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Massive C^*-algebras, Winter 2021, I. Farah, Lecture 9

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$$\mathrm{Th}(A_n) \xrightarrow[\mathcal{U}]{} \varphi$$

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$\mathcal{M} \to \mathbb{R}$

$\varphi \to \varphi^A$

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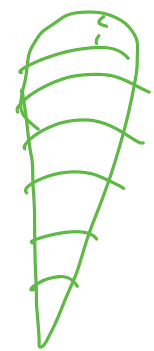
4. If $\chi(C) = \aleph_1$ and C is countably saturated, then C is the union of an increasing chain of separable elementary submodels C_α such that $C \cong (C_\alpha)_{\mathcal{U}}$ for all α. (Notably, in some cases it is possible to choose C_α's so that they are nonisomorphic. This follows from the fact that there is no universal separable C^*-algebra (Junge–Pisier).)



Automorphisms of the Calkin algebra $\mathcal{Q}(H)$

$\pi : B(H) \to Q(H)$ $\quad B(H)/K(H)$

If $a \in \mathcal{B}(H)$ we'll write $\dot{a}$ for $\pi(a)$ (this slide only).

$aa^* = a^*a$

Thm (Brown–Douglas–Fillmore, 1970's) *If a and b are normal operators in $\mathcal{B}(H)$, the following are equivalent.*

1. *There is* $\Phi \in \mathrm{Aut}(\mathcal{Q}(H))$, $\Phi(\dot{a}) = \dot{b}$.
2. *There is a unitary $\dot{u}$ in $\mathcal{Q}(H)$, $\dot{u}\dot{a}\dot{u}^* = \dot{b}$.*
3. $\mathrm{sp}(\dot{a}) = \mathrm{sp}(\dot{b})$.

$C^*(\dot{a}) \cong C(\mathrm{sp}(\dot{a}))$

Fredholm operators (see the references given in §C.6)

An operator a is *Fredholm* if $\dot{a}$ is invertible in $\mathcal{Q}(H)$.

Atkinson's Thm

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Prop ≈C.6.5 *If a is Fredholm and $\pi(a) = \pi(b)$, then b is Fredholm and*

$$\dim\ker(a) - \dim\ker(a^*) = \dim\ker(b) - \dim\ker(b^*)$$

The *Fredholm index* of a Fredholm operator a is

$$\operatorname{index}(a) := \dim\ker(a) - \dim\ker(a^*).$$

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Fact. $\mathrm{index}(a^*) = -\,\mathrm{index}(a)$, $\mathrm{index}(ab) = \mathrm{index}(ba)$.

Thus $\mathrm{GL}(\mathcal{Q}(H)) \to \mathbb{Z} : \dot{a} \mapsto \mathrm{index}(a)$ is a group homomorphism, and $\mathrm{index}(uau^*) = \mathrm{index}(\dot{a})$ for all $\dot{u} \in \mathcal{U}(\mathcal{Q}(H))$ and Fredholm a.

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Fact

$s^*s = 1_{\mathcal{B}(H)}$, *but* $ss^* \neq 1_{\mathcal{B}(H)}$, *but* $\dot{s}\dot{s}* = \dot{s}^*\dot{s} = 1_{\mathcal{Q}(H)}$.

index$(S) = 0 - 1 = \underline{-1}$

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Fact

The unilateral shift is essentially normal, but not normal.

A model-theoretic approach to the BDF question?

Question *(Brown–Douglas–Fillmore) Is there* $\Phi \in \mathrm{Aut}(\mathcal{Q}(H))$ *such that* $\Phi(\dot{s}) = \dot{s}^*$*? (I.e., is there a K-theory reversing automorphism of* $\mathcal{Q}(H)$*?)*

