Massive C^*-algebras, Winter 2021, I. Farah, Lecture 7

Today we'll prove a version of Keisler' 1960s result that in some model of ZFC all ultrapowers of a fixed separable C^*-algebra associated with nonprincipal ultrafilters on $\mathbb{N}$ are isomorphic, unless ZFC is inconsistent. The last part of the previous sentence can be safely ignored for the purposes of this course.

(We are covering parts of Chapter 16, §16.6 and §16.7 in particular.)

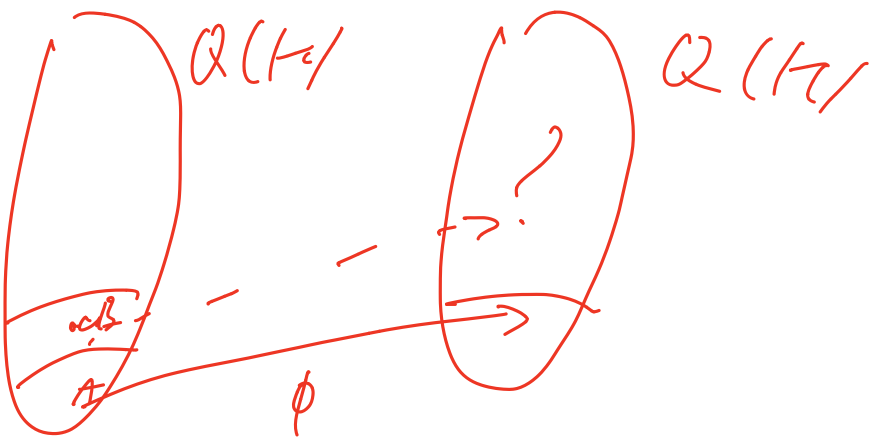
1. We'd like to extend the back-and-forth method to the uncountable.
2. Cantor's theorem fails for uncountable dense linear orderings—there are both trivial and nontrivial counterexamples.

The plan for today's lecture:

1. Explore an obstruction for extending a partial isomorphism.
2. Introduce $\aleph_1$ and the Continuum Hypothesis.
3. Use model theory (with a pinch of set theory) to analyze what an isomorphism between nonseparable C^*-algebras looks like.

Let's see what obstructions one can encounter when trying to extend partial isomorphisms.

Example (J. McCarthy) *There are separable* C^**-subalgebras* $A \leq B \leq \mathcal{Q}(H)$ *and a* **-isomorphism*[1] $\Phi\colon A \to \mathcal{Q}(H)$ *that does not have an extension to a* **-isomorphism of* B *into* $\mathcal{Q}(H)$.



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Proof: If u is a unitary, then $\mathrm{C}^*(u) \cong C(\mathrm{sp}(u))$, and $\mathrm{sp}(u)$ is a closed subset of $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$.

$u^*u = uu^* = 1$

$z\bar{z} = 1$

$\mathrm{sp}(u) = \mathbb{T}$

$A = C^*(u)$

[1] I'll follow the operator-algebraic convention: An isomorphism is not necessarily onto.

$$\boxed{\exists u \in Q(H) \text{ unitary},}$$
$\forall v \in Q(H) \quad v^2 \neq u.$

Take u, :

$$A = C^*(u^2) \cong C^*(w)$$

$$\phi(u^2) = u$$

$$B = C^*(w)$$

$$\underline{\phi(w)}$$

$$\mathrm{sp}(u^2) = \{ t^2 \mid t \in \mathrm{sp}(w) \}$$

$$= \mathbb{T}$$

$$\mathcal{U}(Q(H)) \to (\mathbb{Z}, +)$$

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Proof: Zorn's lemma.
We'll write $\aleph_1$ to denote the unique uncountable well-ordering all of whose proper initial segments are countable (this is $\aleph_1$ as an *ordinal*; its elements are (identified with) *countable ordinals*).
Thus the set of all countable ordinals therefore comes with a well-ordering of type $\aleph_1$.

