Massive C^*-algebras, Winter 2021
Ilijas Farah. Lecture 5, January 25

Last class I said the following:

Prop (Exercise 15.6.4) *Suppose C is infinite-dimensional and countably degree-1 saturated.*

1. *Then C is non-separable.*
2. *Every masa (maximal abelian C^*-subalgebra) in C is nonseparable.*
3. *C is not a von Neumann algebra.*

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. . . and all this is correct. The doubts I expressed were caused by the following.

Prop *There exists a countably degree-1 saturated, infinite-dimensional C^*-algebra C whose center $Z(C)$ is separable and infinite-dimensional. (Hence $Z(C)$ is not countably degree-1 saturated.)*

Thm (Voiculescu) *If A is a separable unital C^*-subalgebra of $\mathcal{Q}(H)$, then $(A' \cap \mathcal{Q}(H))' = A$.*

$C(X)$

$C([0,1]) \hookrightarrow \mathcal{Q}(H)$

$C = C([0,1])' \cap \mathcal{Q}(H)$

$Z(C) = C([0,1])$

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Quoting from Brown–Ozawa, *C^*-algebras and finite-dimensional approximations*, Amer. Math. Soc., 2008:

1.7. Voiculescu's Theorem

Voiculescu's Theorem is analogous to the Hahn-Banach Theorem in two ways: It gets used all of the time; and it really refers to a collection of related results and corollaries.[4] Here, we collect all the forms we need, though we only prove those which haven't yet appeared in a book.

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Question *[Pedersen] If C is the corona of a σ-unital C^*-algebra and C is simple, does every separable unital C^*-subalgebra A of C satisfy $(A' \cap C)' = A$?*

(When is a corona simple? See the early work of Huaxin Lin.)

Ultrafilters, ultraproducts, ultrapowers

Def *Given a set* $\mathbb{J}$ $= \mathbb{N}$*, a filter* $\mathcal{U}$ *on* $\mathbb{J}$ *is ultrafilter if for every* $Y \subseteq \mathbb{J}$ *exactly one of* Y *and* $\mathbb{J} \setminus Y$ *belongs to* $\mathcal{U}$*.*

$\{ Y \subseteq J \mid j \in Y \}$

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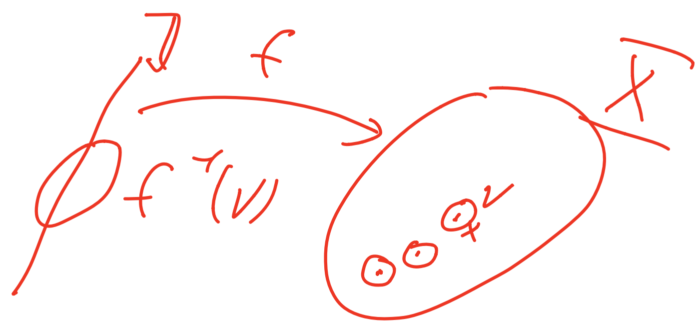
Lemma

If X is a compact Hausdorff space, $\mathcal{U}$ is an ultrafilter on $\mathbb{J}$, and $f : \mathbb{J} \to X$, then there exists a unique $x \in X$ such that $f^{-1}(V) \in \mathcal{U}$ for every open $V \ni x$.
We write $x = \lim_{j \to \mathcal{U}} f(j)$.

$\{\bar{V} \mid V \subseteq X, \text{open}, f^{-1}(V) \in \mathcal{U}\}$



Ultraproducts in analysis

Def D.2.14, C.7.1 *Suppose $\mathcal{U}$ is an ultrafilter on an index set $\mathbb{J}$, A_j, for $j \in \mathbb{J}$, are C^*-algebras. Then*

$$c_{\mathcal{U}} = \{a \in \prod_j A_j : \lim_{j \to \mathcal{U}} \|a_j\| = 0\}$$

is a two-sided, self-adjoint, norm-closed ideal of $\prod_j A_j$, and the quotient

$$\prod_{\mathcal{U}} A_j := \prod_j A_j / c_{\mathcal{U}}$$

is the (norm) ultraproduct associated to $\mathcal{U}$. If all A_j are equal to some A, the ultraproduct is denoted $A_{\mathcal{U}}$[1] *and called ultrapower.*

Exercise. If A_j is unital for all $j \in \mathbb{J}$, then $\mathcal{M}(c_{\mathcal{U}}) \cong \prod_j A_j$ and $\prod_{\mathcal{U}} A_j$ is isomorphic to the corona of $c_{\mathcal{U}}$.

