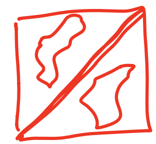
# Massive C*-algebras, Winter 2021, I. Farah, Lecture 13

Today we continue the proof that $\mathrm{OCA_T}$ implies implies that all automorphisms of the Calkin algebra are inner. More precisely, we will prove Theorem 17.8.2 (this is arguably the most elegant non-ZFC part of the proof of Theorem 17.8.5, that $\mathrm{OCA_T}$ implies all automorphisms of $\mathcal{Q}(H)$ are inner).



$X^2 \setminus \Delta_X$

Recall:

**$\mathrm{OCA_T}$** Whenever $\mathrm{X}$ is a separable metrizable space and $[\mathrm{X}]^2 = L_0 \sqcup L_1$ is an open colouring, one of the following alternatives applies.

1. There exists an uncountable $L_0$-homogeneous $\mathrm{Y} \subseteq \mathrm{X}$.
2. There are $L_1$-homogeneous sets $\mathrm{X}_n$, for $n \in \mathbb{N}$, such that $\bigcup_n \mathrm{X}_n = \mathrm{X}$.

$\mathfrak{b} = \aleph_1$ $\quad$ $(\mathbb{N}^{\mathbb{N}}, \leq^*)$

Our next objective is to prove the following.

Prop ≈9.5.7 OCA$_{\mathrm{T}}$ *implies that every* $\mathcal{E} \subseteq \mathrm{Part}_{\mathbb{N}}$ *of cardinality* $\aleph_1$ *is* $\leq^*$*-bounded.*

The proof of this Proposition requires some preparations.

# A closer look at $\mathrm{Part}_{\mathbb{N}}$

Recall that $\mathrm{Part}_{\mathbb{N}}$ is the set of all partitions $\mathsf{E}$ of a cofinite subset of $\mathbb{N}$ into finite intervals:

$$\mathsf{E} = \langle E_j : j \in \mathbb{N} \rangle$$

where $E_j = [f_{\mathsf{E}}(j), f_{\mathsf{E}}(j+1))$ and $f_E \in \mathbb{N}^{\mathbb{N}}$ is increasing.
We topologize $\mathrm{Part}_{\mathbb{N}}$ by indentifying it with a closed subspace of $\mathbb{N}^{\mathbb{N}}$, via $\mathsf{E} \mapsto f_{\mathsf{E}}$.

$$f_E(j) = \min E_j$$

$$\mathrm{Part}_N \cong \mathbb{N}^{\uparrow \mathbb{N}} : E \to f_E$$

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On $\mathrm{Part}_{\mathbb{N}}$ we defined $\mathsf{E} \leq^* \mathsf{F}$ if $(\forall^\infty m)(\exists n) E_n \subseteq F_m$, or equivalently, if

$$\boxed{(\forall^\infty i)(\exists j) E_i \cup E_{i+1} \subseteq F_j \cup F_{j+1}.}$$

$\bigcup \{E_i \mid \exists j \ \ E_i \cup E_{i+1} \subseteq F_j \cup F_{j+1}\}$ is cofinite.

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$F_\sigma$

$$(\forall^\infty i)(\exists j) E_i \cup E_{i+1} \subseteq F_j \cup F_{j+1}.$$

For $m \geq 0$, on $\mathrm{Part}_{\mathbb{N}}$ define $\mathsf{E} \leq^m \mathsf{F}$ if

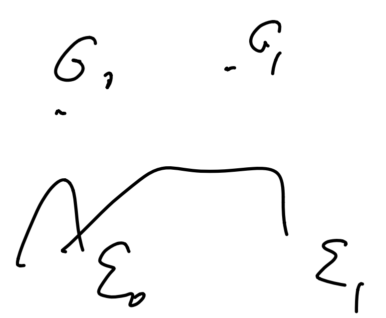
closed

$$(\forall i \geq m)(\exists j) E_i \cup E_{i+1} \subseteq F_j \cup F_{j+1}$$

and write $\leq$ for $\leq^0$.

**Lemma** If $E \in \mathrm{Part}_{\mathbb{N}}$ and $\mathcal{E} \subseteq \mathrm{Part}_{\mathbb{N}}$, and $\{F \in \mathcal{E} \mid E \leq^* F\}$ is unbounded in $(\mathrm{Part}_{\mathbb{N}}, \leq^*)$, then $\exists m \in \mathbb{N}$ such that $\mathcal{E}_m = \{F \in \mathcal{E} \mid E \leq^m F\}$ is unbounded in $(\mathrm{Part}_{\mathbb{N}}, \leq^*)$.