The Continuum Hypothesis

Def *Sets X and Y have the same cardinality ($|X| = |Y|$) if there is a bijection $f : X \to Y$.*

We'll write $\aleph_1$ to denote the least uncountable cardinal (this is $\aleph_1$ as a *cardinal*).
Write $\mathfrak{c} := |\mathbb{R}|$.

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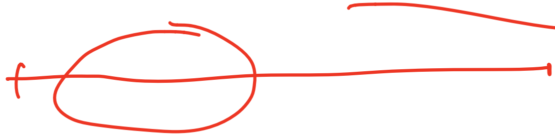
CH will stand for either of the following:

1. For every $X \subseteq \mathbb{R}$, if X is uncountable then $|X| = \mathfrak{c}$.
2. Every set of cardinality $\mathfrak{c}$ has a well-ordering such that every proper initial segment is countable (i.e., a well-ordering of *type* $\aleph_1$).



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These two assertions are equivalent, but (1) $\Rightarrow$ (2) requires (some form of the) Axiom of Choice.

The *density character* χ of a topological space is the least cardinality of a dense subset.
$\chi(A) = \aleph_0 \Leftrightarrow A$ is separable.

Lemma *If A is separable and infinite-dimensional, then $A^{\mathcal{U}}$ has density character $\mathfrak{c}$. ($\mathcal{U}$ stands for a nonprincipal ultrafilter on $\mathbb{N}$.)*

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Lemma *If A is separable and infinite-dimensional, then $A^{\mathcal{U}}$ has density character $\mathfrak{c}$. ($\mathcal{U}$ stands for a nonprincipal ultrafilter on $\mathbb{N}$.)*

Proof: Let $\{0,1\}^{<\mathbb{N}}$ denote the (countable) set of finite binary sequences.

$|\{0,1\}^{<\mathbb{N}}| = \aleph_0$ $\langle\rangle, 0, 1, 00, 01, \ldots$

A_1 is not totally bdd, i.e.,

$\exists \varepsilon > 0$ and $D \subseteq A_1$, $|D| = \aleph_0$,

$\|d - c\| > \varepsilon$, $\forall d, c$ in D (distinct).

Enumerate D as d_s, $s \in \{0,1\}^{<\mathbb{N}}$.

For $f \in \{0,1\}^{\mathbb{N}}$, $f \restriction n \in \{0,1\}^{<\mathbb{N}}$, $\forall n$.

Let $a_f = (d_{f\restriction n})_{n=0}^{\infty} \in \ell_\infty(A)$

if $f \neq g$ then

$$\{n \mid \|d_{f\restriction n} - d_{g\restriction n}\| < \varepsilon/2\} \notin \mathcal{U}$$

$$\boxed{\|a_f/\mathcal{U} - a_g/\mathcal{U}\| > \varepsilon/2}$$

$a_f/\mathcal{U}$, $f \in \{0,1\}^{\mathbb{N}}$ is a subset of $A^{\mathcal{U}}$ of card. $\mathfrak{c}$. $(2^{\aleph_0})$

So $\chi(A^{\mathcal{U}}) \geq \mathfrak{c}$

$|A^{\mathcal{U}}| \underset{(!)}{\leq} |\ell_\infty(A)| = \mathfrak{c}$

$$\chi(A^{\mathcal{U}}) = |A^{\mathcal{U}}| = \mathfrak{c}$$

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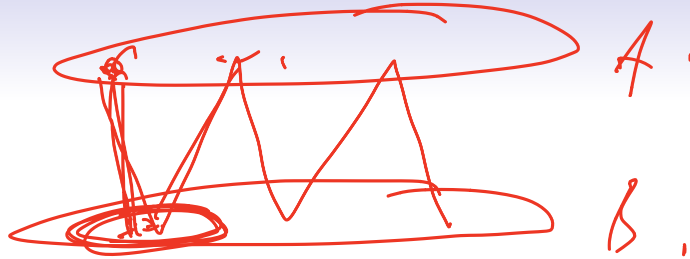
We will now proceed to see what an isomorphism between algebras of density character $\aleph_1$ *has* to look like.