[1] Or $A^{\mathcal{U}}$; we'll get back to the choice of the notation.

Languages, 1: terms

Chang–Keisler: Model theory = logic + universal algebra
Continuous model theory ≈ functional analysis

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Suppose A is a C^*-algebra, $n \geq 1$, and x_j, $j \in \mathbb{N}$, are non-commuting variables.
Each variable belongs to a *sort*: $x_{2^k(2j+1)}$ ranges over the k-ball.
$A[\bar{x}]$: The algebra of *-polynomials in $\bar{x}$ with coefficients in A.

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$A[\bar{x}]$: The algebra of *-polynomials in $\bar{x}$ with coefficients in A.

If $A \leq B$ (with $1_A = 1_B$ if A is unital) then $P(\bar{x}) \in A[\bar{x}]$ defines the evaluation function

$$B^{\mathbb{N}} \mapsto B : \bar{b} \mapsto P(b_0, \dots, b_{n-1}).$$

Def *The elements of $A[\bar{x}]$ are called terms over A.*

(We'll eventually expand the language, but for now this will do.)

Languages, 2: formulas

Def D.2.2 *The space $\mathfrak{F}_A$ of formulas over A is defined recursively:*

1. *The atomic formulas are expressions of the form $\|P(\bar{x})\|$, for $P(\bar{x})$ a term over A.*
2. *If $n \geq 1$, $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function, and $\varphi_0, \ldots, \varphi_{n-1}$ are formulas then $f(\varphi_0, \ldots, \varphi_{n-1})$ is a formula.*
3. *If φ is a formula, x is a variable of the appropriate sort, and $k < \infty$, then both $\inf_{\|x\| \leq k} \varphi$ and $\sup_{\|x\| \leq k} \varphi$ are formulas.*

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The space $\mathfrak{F}_A$ of formulas over A has an algebra structure.

If $A \leq B$ and $\varphi(\bar{x}) \in \mathfrak{F}_A$, define the interpretation (i.e., evaluation) φ^B from (an appropriate sort of) B into $\mathbb{R}$.

Examples of formulas over $\mathbb{C}$

Example

1. $\|[x,y]\|$.
2. $\sup_{\|y\|\leq 1} \|[x,y]\|$.
3. $\sup_{\|x\|\leq 1} \sup_{\|y\|\leq 1} \|[x,y]\|$.
4. $\sup_{\|x\|\leq 1} |\|x\|^2 - \|xx^*\||$.

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3. $\sup_{\|x\| \leq 1} \sup_{\|y\| \leq 1} \|[x, y]\|$.
4. $\sup_{\|x\| \leq 1} |\|x\|^2 - \|xx^*\||$.
5. We can expand the language by continuous functional calculus $\inf_{\|y\| \leq k} \|x - \exp(i\pi y^* y)\|$.

exp(ia) exp(ib)

$\sum \frac{x^n}{n!}$

Induction/recursion on complexity of the formula

In order to prove that all formulas in $\mathfrak{F}_A$ have a certain property $\mathbb{P}$, it suffices to prove the following:

1. $\mathbb{P}(\varphi)$ for every atomic φ.
2. If $\mathbb{P}(\varphi_0), \ldots, \mathbb{P}(\varphi_{n-1})$ holds and g is continuous, then $\mathbb{P}(g(\varphi_0, \ldots, \varphi_{n-1}))$ holds.
3. If $\mathbb{P}(\varphi)$ holds and x is a variable, then $\mathbb{P}(\sup_{\|x\| \leq k} \varphi)$ holds and $\mathbb{P}(\inf_{\|x\| \leq k} \varphi)$ holds.

This is *induction on complexity of the formula*. Similarly, if one needs to define something for all formulas, this is usually done by *recursion on complexity of the formula*.

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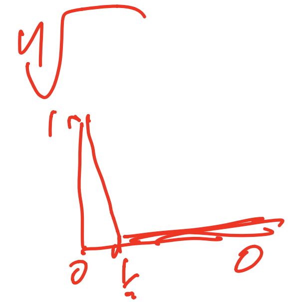
Def *If $\varphi(\bar{x})$ is in $\mathfrak{F}_A$, $A \leq B$, $\bar{b}$ in B of the same 'sort' as $\bar{x}$, define the interpretation $\varphi^B(\bar{b})$ by recursion on complexity of φ.*