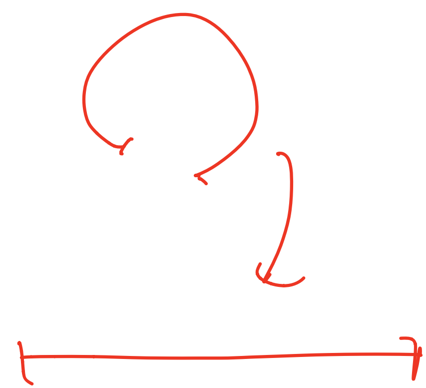
$K_0(Q(H)) = 0$

$K_1(Q(H)) = \mathbb{Z}$

$\boxed{\dot{u}\dot{s}\dot{u}^* \neq \dot{s}^*}$

$sp(\dot{s}) = \mathbb{T}$

$sp(\dot{s}^*) = \mathbb{T}$



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Does $\mathcal{Q}(H)$ have outer automorphisms?

-1

Question

$-\frac{1}{2}$

1. *Is* $\mathrm{type}_{\mathcal{Q}(H)}(\dot{s}/\emptyset) = \mathrm{type}_{\mathcal{Q}(H)}(\dot{s}^*/\emptyset)$*?*
2. *If a is Fredholm, can one recover* $\mathrm{index}(a)$ *from* $\cancel{\mathrm{type}_{\mathcal{Q}}(H)(\dot{a})}$*?*

sign? $\mathrm{type}_{\mathcal{Q}(H)}(\dot{a}/\emptyset)$?

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Remark

1. A negative answer to (1) would imply that $\Phi(\dot{s}) \neq \Phi(\dot{s}^*)$ for all $\Phi \in \mathrm{Aut}(\mathcal{Q}(H))$
2. Since $\mathcal{Q}(H)$ is not countably saturated, and even not countably homogeneous (F.–Hirshberg), a positive answer to (2) would be inconclusive.

A short intermission

If a is Fredholm, one *can* recover $|\mathrm{index}(a)|$ from $\mathrm{type}_{\mathcal{Q}}(H)(\dot{a})$.

Exercise. For every $u \in \mathcal{U}(\mathcal{Q}(H))$ either $u = \exp(ia)$ for some $0 \leq a \leq 2\pi$, or there is $m \in \mathbb{N}$ such that u has an n-th root iff $n|m$ for all $n \geq 2$.

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For every *supernatural number* x (i.e., a formal product $x = \prod_{p \text{ prime}} p^{k(p)}$, $0 \leq k(p) \leq \infty$) there exists $u \in \mathcal{U}(\mathcal{Q}(H)_{\mathcal{U}})$ such that u has an n-th root if and only if $n|x$, for all $n \geq 2$.

A deep and beautiful theorem of W.H. Woodin suggests that if there is an outer automorphism of $\mathcal{Q}(H)$ in some model of ZFC then there is an outer automorphism of $\mathcal{Q}(H)$ in every model of ZFC that satisfies CH. (Similarly for a K-theory reversing automorphism of $\mathcal{Q}(H)$.)

Σ^2_1

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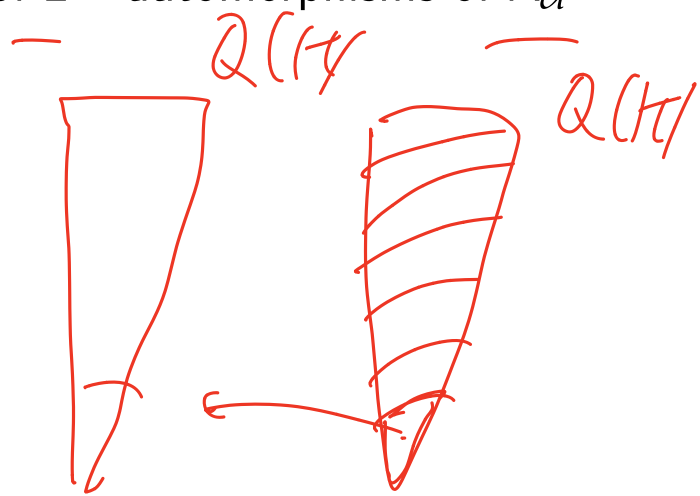
Thm (Phillips–Weaver, 2008) *CH implies that $\mathcal{Q}(H)$ has $2^{\aleph_1}$ outer automorphisms.*

