\mathrm{X}}[\mathrm{E}]$ for some infinite $\mathrm{X} \subseteq \mathbb{N}$.*

Proof: Recursively find an increasing sequence $(n(j))_j$, $s(j) \in \mathrm{D}_{(n(j),n(j+1))}$ (with $n(0) := 0$), and an increasing sequence of finite-rank projections $(r_j)_j$ so that for all j, all a and b in $\mathrm{D}_{[0,n(j)]}$, and all c and d in $\mathrm{D}_{[n(j+1),\infty)}$:

1. $\|(\Theta(a+s(j)+c) - \Theta(b+s(j)+c))(1-r_j)\| \leq 2^{-j}$,
2. $\|(1-r_j)(\Theta(a+s(j)+c) - \Theta(b+s(j)+c))\| \leq 2^{-j}$,
3. $\|(\Theta(a+s(j)+c) - \Theta(a+s(j)+d))r_j\| \leq 2^{-j}$,
4. $\|r_j(\Theta(a+s(j)+c) - \Theta(a+s(j)+d))\| \leq 2^{-j}$.

$u(j)$, $j \le k$, $s(j)$, $j < k$, r_j, $j < k$,



a, b

$r'_k \le r''_k \le$

$\theta(a+s) \neq^{k}_{2^{-j}} \theta(b+s)$

$X = \{ u(i) \mid i \in N \}$

$D_X \ni a \longrightarrow \theta(a + \sum s(i))$