On B^n consider the norm



$$\|\bar{x}\| := \max_{i<n} \|x_j\|.$$

Lemma D.2.3 *To every term $P(\bar{x})$ over A and every formula $\varphi(\bar{x})$ over A one can associate a uniform continuity modulus so that if $A \leq B$ then the interpretations P^B and φ^B satisfy this uniform continuity modulus, and their ranges are bounded subsets of B and $\mathbb{R}$, respectively.*

The fundamental theorem of ultraproducts

Thm 16.2.8, Łoś's Theorem *If* $A \leq A_j$ *for all* $j \in \mathbb{J}$, $\mathcal{U}$ *is an ultrafilter on* $\mathbb{J}$, $\varphi(\bar{x})$ *is a formula over* A, *and* $C := \prod_{\mathcal{U}} A_j$, *then* $\varphi^C(\bar{a}) = \lim_{j \to \mathcal{U}} \varphi^{A_j}(\bar{a}_j)$ *for every* $\bar{a} = (\bar{a}_j)_{j \in \mathbb{J}}$ *in* $\prod_{\mathcal{U}} A_j$ *of the appropriate sort.*

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Proof by induction on complexity:

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$\lim_{j\to\mathcal{U}} f(\varphi_0 \dots \varphi_{n-1})$

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$\lim_{j \to \mathcal{U}} \inf_{\|x\| \leq k} \varphi(x) = r$ fix $\varepsilon > 0$

$$Y = \{ j \mid | \inf_{\|x\| \le u} \psi^{A_j}(x_j) - r | < \varepsilon \} \in \mathcal{U}$$

For $i \in Y$ fix $x_i \in A_i$, $\|x_i\| \le h$

$$\underline{|\psi^{A_i}(x_i) - r| < \varepsilon} \qquad x = (x_i)$$

$$\psi^{C}(x) = \lim_{i \to u} \psi^{A_i}(x_i) \approx \underline{\underline{r}}$$

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$\sup \psi = -\inf(-\psi)$

Type as a functional

The algebra $\mathfrak{F}_A$ of formulas over A can be endowed with a seminorm,

$$\|\varphi(\bar{x})\| = \sup_{B,\bar{b}} |\varphi^B(\bar{b})|.$$

(The sup is taken over all $A \leq B$ and $\bar{b}$ in B of the appropriate sort.)

Let $\mathfrak{W}_A$ be the Banach algebra obtained by quotienting and completing $\mathfrak{F}_A$ with respect to $\|\cdot\|$.

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Def 16.1.4, roughly *If $A \leq C$ and $\bar{b} \in C^{\mathbb{N}}$, the type of $\bar{b}$ is the evaluation character (with $\bar{x}$ of the appropriate sort) on $\mathfrak{F}_A^{\bar{x}} := \{\varphi(\bar{x}) | \varphi(\bar{x}) \in \mathfrak{F}_A\}$:*

$$\mathfrak{F}_A^{\bar{x}} \mapsto \mathbb{R} \colon \varphi(\bar{x}) \mapsto \varphi^C(\bar{b}).$$

It is denoted $\mathrm{type}_C(\bar{b}/A)$.

The type of $\bar{b}$ over A codes all first-order properties of $\bar{b}$ with parameters in A.

Def *If $B \leq C$, we say that B is an elementary submodel of C, and write $B \prec C$, if $\varphi^B(\bar{b}) = \varphi^C(\bar{b})$ for all $\varphi \in \mathfrak{F}_B$. (Equiv., for all $\varphi \in \mathfrak{F}_A$, for a fixed $A \leq B$.)*

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Thm (Łoś) $A \preceq A_{\mathcal{U}}$.

Type as a set of conditions

Def 16.1.2 *A condition (over A) is an expression of the form* $\varphi(\bar{x}) = r$ *for* $r \in \mathbb{R}$ *and* $\varphi(\bar{x})$ *in* $\mathfrak{F}_A$.
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If $A \leq C$, a type $\mathrm{t}(\bar{x})$ is realized in C if there exists $\bar{b}$ of the appropriate sort in C such that every condition in $\mathrm{t}(\bar{x})$ is satisfied by $\bar{b}$.

$\varphi^C(\bar{b}) = r$

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A type $\mathrm{t}(\bar{x})$ *is approximately realized (or satisfiable) in C if for every finite subset* $\mathrm{t}_0(\bar{x})$ *of* $\mathrm{t}(\bar{x})$ *and every* $\varepsilon > 0$ *there exists* $\bar{b}$ *of the appropriate sort in C such that for every condition* $\varphi(\bar{x}) = r$ *in* $\mathrm{t}_0(\bar{x})$ *we have* $|\varphi^C(\bar{b}) - r| < \varepsilon$. *Such* $\bar{b}$ *is a partial realization of* $\mathrm{t}(\bar{x})$.