Elliott

Approximate intertwining is of no use with nonseparable structures.

Lemma *If x_α, for $\alpha < \aleph_1$, is a Cauchy net in a metric space then it is eventually constant, and therefore convergent.*

$$\exists \alpha_n < \aleph_1 \quad \forall \beta > \alpha_n \quad \|x_\beta - x_{\alpha_n}\| < \frac{1}{n}$$

$$\alpha = \sup_n \alpha_n < \aleph_1$$

$$\forall \beta > \alpha \quad \|x_\beta - x_\alpha\| = 0$$

Approximate intertwining is of no use with nonseparable structures.

Lemma *If* x_α, *for* $\alpha < \aleph_1$, *is a Cauchy net in a metric space then it is eventually constant, and therefore convergent.*
Therefore, if $\Phi_\alpha\colon A \to B$, *for* $\alpha < \aleph_1$, *is a point-norm convergent net of* **-homomorphisms, then* $(\Phi_\alpha(a))_\alpha$ *is eventually constant for every* $a \in A$.

A model theory refresher

Recall that $\mathfrak{F}_A$ is the algebra of formulas over A. If $A \leq C$ and $\bar{b}$ is in C^n, then $\mathrm{type}_C(\bar{b}/A)$ is the functional $\varphi \mapsto \varphi^C(\bar{b})$ on

$$\mathfrak{F}_A = \{\varphi(\bar{x}) \in \mathfrak{F}_A | \bar{x} \text{ is of the same sort as } \bar{b}\}.$$

Def *If $B \leq C$, we say that B is an elementary submodel of C, and write $B \preceq C$, if $\varphi^B(\bar{b}) = \varphi^C(\bar{b})$ for all $\varphi \in \mathfrak{F}_B$. (Equiv., for all $\varphi \in \mathfrak{F}_A$, for a fixed $A \leq B$.)*

In other words, if $B \leq C$ then $B \preceq C$ if $\mathrm{type}_C(\bar{b}/\emptyset) = \mathrm{type}_B(\bar{b}/\emptyset)$ for all $\bar{b}$ in B.

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Exercise. If $A \preceq B$ and u is a unitary in A, then u has a square root in A if and only if it has a square root in B.

Exercise. (Requires familiarity with the Cuntz–Pedersen nullset.) If $A \prec C$ then every tracial state of A has an extension to a tracial state of C.

Some $\Phi\colon A \to B$ is an *elementary embedding* if

$$\psi^A(\bar{a}) = \psi^B(\Phi(\bar{a}))$$

$\in A$

for every formula ψ and every $\bar{a}$ of the appropriate sort.
(Equivalently, Φ is an elementary embedding if it is injective and $\Phi[A] \preceq B$.)

(C*-alg, elem. emb.)

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$\sup \sup \|[x,y]\|$
$\|x\| \leq 1, \|y\| \leq 1$

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(Equivalently, Φ is an elementary embedding if it is injective and $\Phi[A] \preceq B$.)
A formula with no free variables is a *sentence*. The *theory of A* is

$$\mathrm{Th}(A) = \{\varphi \in \mathfrak{F}_\emptyset, |\varphi \text{ is a sentence and } \varphi^A = 0\}.$$

We can identify it as the functional on the algebra of all sentences, i.e., with the type of the empty sequence over the empty set.
We say that $A \equiv B$ (A is elementarily equivalent to B) if $\mathrm{Th}(A) = \mathrm{Th}(B)$.

$A \to (\mathfrak{F}^{\langle\rangle}_A)^*$

smooth

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Exercise. For all A and B, $A \equiv B$ if and only if for every type t over $\emptyset$, t is approximately finitely satisfiable in A if and only if it is approximately finitely satisfiable in B.

A C^*-algebra is UHF if it is unital and an inductive limit of full matrix algebras, $M_n(\mathbb{C})$.
A C^*-algebra is AF if it is an inductive limit of finite-dimensional C^*-algebras.