The proof resembles the construction of $2^{\aleph_1}$ automorphisms of $A_{\mathcal{U}}$ using CH, with two differences:



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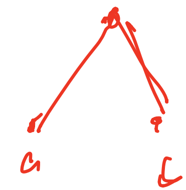
I'll present an (arguably) simpler proof of the Phillips–Weaver Theorem

A warm-up: σ-directed posets

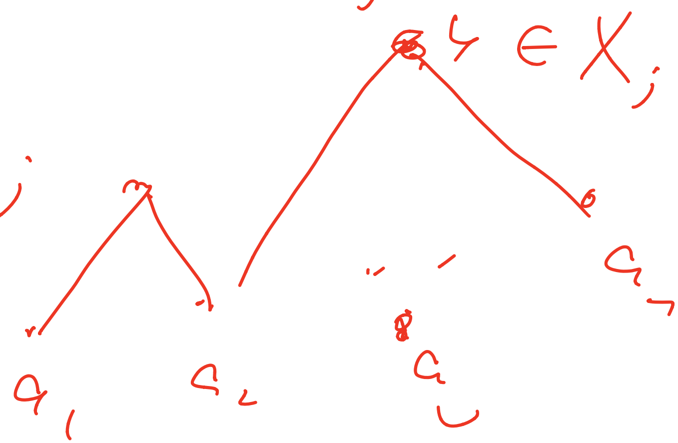


upwards

Lemma 8.5.6 *If a directed poset $\mathbb{P}$ is partitioned into finitely many pieces, then at least one of them is cofinal.*

Pf: $\mathbb{P} = \bigcup_{j \le n} X_j$ Fix $a_j \in \mathbb{P}$

$\forall x \in X_j \quad a_j \not\leq x_j$



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$\aleph_1$ -directed

Def *A partial ordering is σ-directed if every countable subset is bounded above. (Caveat: This is strictly weaker than 'every countable subset has a supremum'.)*

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Lemma 9.5.2 *If a σ-directed poset is partitioned into countably many pieces, then at least one of them is cofinal.*

Stratifying $\mathcal{Q}(H)$; the poset $\mathrm{Part}_{\mathbb{N}}$ (§9.7)

$\mathbb{N}^{\mathbb{N}}, \leq^*$

te

Let $\mathrm{Part}_{\mathbb{N}}$ denote the set of all partitions E of a cofinal subset of $\mathbb{N}$ into finite intervals:

$$\mathrm{E} = \langle E_j : j \in \mathbb{N} \rangle$$

where $E_j = [n(j), n(j+1))$ and $n(0) < n(1) < n(2) < \ldots$ are in $\mathbb{N}$.