**Pf** Otherwise, fix $G_m \in \mathrm{Part}_{\mathbb{N}}$ which is an upper bound for $\mathcal{E}_m$ (for $m$).




Let $G \in \mathrm{Part}_{\mathbb{N}}$ be such that $G_m \leq^* G$ $\forall m$. Then $G$ is an upper bound for $\mathcal{E}$. $\square$

Here is an equivalent definition of the topology on $\mathrm{Part}_{\mathbb{N}}$.

**Lemma** *For* $\mathrm{e} = \langle E_0, \ldots, E_{n-1} \rangle$, *the set*

$\in \mathrm{Part}_{\mathbb{N}}$

$$[\mathrm{e}] := \{\mathrm{F} \mid F_j = E_j \textit{ for all } j < n\} = \{F \mid e \subseteq F\}$$

*is the open ball of diameter* $1/(n+2)$ *centered at any* $\mathrm{F} \in [\mathrm{e}]$. *The sets of the form* $[\mathrm{e}]$ *form a basis for the topology on* $\mathrm{Part}_{\mathbb{N}}$.

$F, G \in [e]$

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**Def** *Some* $\mathcal{E} \subseteq \mathrm{Part}_{\mathbb{N}}$ *is everywhere unbounded if for every* $\mathsf{e}$, $[\mathsf{e}] \cap \mathcal{E} \neq \emptyset$ *if and only if* $[\mathsf{e}] \cap \mathcal{E}$ *is* $\leq^*$*-unbounded.*

**Lemma** *If* $\mathcal{E} \subseteq \mathrm{Part}_{\mathbb{N}}$ *is* $\leq^*$*-unbounded, then it has an everywhere unbounded subset.*

Pf let $\mathcal{F} = \{ e \mid [e] \cap \mathcal{E}$ is not $\leq^*$-unbdd$\}$

Let $\mathcal{E}' = \mathcal{E} \setminus \bigcup [e]$

Let $\mathcal{E}' = \mathcal{E} \setminus \bigcup_{e \in \mathcal{F}} [e]$.

Claim $\mathcal{E}'$ is everywhere unbdd.

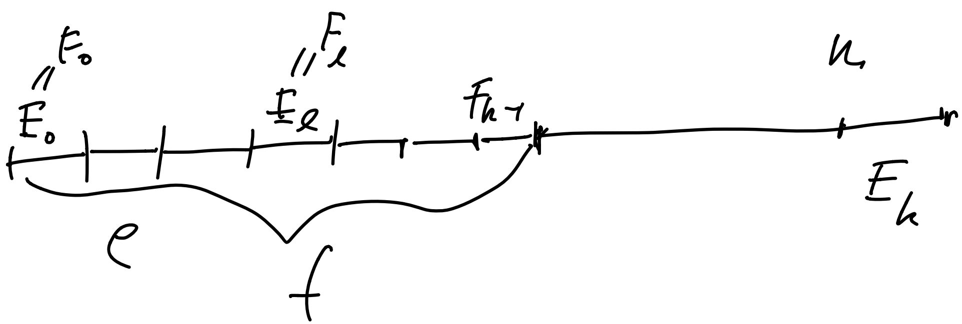
Pf Fix $f$, $[f] \cap \mathcal{E}' \neq \emptyset$.

Note $\mathcal{E}_0 = \bigcup_{e \in \mathcal{F}} ([e] \cap \mathcal{E})$ is a ctble union of bounded sets, hence bounded.

So $[f] \cap \mathcal{E}' = \underset{\text{unbounded}}{\mathcal{E}} \setminus \underset{\text{bounded}}{\mathcal{E}_0}$

is unbounded (the poset is directed).