$M_{n(1)}(\mathbb{C}) \oplus M_{n(2)}(\mathbb{C}) \oplus \cdots \oplus M_{n(k)}(\mathbb{C})$

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Exercise. If A and B are (separable) UHF algebras, then $A \cong B$ if and only if $A \equiv B$.

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Fact

There are unital, separable, AF algebras A and B such that $A \equiv B$ and $A \ncong B$.

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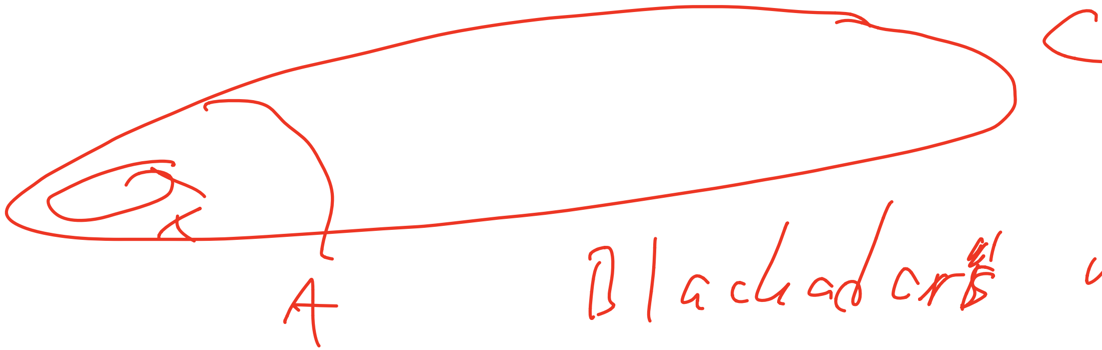
Exercise. If A and B are (separable) UHF algebras, then $A \cong B$ if and only if $A \equiv B$.

Fact

There are unital, separable, AF algebras A and B such that $A \equiv B$ and $A \not\cong B$.

The proof of this fact is purely existential; no concrete example of a pair of such algebras is known. (Analogous remark applies to the Kirchberg algebras.)

Lemma (Downward Löwenheim–Skolem Theorem) *If C is a nonseparable C^*-algebra and $X \subseteq C$ is countable, then there is a separable $A \prec C$ such that $X \subseteq A$.*



Blackadar's method

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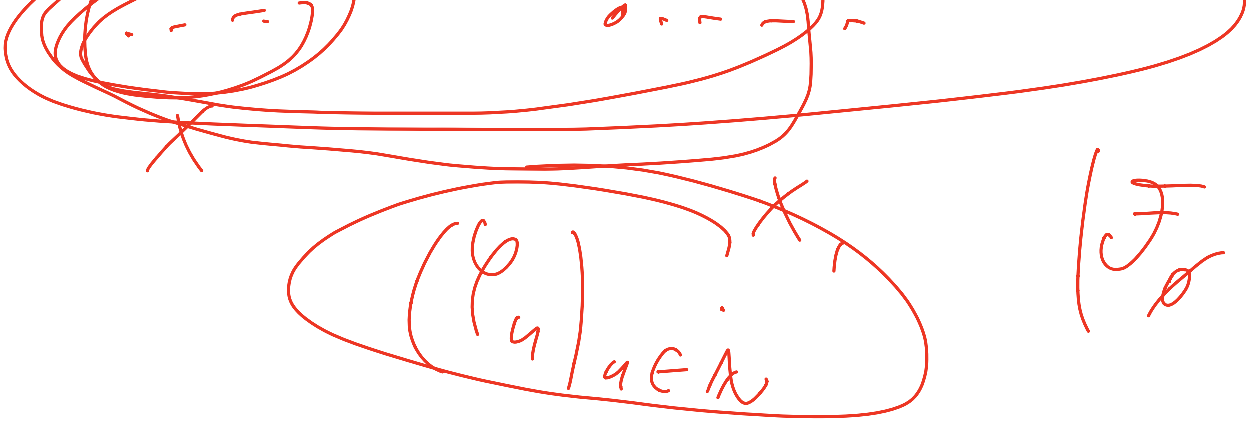
The proof relies on the following:

$B \prec A$

Thm D.1.3 (The Tarski–Vaught test) *If $B \leq A$ then $B \preceq A$ if and only if for every formula $\varphi(x, \bar{z})$ and every $\bar{b}$ in B of the appropriate sort,*

$$\inf_{x \in A, \|x\| \leq 1} \varphi^A(x, \bar{b}) \geq \inf_{x \in B, \|x\| \leq 1} \varphi^A(x, \bar{b}).$$



$|\mathcal{F}_\sigma| = c$

$(\bar{x}_u)_{u\in\mathcal{U}}$

$\varphi_u(\bar{x}_u, z)$

$$\|\varphi\| = \sup_{A,\bar{a}} \|\varphi^A(\bar{a})\|$$

$\mathcal{F}_\sigma$ is $\|\cdot\|$-sep.

$\mathcal{F}_A \models_A$ sep.

$g:\mathbb{R}^n \to \mathbb{R}$

$\mathrm{supp}(g)$ cpt

Stone–Weierstrass,

w.l.o.g., g is a polynomial

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$C_0 \prec C_1 \prec C_2 \prec \dots$

Lemma *If C is a C^*-algebra of density character $\aleph_1$, then $C = \bigcup_{\alpha < \aleph_1} C_\alpha$ for a continuous $\aleph_1$-chain of separable elementary submodels C_α, for $\alpha < \aleph_1$.*

C

$C_\beta = \bigcup_{\alpha < \beta} C_\alpha$

(*Continuous* means that $C_\beta = \lim_{\alpha < \beta} C_\alpha$ for every limit ordinal β.)

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club

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Exercise. If A has density character $\aleph_1$ and it is the union of a continuous chain $(A_\alpha)_{\alpha < \aleph_1}$ of separable substructures, then $\mathcal{C}' := \{A_\alpha | A_\alpha \prec A\}$ is a continuous chain of separable substructures and $A = \bigcup \mathcal{C}'$.

What an isomorphism *has* to look like

$\Phi : A^{\mathcal{U}} \to A^{\mathcal{U}}$

$\phi : A \to B$

Lemma *Suppose that A and B have density character $\aleph_1$ and Φ is an isomorphism from A onto B. Then A and B can be represented as increasing unions of countable chains of separable elementary substructures, $A = \bigcup_\alpha A_\alpha$, $B = \bigcup_\alpha B_\alpha$, so that $\Phi[A_\alpha] = B_\alpha$ for all α.*

Pf. Let $A = \bigcup_\alpha A_\alpha^0$, $B = \bigcup_\alpha B_\alpha^0$

(sep. elem., ctns chain)

$\exists \alpha(0)$ Fix $\beta < \delta$,

$\phi[A$] $\subseteq B$

$\phi[A_B] \subseteq B_{\alpha(0)}$

$\phi^{-1}[B_{\alpha(0)}] \subseteq A_{\alpha(1)}$ $\quad (\alpha(1) > \alpha(0))$

$\phi[A_{\alpha(1)}] \subseteq B_{\alpha(2)}$

$\vdots$

Find $\alpha(n)$, $n \in \mathbb{N}$ $\nearrow$

$$B_{\alpha(2k-2)} \subseteq \phi[A_{\alpha(2k+1)}] \subseteq B_{\alpha(2k)}$$

$\forall k \geq 1$

$$\bigcup_k B_{\alpha(2k)} = \bigcup \phi[A_{\alpha(2k+1)}]$$

$$= B_{\sup \alpha(k)} = \phi[A_{\sup \alpha(k)}]$$

This proves that

$\{\gamma < \aleph_1 \mid \phi[A_\gamma] = B_\gamma\}$

is unbounded in $\aleph_1$

It is also closed (under sups of cble subsets)