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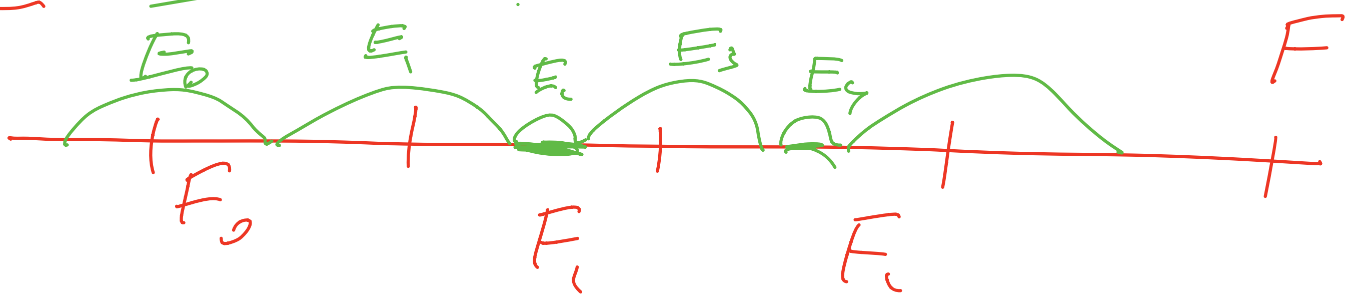
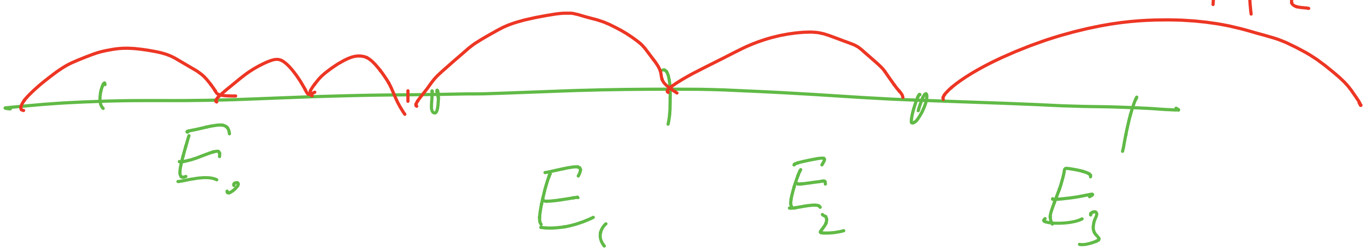
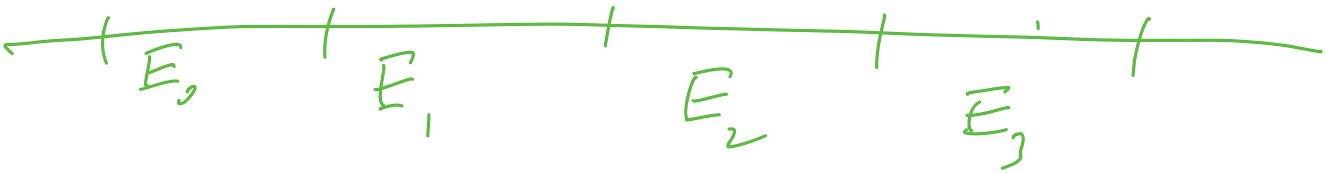



E_0
E_1
E_2
E_3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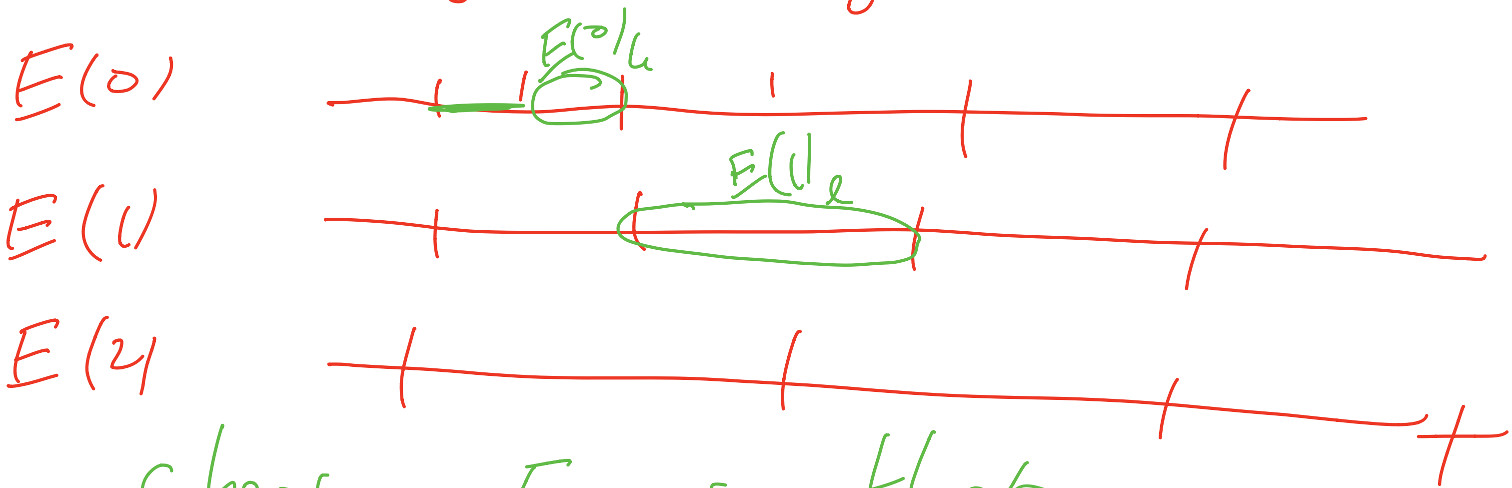
Lemma (≈9.7.1) *The poset* $(\mathrm{Part}_{\mathbb{N}}, \leq^*)$ *is σ-directed.*