**Lemma ≈9.7.9** *If $\mathcal{E}$ is everywhere unbounded and $[\mathrm{e}] \cap \mathcal{E} \neq \emptyset$, then there is $\mathrm{f} = \langle F_0, \ldots, F_{k-1} \rangle$ which extends e and is such that for every $m$ there is $\mathrm{E} \in [\mathrm{f}] \cap \mathcal{E}$ for which $\max(E_k) > m$.*




If

Assum. not: $\forall f$ extending $e$ $\equiv$ length$(f) = k$

if $f = \langle F_0, \ldots, F_{k-1} \rangle$ then

$g(f) = \sup \{ \max E_k \mid E \in [f] \cap \mathcal{E} \} < \infty$

Hence, $\forall f$ as above, the set

$\{ f' \mid f'$ extends $f$, $\mathrm{length}(f') = \mathrm{length}(f)+1$, $[f'] \cap \mathcal{E} = \emptyset \}$

is finite.

$g(f) = \sup \{ \max E_k \mid E \in [f] \cap \mathcal{E} \}$

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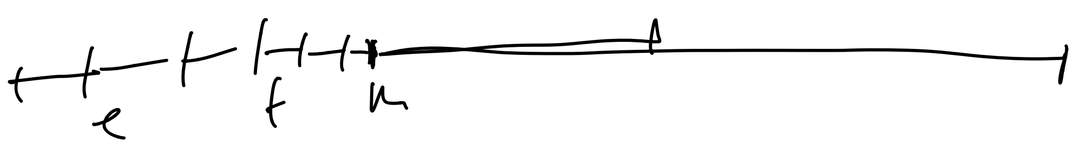
__Fact__ $\forall n \geq \max(E_{n-1})$

$\langle E_0 \ldots F_e \ldots \rangle$

$X_n = \bigcup \{ F_e \mid$ some $f''$ extends $e$, $[f''] \cap \mathcal{E} \neq \emptyset$, $n \in F_e \}$ is finite.



__If__ otherwise, fix minimal such $n$.



then $m = \min F_\ell$, for as many

$\ell$ as above, and there

is $f$ extending $e$, $F = \langle \ldots, F_\ell \rangle$,

$\max F_\ell + 1 = n$.

Define $E \in [e] \cap \mathcal{E}$, recursively:

$E$ extends $e$, if $E_j$ has been

defined, let $E_{j+1} = \overbrace{X_{\max E_j + 1}}^{Y} \cup X_{\max \cancel{Y} + 1}$

Then $E \not\leq^* F$, $\forall F \in \mathcal{E} \cap [e]$

**Lemma ≈9.7.9** *If $\mathcal{E}$ is everywhere unbounded and* $[\mathrm{e}] \cap \mathcal{E} \neq \emptyset$, *then there is* $\mathrm{f} = \langle F_0, \ldots, F_{k-1} \rangle$ *which extends* e *and is such that for every $m$ there is* $\mathrm{E} \in [\mathrm{f}] \cap \mathcal{E}$ *for which* $\max(E_k) > m$.

We say that f as in Lemma is *infinitely branching for $\mathcal{E}$*.

We can now start the proof of

Prop $\approx$9.5.7 $\mathsf{OCA_T}$ *implies that every* $\mathcal{E} \subseteq \mathrm{Part}_{\mathbb{N}}$ *of cardinality* $\aleph_1$ *is* $\leq^*$*-bounded.*

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Proof: Assume the contrary. Since every countable subset of $\mathrm{Part}_{\mathbb{N}}$ is bounded, there is an unbounded $\underbrace{X = \{\mathrm{E}(\alpha) | \alpha < \aleph_1\}}$ such that $\alpha < \beta$ implies $\mathrm{E}(\alpha) \leq^* \mathrm{E}(\beta)$ and which is $\leq^*$-unbounded in $\mathrm{Part}_{\mathbb{N}}$.

$Y = \{F(\alpha) | \alpha < \aleph_1\}$

Find $E(\alpha)$, $F(\beta) \leq^* E(\alpha)$, $\forall \beta < \alpha$

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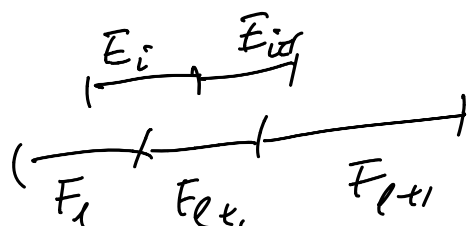
$E \leq F \iff \forall i \; \exists j \; E_i \cup E_{i+1} \subseteq F_j \cup F_{j+1}$

$$L_0 := \{\{\mathrm{E}, \mathrm{F}\} | \mathrm{E} \not\leq \mathrm{F} \text{ and } \mathrm{F} \not\leq \mathrm{E}\}$$

open ( $\exists i \; \forall j \; E_i \cup E_{i+1} \not\subseteq F_j \cup F_{j+1}$