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Lemma
If $A \leq C$, $b \in C^n$, *then*
$\ker(\mathrm{type}_C(\bar{b}/A)) = \{\varphi(\bar{x}) - r : \varphi^C(\bar{b}) = r\}$.

(Full) countable saturation

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An ultrafilter $\mathcal{U}$ on a set X is *countably incomplete* if there exists a partition of X into countably many sets X_j, for $j \in \mathbb{N}$, neither of which belongs to $\mathcal{U}$.
Every nonprincipal ultrafilter on $\mathbb{N}$ is countably incomplete.

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Thm 16.4.1 *Suppose that* $\mathcal{U}$ *is a countably incomplete ultrafilter on* $\mathbb{J}$ *and that* A_j*, for* $j \in \mathbb{J}$*, are* C^**-algebras. Then the ultraproduct* $C := \prod_{\mathcal{U}} A_j$ *is countably saturated.* This is a relative to Kirchberg's ε-test.

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Kirchberg's ε-test.
Proof: Fix a type $\varphi_n(\bar{x}) = r_n$, for $n \in \mathbb{N}$. Fix $\bar{b}(n)$ such that all $k \leq n$ satisfy $|\varphi_k^C(\bar{b}(n)) - r_k| < \frac{1}{n}$.

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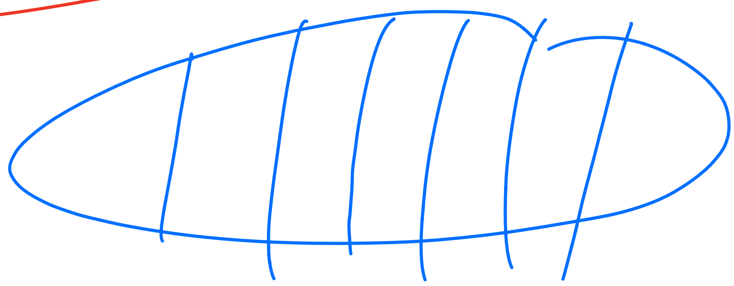
By Łoś, there is $Y_n \in \mathcal{U}$ such that all $j \in Y_n$ and all $k \leq n$ satisfy

$$|\varphi_k^{A_j}(\bar{b}(n)_j) - r_k| < \frac{1}{n}.$$



Def. $\bar{b} \in \bigcap_j A_j$

we want:

$\bar{b}_j = \bar{b}(1)_j$, $j \in Y_1 \setminus Y_2$

$\bar{b}_j = \bar{b}(2)_j$, $j \in Y_2 \setminus Y_3$

⋮

$\bar{b}_j = \bar{b}(n)_j$, $j \in Y_n \setminus Y_{n+1}$

If $\bigcap_j Y_j = \emptyset$, then

$\forall k$, $\varphi_k^{C}(\bar{b}) = \lim_{i \to u} \varphi_k^{A_i}(\bar{b}_i)$

$= r_k$

To assure $\bigcap_j Y_j = \emptyset$,

Let $Y_1' = Y_1 \cup X_1$, $Y_n' = \left(Y_n \cup \left(\bigcup_{i \le n} X_i\right)\right) \setminus \bigcup_{j<n} Y_j$

Fix $k \in \mathbb{N}$

$$\varphi_k^{\Pi_\mathcal{U} A_i}(\bar{b}) = \lim_{i \to \mathcal{U}} \varphi_k^{A_i}(\bar{b}_i)$$

$$\lim_{i \to \mathcal{U}} s_i = r_k$$

$$\Leftrightarrow \forall \varepsilon > 0$$

$$\{ i \mid |s_i - r_k| < \varepsilon \} \in \mathcal{U}$$

$$Z_j = \bigcup_{n \geq j} X_n \in \mathcal{U}$$

$$Y_j' = Y_j \cap Z_j$$

Coro *If $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$ and A_n, for $n \in \mathbb{N}$, are* C^**-algebras,* $C := \prod_{\mathcal{U}} A_n$*, and* $B \leq C$ *is separable, then each one of C and $B' \cap C$ is countably degree-1 saturated. It is therefore SAW*, CRISP, every uniformly bounded reporesentation of an amenable group into it is uniformizable, it allows 'discontinuous functional calculus', it is essentially non-factorizable, satisfies the conclusion of Kasparov's Technical Theorem, etc.*