Pf Fix $E(n)$, $n \in \mathbb{N}$

Let $F_0 = E(0)$



choose F_1 so that

$$F_1 \supseteq E(0)_k, \quad F_1 \supseteq E(1)_\ell$$

for some k, ℓ

If $F_0, \ldots, F_n$ have been chosen

$$\left(\max F_j + 1 = \min F_{j+1}, \quad \forall \ell \le j \ \exists k \ F_j \supseteq E(\ell)_k \right)$$

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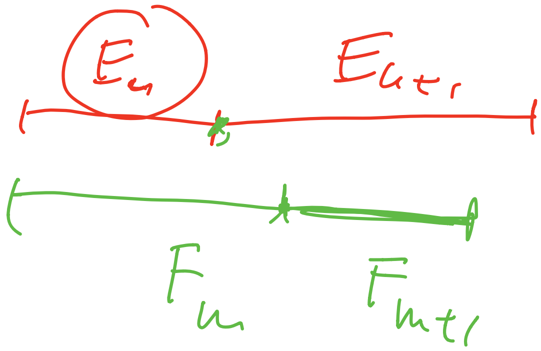
The orders $\leq^*$ *and* $\ll^*$ *agree on* $\mathrm{Part}_{\mathbb{N}}$.

Pf $\Leftarrow$ Suppose $E \leq^* F$.

$\forall m' \geq m$

Fix m $\forall$ $\exists n$ $E_n \subseteq F_m'$



$\forall n$
$\min E_n$
$\geq$
$\min F_m$
$\exists i$ $E_n \cup E_{n+1}$
$\subseteq F_j \cup F_{j+1}$

If $E_n \cup E_{n+1} \not\subseteq F_m \cup F_{m+1}$

then $E_{n+1} \not\subseteq F_{m+1}$

actually, $E_{n+1} \supseteq F_{m+1}$

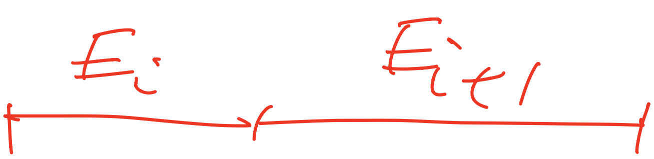
so $E_n \not\subseteq F_{m+1}$, $\forall n$

So $E \leq^* F \Rightarrow E \ll^* F$.

Assume $E \ll^* F$, i.e.,

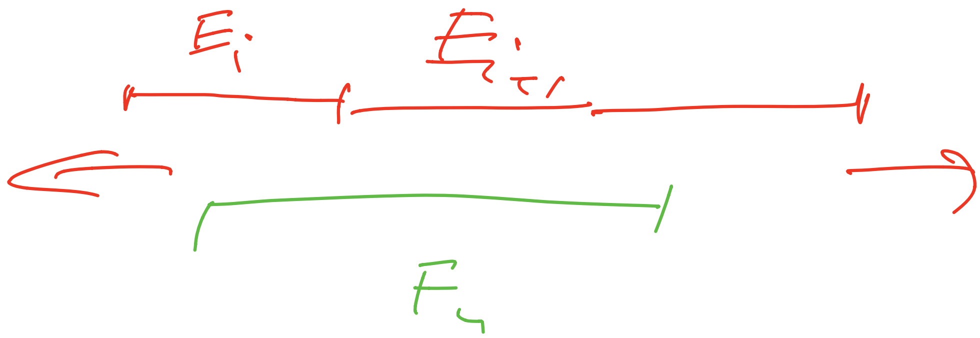
$(\forall^\infty i)(\exists j)\ E_i \cup E_{i+1} \subseteq F_j \cup F_{j+1}$ (*)