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$$L_0 := \{\{\mathsf{E}, \mathsf{F}\} |\, \mathsf{E} \not\leq \mathsf{F} \text{ and } \mathsf{F} \not\leq \mathsf{E}\}$$

Claim. $L_0$ *is open.*

(Proof of Prop 9.5.7, continued.)

$$L_0 := \{\{E, F\} \mid E \not\leq F \text{ and } F \not\leq E\}$$

Claim. $\mathrm{Part}_{\mathbb{N}}$ *cannot be covered by countably many $L_1$-homogeneous sets.*

$\{E, F\} \in L_1$ if $E \leq F$ or $F \leq E$

$\{E(\alpha), E(\beta)\} \in L_1$ if $\underline{E(\alpha) \leq E(\beta)}$

$(\alpha < \beta)$

(Proof of Prop 9.5.7, continued.)

$$L_0 := \{\{\mathsf{E},\mathsf{F}\} \mid \mathsf{E} \not\leq \mathsf{F} \text{ and } \mathsf{F} \not\leq \mathsf{E}\}$$

Claim. $\mathrm{Part}_{\mathbb{N}}$ *cannot be covered by countably many $L_1$-homogeneous sets.*

Proof: $\mathrm{Part}_{\mathbb{N}}$ is identified with a closed subspace of the Polish space $\mathbb{N}^{\mathbb{N}}$.

(Proof of Prop 9.5.7, continued.)

$$L_0 := \{\{E, F\} \mid E \not\leq F \text{ and } F \not\leq E\}$$

Claim. $\text{Part}_{\mathbb{N}}$ *cannot be covered by countably many $L_1$-homogeneous sets.*

Proof: $\text{Part}_{\mathbb{N}}$ is identified with a closed subspace of the Polish space $\mathbb{N}^{\mathbb{N}}$. Every $L_1$-homogeneous set is nowhere dense.

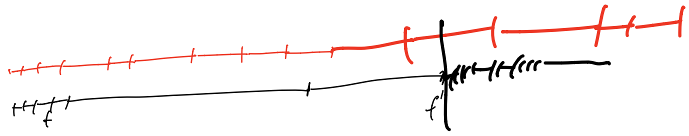
Fix $Y \subseteq X$ $L_1$-hom.

w/o, $\exists E \in Y$ such that

$\exists^\infty i \quad |E_i| \geq 2$

Fix $X$ such $E$.

Fix $f$



(Proof of Prop 9.5.7, continued.)

$$L_0 := \{\{E, F\} \mid E \not\leq F \text{ and } F \not\leq E\}$$

**Claim.** $\mathrm{Part}_{\mathbb{N}}$ *cannot be covered by countably many* $L_1$*-homogeneous sets.*

Proof: $\mathrm{Part}_{\mathbb{N}}$ is identified with a closed subspace of the Polish space $\mathbb{N}^{\mathbb{N}}$. Every $L_1$-homogeneous set is nowhere dense. Thus the Baire Category Theorem implies Claim.

Claim X cannot $L_1$ covered

by ctbly many $L_1$-homogeneous

[illegible]

sets.

Pf Every $C_1$-hom set is bounded (coming up next).

(Proof of Prop 9.5.7, continued.)

$$L_0 := \{\{E, F\} \mid E \not\leq F \text{ and } F \not\leq E\}$$

Assume

~~By the last Claim and $OCA_T$~~, $X$ has an uncountable $L_0$-homogeneous subset.

(Proof of Prop 9.5.7, continued.)

$$L_0 := \{\{\mathsf{E}, \mathsf{F}\} \mid \mathsf{E} \not\leq \mathsf{F} \ \text{ and } \ \mathsf{F} \not\leq \mathsf{E}\}$$

By the last Claim and $\mathsf{OCA_T}$, $X$ has an uncountable $L_0$-homogeneous subset. By the choice of $X$, every uncountable subset of $X$ is $\leq^*$-unbounded; so we have an unbounded, $L_0$-homogeneous set.

Claim. *Every $L_0$-homogeneous subset of $X$ is bounded.*

(Proof of Prop 9.5.7, continued.)