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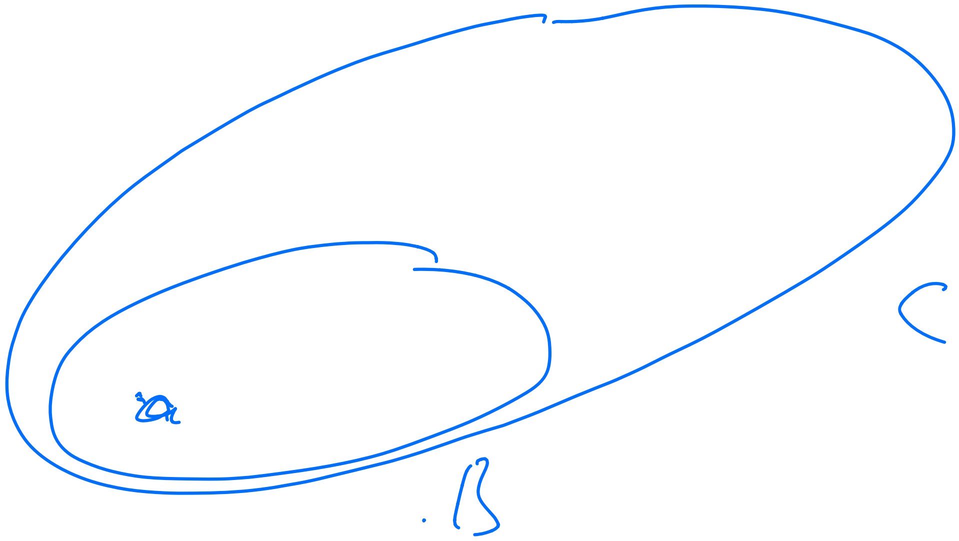
Coro (Effros–Rosenberg, Kirchberg) *For every separable B the following are equivalent*

1. *For all $F \Subset B$, for all $\varepsilon > 0$, $M_2(\mathbb{C}) \hookrightarrow B$ so that it ε-commutes with all $b \in F$.*
2. $M_2(\mathbb{C}) \hookrightarrow B_{\mathcal{U}} \cap B'$.
3. $M_2(\mathbb{C}) \hookrightarrow B_{\mathcal{U}} \cap C'$ *for every separable* $C \leq B_{\mathcal{U}}$.
4. $\bigotimes_{\mathbb{N}} M_2(\mathbb{C}) \hookrightarrow B_{\mathcal{U}} \cap B'$
5. $\bigotimes_{\aleph_1} M_2(\mathbb{C}) \hookrightarrow B_{\mathcal{U}} \cap B'$.



$B \prec C \qquad \exists_{\mathbb{C}}$

$A \leq B$

$\underline{\underline{\exists_{\{a\}}}}$

$type_B(a) = type_{\mathbb{C}}(a)$

$\varphi^B(\bar{x}, a) = \varphi^C(\bar{x}, a)$

$\varphi^B(\bar{x}, y) = \varphi^C(\bar{x}, y)$

$\mathbb{C} \cdot 1_A \simeq \mathbb{C} \qquad \lambda \in \mathbb{C}$

$\underline{\underline{\mathbb{C}}} \leq A \qquad \frac{\lambda \cdot a}{\underline{P(\bar{x})}}$

$\varphi[x]$

A formula φ is *quantifier-free* if it does not involve quantifiers sup or inf; that is, $\varphi = f(\psi_0, \ldots, \psi_{n-1})$ for atomic formulas ψ_j, $j < n$. Quantifier-free formulas over A form an algebra.

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Quantifier-free formulas over A form an algebra.
Quantifier-free conditions, quantifier-free types, and quantifier-free saturation are defined in a natural way.
For $n \geq 2$, degree-n conditions, degree-n types, and degree-n saturation are defined in a natural way.

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Fact

Saturation $\Rightarrow$ quantifier-free saturation $\Rightarrow \ldots \Rightarrow$ degree-$n+1$ saturation $\Rightarrow$ degree-n saturation $\Rightarrow$ $\ldots \Rightarrow$ degree-2 saturation $\Rightarrow$ degree-1 saturation

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Q: Which, if any, of these arrows are reversible?

Prop *If C is countably saturated and $A \leq C$ is separable, then $A' \cap C$ is countably quantifier-free saturated but not necessarily countably saturated.*

A proof can be found in today's lecture.

Notably, the proofs of Łoś's Theorem and countable saturation of ultraproducts have nothing to do with C^*-algebras. They are general theorems of model theory, applicable to arbitrary (appropriately defined) metric structures.
Let's take a look at a relevant example.