Fix i large enough



Fix F_n, $\min F_n \geq \min E_i$

Assume $E_n \not\subseteq F_n$, $\forall n$



Fix min i, $E_i \cap F_n \neq \emptyset$

Then $E_i \not\subseteq F_n$

$E_{i+1} \not\subseteq F_n$

$E_i \cup E_{i+1} \not\subseteq F_n \cup F_{n+1}$

$\not\subseteq F_{n-1} \cup F_n$

So $E_i \cup E_{i+1} \not\subseteq F_j \cup F_{j+1}$

$\forall j$.

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The poset $(\mathrm{Part}_{\mathbb{N}}, \lll^*)$ *is not directed (so forget it).*

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$\mathsf{E} \lll^* \mathsf{F}$ *if* $(\forall^\infty n)(\exists m) E_n \subseteq F_m$.
$\mathsf{E} \ll^* \mathsf{F}$ *if* $(\forall^\infty i)(\exists j) E_i \cup E_{i+1} \subseteq F_j \cup F_{j+1}$.

Lemma (≈9.7.1) *The poset* $(\mathrm{Part}_{\mathbb{N}}, \leq^*)$ *is* σ*-directed.*
The orders $\leq^*$ *and* $\ll^*$ *agree on* $\mathrm{Part}_{\mathbb{N}}$.
The poset $(\mathrm{Part}_{\mathbb{N}}, \lll^*)$ *is not directed (so forget it).*

Von Neumann Algebras $\mathcal{D}[E]$ (§9.7.1).

Von Neumann Algebras $\mathcal{D}[\mathrm{E}]$ (§9.7.1).

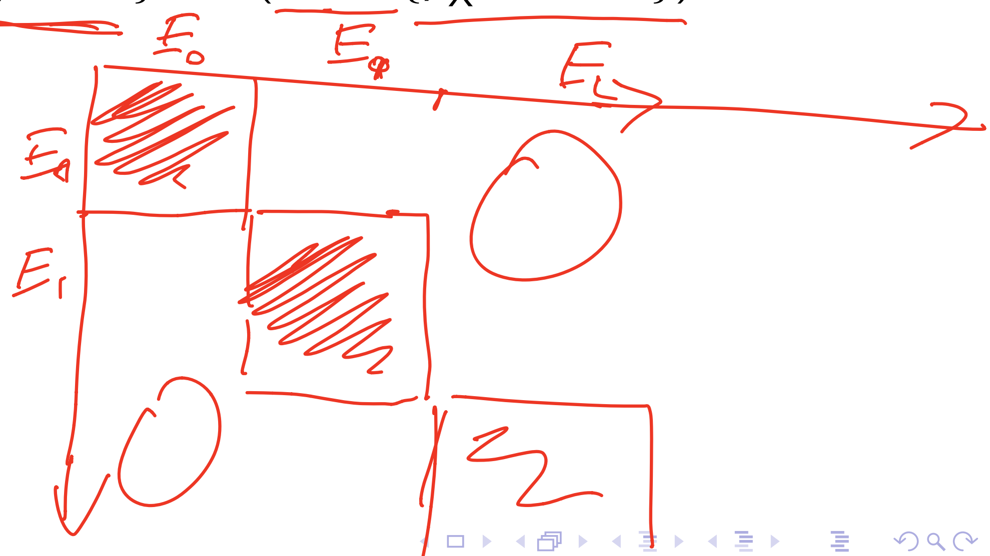
Def 9.7.5 *Consider H with an orthonormal basis (ξ_n). For $\mathrm{E} \in \mathrm{Part}_{\mathbb{N}}$ and $\mathrm{X} \subseteq \mathbb{N}$ let*