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Claim. *Every $L_0$-homogeneous subset of $X$ is bounded.*

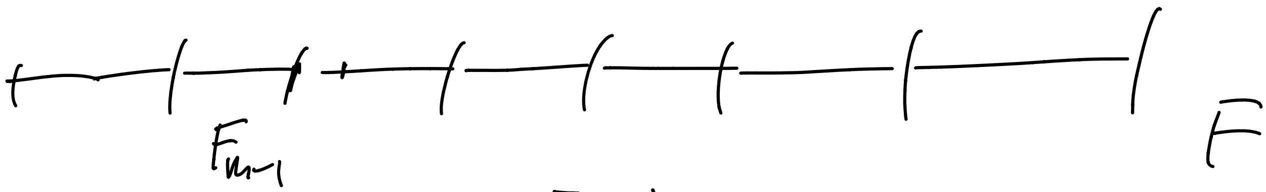
Proof: Suppose $Y \subseteq X$ is unbounded, and let $Z \subseteq Y$ be everywhere unbounded. Fix a countable dense $Z_0 \subseteq Z$.

Fix $F \in Z$, $E \leq^* F$, $E \in Z_0$

There is $m$ such that

$\mathcal{F} = \{E(\alpha) \in \mathcal{E} \mid F \le E(\alpha)\}$ is unbounded.

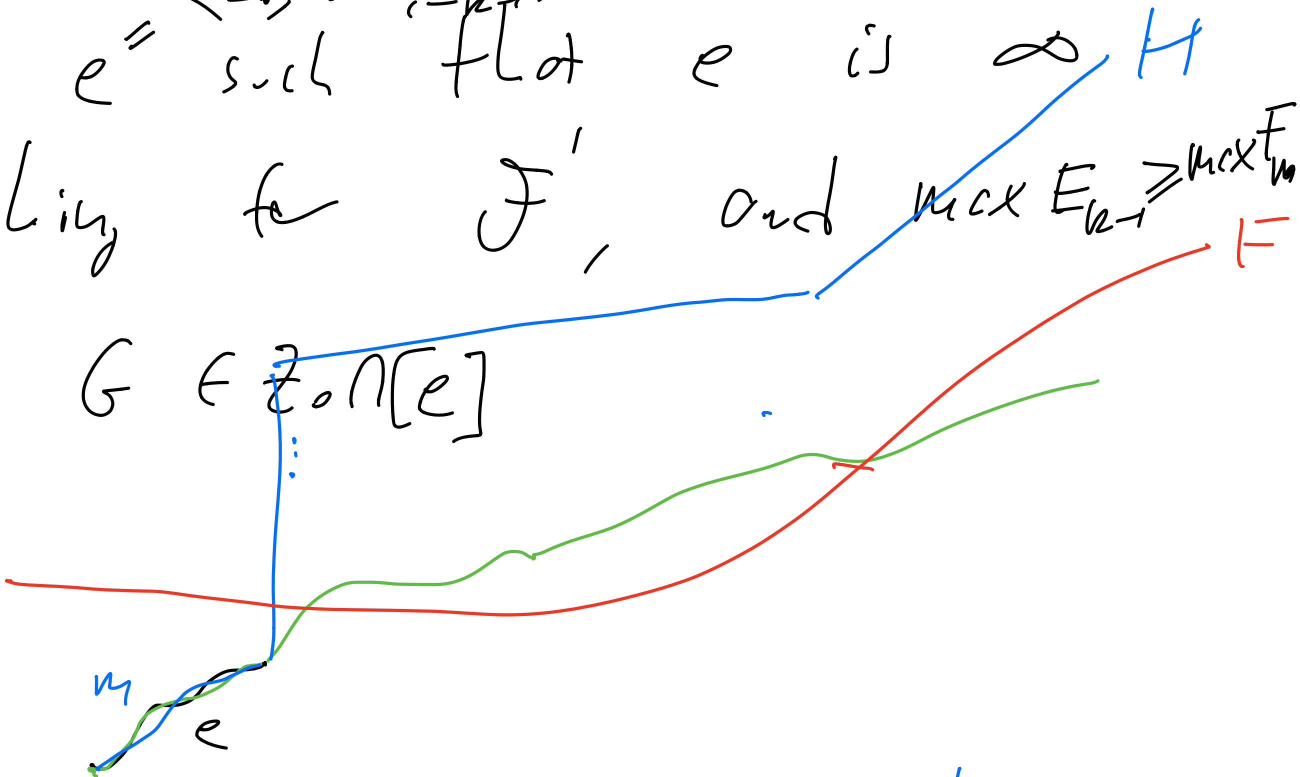
Fix $\mathcal{F}' \subseteq \mathcal{F}$, everywhere unbounded.