Tracial ultraproducts

Def *A state* τ *on a* C^**-algebra is a positive functional of norm 1. It is tracial if* $\tau(ab) = \tau(ba)$ *for all* a *and* b *in* A*.*
$T(A) := \{\tau | \tau \text{ is a tracial state on } A\}$.

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Def *A unital* C^**-algebra A is finite if there is no* $v \in A$ *such that* $v^*v = 1_A$ *and* $vv^* < 1_A$.
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if $T(A) \neq \emptyset$ *then A is stably finite.*

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if $T(A) \neq \emptyset$ then A is stably finite.

The converse is an open problem (deep partial results by Haagerup, Kirchberg, Haagrup–Thornbjørsen.)
(Note that 'A is not finite' is equivalent to $\psi^A = 0$, with ψ defined as

$$\inf_{\|x\| \leq 1} \|1 - x^*x\| + \|x^*xxx^* - x^*x\| + |1 - \|x^*x - xx^*\||.$$

Lemma *If* $\tau \in T(A)$ *then*

$$\|a\|_{2,\tau} := \tau(a^*a)^{1/2}$$

is a seminorm on A *and* $J_\tau := \{a | \|a\|_{2,\tau} = 0\}$ *is an ideal of* A.
If $T(A) \neq \emptyset$, *then*

$$\|a\|_{2,u} := \sup_{\tau \in T(A)} \|a\|_{2,\tau}$$

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Exercise. If A is abelian, then $\|\cdot\|$ and $\|\cdot\|_{2,u}$ agree on A.

Caveat: $\|\cdot\|_{2,u}$ is uniformly continuous with respect to $\|\cdot\|$, but not vice versa, except in very specific situations.

Tracial ultraproduct

Def D.2.14, C.7.1 *Suppose* $\mathcal{U}$ *is an ultrafilter on an index set* $\mathbb{J}$, A_j, *for* $j \in \mathbb{J}$, *are unital* C^**-algebras with* $T(A_j) \neq \emptyset$. *Then*

$$J_{\mathcal{U}} := \{a \in \prod_j A_j : \lim_{j \to \mathcal{U}} \|a_j\|_{2,u} = 0\}$$

is a two-sided, self-adjoint, norm-closed ideal of $\prod_j A_j$, *and the quotient*

$$\prod^{\mathcal{U}} A_j := \prod_j A_j / c_{\mathcal{U}}$$

is the (tracial) ultraproduct associated to $\mathcal{U}$. *If all* A_j *are equal to some* A, *the tracial ultraproduct is denoted* $A^{\mathcal{U}}$ *and called tracial ultrapower.*

(See e.g., C. Schafhauser *A new proof of the Tikuisis–White–Winter theorem*, Crelle, 2020 or Castillejos et. al., *Nuclear dimension of simple* C^**-algbras*, Inv. Math. 2020)

Formulas, revisited

Recall that $A[\bar{x}]$ is the algebra of *-polynomials in $\bar{x}$ with coefficients in A, called *terms*. Suppose $T(A) \neq \emptyset$ and A is unital.

Def D.2.2 *Formulas over A are defined recursively:*

1. *The atomic formulas are expressions of the form $\|P(\bar{x})\|_2$ for $P(\bar{x})$ a term over A.*
2. *If $n \geq 1$, $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function, and $\varphi_0, \ldots, \varphi_{n-1}$ are formulas then $f(\varphi_0, \ldots, \varphi_{n-1})$ is a formula.*
3. *If φ is a formula, x is a variable, and $k < \infty$, then both $\inf_{\|x\| \leq k} \varphi$ and $\sup_{\|x\| \leq k} \varphi$ are formulas.*

The space $\mathfrak{F}_{A,t}$ of formulas over A has an algebra structure.

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Def *If* $\varphi(\bar{x})$ *is in* $\mathfrak{F}_{A,t}$, $A \leq B$, $T(B) \neq \emptyset$, $\bar{b}$ *in* B *of the same 'sort' as* $\bar{x}$, *define the interpretation* $\varphi^B(\bar{b})$ *by recursion on complexity of* φ.