$$p^{\mathrm{E}}_{\mathrm{X}} := \mathrm{proj}_{\overline{\mathrm{span}}\{\xi_i : i \in \bigcup_{n\in \mathrm{X}} E_n\}},$$

and let

$$\mathcal{D}[\mathrm{E}] := \{a \in \mathcal{B}(H) : (\forall m)(\forall n)((a\xi_m|\xi_n) \neq 0 \text{ implies } (\exists j)\{m,n\} \subseteq E_j)\},$$

$$\mathcal{A}[\mathrm{E}] := \{\textstyle\sum_n \lambda_n p^{\mathrm{E}}_{\{n\}} | (\lambda_n) \in \ell_\infty\} \qquad (= \mathrm{W}^*\{p^{\mathrm{E}}_{\mathrm{X}} : \mathrm{X} \subseteq \mathbb{N}\}).$$



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Lemma $\mathcal{D}[\mathsf{E}]$ *is a von Neumann (i.e., WOT-closed, self-adjoint) subalgebra of $\mathcal{B}(H)$, and $\mathcal{A}[\mathsf{E}]$ is its centre.*

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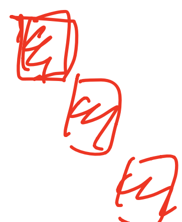
Lemma *$\mathcal{D}[\mathsf{E}]$ is a von Neumann (i.e., WOT-closed, self-adjoint) subalgebra of $\mathcal{B}(H)$, and $\mathcal{A}[\mathsf{E}]$ is its centre.*

Proof: $\mathcal{D}[\mathsf{E}] \cong \prod_n M_{k(n)}(\mathbb{C})$, with $k(n) := |E_n|$, and $\mathcal{A}[\mathsf{E}] = \prod_n \mathbb{C}1_{k(n)}$.



Von Neumann Algebras $\mathcal{D}[\mathsf{E}]$ (§9.7.1).

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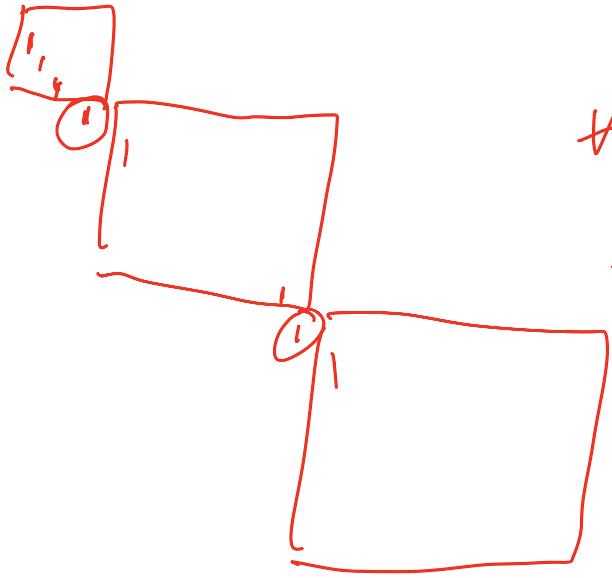
Fact

The unilateral shift $\dot{s}$ is not in $\mathcal{D}[\mathsf{E}]/(\mathcal{K}(H) \cap \mathcal{D}[\mathsf{E}])$ for any $\mathsf{E} \in \mathrm{Part}_{\mathbb{N}}$.