$\langle E_0, \dots, E_{k-1} \rangle$

Find $e =$ such that $e$ is $\infty$ branching for $\mathcal{F}'$, and $\max E_{k-1} \ge \max F_{m-1}$

Fix $G \in \mathcal{F}_0 \cap [e]$




Find $H \in [e] \cap \mathcal{F}'$, with the next interval (after $e$) long enough so that $H_i \cup H_{i+1} \not\subseteq G_i \cup G_{i+1}$, $\forall i$.

(Proof of Prop 9.5.7, continued.)

$$L_0 := \{\{\mathsf{E}, \mathsf{F}\} \,|\, \mathsf{E} \not\leq \mathsf{F} \text{ and } \mathsf{F} \not\leq \mathsf{E}\}$$

By the last Claim and $\mathsf{OCA_T}$, $X$ has an uncountable $L_0$-homogeneous subset. By the choice of $X$, every uncountable subset of $X$ is $\leq^*$-unbounded; so we have an unbounded, $L_0$-homogeneous set.

Claim. *Every $L_0$-homogeneous subset of $X$ is bounded.*

Proof: Suppose $Y \subseteq X$ is unbounded, and let $Z \subseteq Y$ be everywhere unbounded. Fix a countable dense $Z_0 \subseteq Z$.
Since $Z$ is well-ordered by $\leq^*$, we can fix $\mathsf{F} \in Z$ such that $\mathsf{E} \leq^* \mathsf{F}$ for all $\mathsf{E} \in Z_0$.

We will need this:

## Corollary

OCA$_T$ *implies that for every uncountable* $\mathcal{E} \subseteq \mathrm{Part}_{\mathbb{N}}$ *there exists* $F \in \mathrm{Part}_{\mathbb{N}}$ *such that*

$$\{E \in \mathcal{E} \mid E \leq F\}$$

*is uncountable.*

Pf. W/o.g., $|\mathcal{E}| = \aleph_1$. By Lemma, find $F$, $E \leq^* F$ $\forall E \in \mathcal{E}$.

Find $n$ s.t. $\mathcal{E}'$

$$\left|\underbrace{\{E \in \mathcal{E} \mid E \leq^n F\}}_{=\mathcal{E}'}\right| = \aleph_1$$

Also, find $e = \langle E_0, \ldots, E_{n-1} \rangle$

so that $|\{E \in \mathcal{E}' \mid E \text{ extends } e\}| = \omega$,

In $F = \langle F_j \mid j \in \mathbb{N} \rangle$ fix $j$, $= \mathcal{E}'$

$\min F_j > \max E_{n-1}$,

Let $F' = \langle \bigcup_{i \leq j} F_i, F_j, F_{j+1}, \ldots \rangle$

Then $E \leq F'$, $\forall E \in \mathcal{E}''$.

We will need this:

**Corollary**

$\mathrm{OCA}_{\mathrm{T}}$ *implies that for every uncountable* $\mathcal{E} \subseteq \mathrm{Part}_{\mathbb{N}}$ *there exists* $\mathrm{F} \in \mathrm{Part}_{\mathbb{N}}$ *such that*

$$\{\mathrm{E} \in \mathcal{E} | \, \mathrm{E} \leq \mathrm{F}\}$$

*is uncountable.*

The following result (or the question) we'll not need, but it would be hard not to mention them.

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**Corollary**

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$$\{\mathrm{E} \in \mathcal{E} \,|\, \mathrm{E} \leq \mathrm{F}\}$$

*is uncountable.*

The following result (or the question) we'll not need, but it would be hard not to mention them.

**Thm** $\mathrm{OCA_T}$ *implies that the smallest cardinality of an* $\leq^*$*-unbounded subset of* $\mathrm{Part}_{\mathbb{N}}$ *is* $\underline{\aleph_2}$.   $\mathfrak{b} = \aleph_2$

**Question** *Does* $\mathrm{OCA_T}$ *imply that* $2^{\aleph_0} = \aleph_2$*?*

# Coherent families of unitaries

We will need the notation from the proof that CH implies $\mathcal{Q}(H)$ has an outer automorphism.