Łoś's Theorem and countable saturation hold for tracial ultraproducts

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Thm 16.2.8, Łoś's Theorem *If $A \leq A_j$ are unital, $T(A_j) \neq \emptyset$ for all $j \in \mathbb{J}$, $\mathcal{U}$ is an ultrafilter on $\mathbb{J}$, $\varphi(\bar{x}) \in \mathfrak{F}_{A,t}$, and $C := \prod^{\mathcal{U}} A_j$, then $\varphi^C(\bar{a}) = \lim_{j \to \mathcal{U}} \varphi^{A_j}(\bar{a}_j)$ for all $\bar{a}$ in $\prod^{\mathcal{U}} A_j$ of the appropriate sort.*

Thm 16.4.1 *Suppose that $\mathcal{U}$ is a countably incomplete ultrafilter on $\mathbb{J}$ and that A_j, for $j \in \mathbb{J}$, are unital C^*-algebras with $T(A_j) \neq \emptyset$. Then the ultraproduct $C := \prod^{\mathcal{U}} A_j$ is countably saturated (with respect to the tracial language $\mathfrak{F}_{C,t}$).*

Łoś's Theorem and countable saturation hold for tracial ultraproducts

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Coro *If* $\mathcal{U}$ *is a nonprincipal ultrafilter on* $\mathbb{N}$ *and* A_n*, for* $n \in \mathbb{N}$*, are unital* C^**-algebras,* $T(A_n) \neq \emptyset$*,* $C := \prod^{\mathcal{U}} A_n$*, then* C *is countably saturated with respect to* $\mathfrak{F}_{C,t}$*. It is therefore SAW*, CRISP,...*

Q: If $a \in C$, $0 \leq a \leq 1$ and $0 \in \mathrm{sp}(a)$, is $a^{\perp} \cap C \neq \{0\}$?

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Q: If $a \in C$, $0 \leq a \leq 1$ and $0 \in \mathrm{sp}(a)$, is $a^{\perp} \cap C \neq \{0\}$?
A: Not necessarily! Let's see why.

Example

Let A be the CAR algebra M_{2^∞}. It has a unique tracial state τ. let $C := A^{\mathcal{U}}$. Choose $a \in A_+$ such that $\mathrm{sp}(a) = [0,1]$ and $\tau^{\mathcal{U}} \restriction \mathrm{C}^*(a) \cong C([0,1])$ is the Lebesgue measure. (I.e., $\tau(f(a)) = \int f \, d\lambda$ for all $f \in C([0,1])$.)

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Recall that $A[\bar{x}]$ is the algebra of *-polynomials in $\bar{x}$ with coefficients in A, called *terms*.

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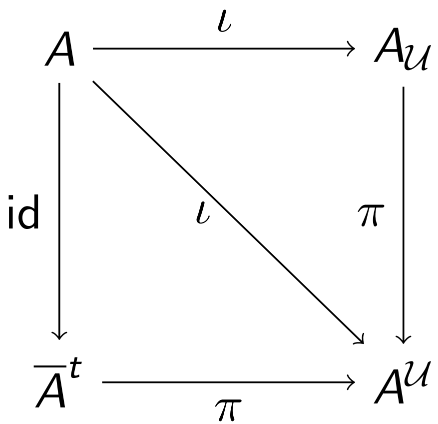
This language describes pairs $(C, C/J)$, where $J = \{a \mid \|a\|_{2,u} = 0\}$ (the quotient map $\pi \colon C \to C/J$ is definable in this language).

Łoś's Theorem and countable saturation hold for tracial ultraproducts



Suppose that A is a separable C^*-algebra, $T(A) \neq \emptyset$. If $D \leq A_{\mathcal{U}}$ is separable and $a \in \pi[D]' \cap A^{\mathcal{U}}$, consider the type with conditions

$$\|a - x\|_2 = 0, \|[d, x]\| = 0, \ d \in D.$$

This type is consistent and "countable".

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This type is consistent and "countable".
So there is $\tilde{a} \in A_\mathcal{U} \cap D'$ such that $\pi(\tilde{a}) = a$.

Prop (Sato, Kirchberg–Rørdam) *If $T(A) \neq \emptyset$ and $D \leq A_\mathcal{U}$ is separable, then $\pi[D' \cap A_\mathcal{U}] = \pi[D]' \cap A^\mathcal{U}$.*

Kirchberg's invariant:
$F(A) = (A_{\mathcal{U}} \cap A')/(A^{\perp} \cap A_{\mathcal{U}})$.

Fact

If A is unital, then $F(A) = A_{\mathcal{U}} \cap A'$.

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Thm (F.–Hart–Sherman) *The answer to either question cannot be decided in ZFC.*

Massive C^*-algebras, Winter 2021
Ilijas Farah. Lecture 5, January 25

Last class I said the following:

Prop (Exercise 15.6.4) *Suppose C is infinite-dimensional and countably degree-1 saturated.*

1. *Then C is non-separable.*
2. *Every masa (maximal abelian C^*-subalgebra) in C is nonseparable.*
3. *C is not a von Neumann algebra.*

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. . . and all this is correct. The doubts I expressed were caused by the following.