Fix E

then $\| P^X_{\{n+1\}} \, s \, P^X_{\{n\}} \| = 1$

& if $a \in D[E]$

then

$\| \dot a - \dot s \| = 1$

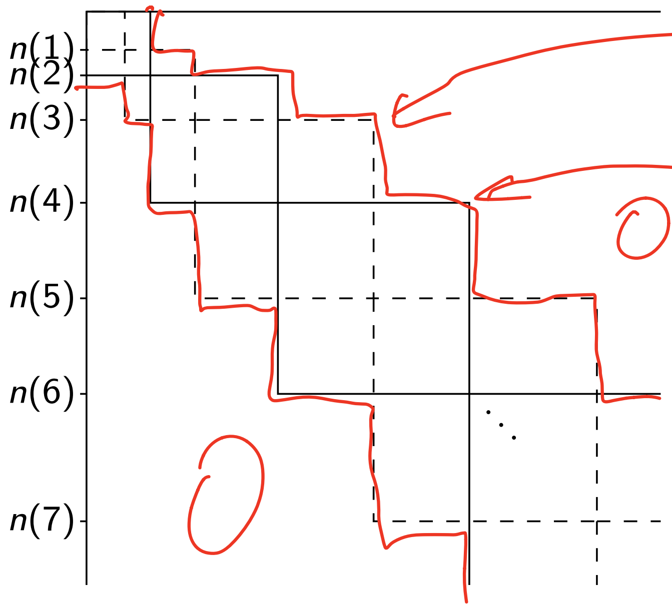
$\left(P^X_{\{n+1\}} \, a \, P^X_{\{n\}} = 0 \right.$

$\left. \text{then} \right)$

For $E \in \mathrm{Part}_{\mathbb{N}}$ define two coarser partitions, E^{even} and E^{odd}, by (with $E_{-1} := \emptyset$)

$$E_n^{\text{even}} := E_{2n} \cup E_{2n+1},$$

$$E_n^{\text{odd}} := E_{2n-1} \cup E_{2n}.$$



$\partial[E^{\text{even}}]$

$\partial[E^{\text{odd}}]$

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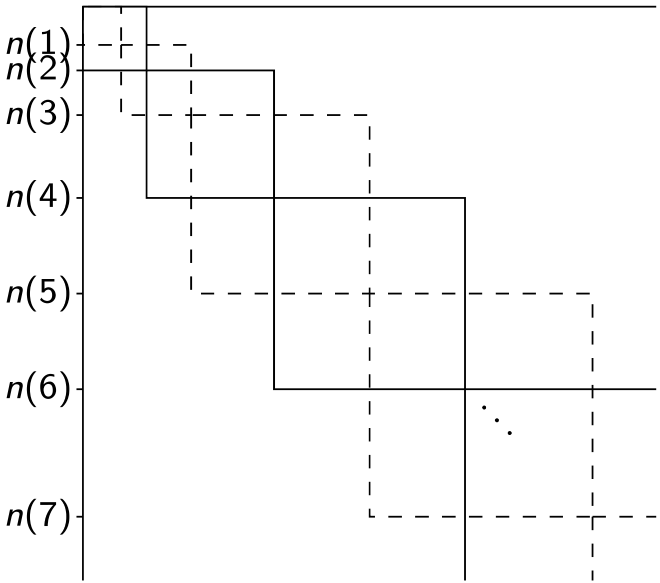

Lemma 9.7.6 *Let H be a Hilbert space with an orthonormal basis ξ_n, for $n \in \mathbb{N}$. For a sequence a_n, for $n \in \mathbb{N}$ in $\mathcal{B}(H)$ there are $\mathrm{E} \in \mathrm{Part}_{\mathbb{N}}$, $a_n^0 \in \mathcal{D}[\mathrm{E}^{\mathrm{even}}]$ and $a_n^1 \in \mathcal{D}[\mathrm{E}^{\mathrm{odd}}]$ such that $a_n - a_n^0 - a_n^1$ is compact for each n.*

Fix $a \in \mathcal{B}(H)$.

We'll find $n(0) < n(1) < \dots$

Fix $n(0) = 1$, $n(1) = 2$

$(E_i = [n(i), n(i+1)))$

$a p^E_{\{0\}}$ is cpct

$p^E_{\{0\}} a$ is cpct.

Find $n(2)$ so that

$(r_k = \mathrm{Proj}_{\mathrm{span}\{\xi_i \mid i \le k\}})$

((r_k) is an approx unit for $K(H)$.)

Find $n(2)$ so that

$$\| r_{n(2)} a p^E_{\{0\}} - a p^E_{\{0\}} \| < 2^{-2}$$

$$\| r_{n(2)} p^E_{\{0\}} a - p^E_{\{0\}} a \| < 2^{-2}$$

(next time)