Prop *There exists a countably degree-1 saturated, infinite-dimensional C^*-algebra C whose center $Z(C)$ is separable and infinite-dimensional. (Hence $Z(C)$ is not countably degree-1 saturated.)*

Thm (Voiculescu) *If A is a separable unital C^*-subalgebra of $\mathcal{Q}(H)$, then $(A' \cap \mathcal{Q}(H))' = A$.*

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Quoting from Brown–Ozawa, *C^*-algebras and finite-dimensional approximations*, Amer. Math. Soc., 2008:

1.7. Voiculescu's Theorem

Voiculescu's Theorem is analogous to the Hahn-Banach Theorem in two ways: It gets used all of the time; and it really refers to a collection of related results and corollaries.[4] Here, we collect all the forms we need, though we only prove those which haven't yet appeared in a book.

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Question *[Pedersen] If C is the corona of a σ-unital C^*-algebra and C is simple, does every separable unital C^*-subalgebra A of C satisfy $(A' \cap C)' = A$?*

(When is a corona simple? See the early work of Huaxin Lin.)

Ultrafilters, ultraproducts, ultrapowers

Def *Given a set* $\mathbb{J}$, *a filter* $\mathcal{U}$ *on* $\mathbb{J}$ *is ultrafilter if for every* $Y \subseteq \mathbb{J}$ *exactly one of* Y *and* $\mathbb{J} \setminus Y$ *belongs to* $\mathcal{U}$.

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Lemma

If X is a compact Hausdorff space, $\mathcal{U}$ is an ultrafilter on $\mathbb{J}$, and $f : \mathbb{J} \to X$, then there exists a unique $x \in X$ such that $f^{-1}(V) \in \mathcal{U}$ for every open $V \ni x$.
We write $x = \lim_{j \to \mathcal{U}} f(j)$.

Ultraproducts in analysis

Def D.2.14, C.7.1 *Suppose* $\mathcal{U}$ *is an ultrafilter on an index set* $\mathbb{J}$, A_j, *for* $j \in \mathbb{J}$, *are* C^**-algebras. Then*

$$c_{\mathcal{U}} = \{a \in \prod_j A_j : \lim_{j \to \mathcal{U}} \|a_j\| = 0\}$$

is a two-sided, self-adjoint, norm-closed ideal of $\prod_j A_j$, *and the quotient*

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is the (norm) ultraproduct associated to $\mathcal{U}$. *If all* A_j *are equal to some* A, *the ultraproduct is denoted* $A_{\mathcal{U}}$[1] *and called ultrapower.*

Exercise.If A_j is unital for all $j \in \mathbb{J}$, then $\mathcal{M}(c_{\mathcal{U}}) \cong \prod_j A_j$ and $\prod_{\mathcal{U}} A_j$ is isomorphic to the corona of $c_{\mathcal{U}}$.

---

[1] Or $A^{\mathcal{U}}$; we'll get back to the choice of the notation.

Languages, 1: terms

Chang–Keisler: Model theory $=$ logic $+$ universal algebra
Continuous model theory $\approx$ functional analysis

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Suppose A is a C^*-algebra, $n \geq 1$, and x_j, $j \in \mathbb{N}$, are non-commuting variables.
Each variable belongs to a *sort*: $x_{2^k(2j+1)}$ ranges over the k-ball.
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$A[\bar{x}]$: The algebra of *-polynomials in $\bar{x}$ with coefficients in A.
If $A \leq B$ (with $1_A = 1_B$ if A is unital) then $P(\bar{x}) \in A[\bar{x}]$ defines the evaluation function

$$B^{\mathbb{N}} \mapsto B : \bar{b} \mapsto P(b_0, \ldots, b_{n-1}).$$

Def *The elements of $A[\bar{x}]$ are called terms over A.*

(We'll eventually expand the language, but for now this will do.)

Languages, 2: formulas

Def D.2.2 *The space $\mathfrak{F}_A$ of formulas over A is defined recursively:*

1. *The atomic formulas are expressions of the form $\|P(\bar{x})\|$, for $P(\bar{x})$ a term over A.*
2. *If $n \geq 1$, $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function, and $\varphi_0, \ldots, \varphi_{n-1}$ are formulas then $f(\varphi_0, \ldots, \varphi_{n-1})$ is a formula.*
3. *If φ is a formula, x is a variable of the appropriate sort, and $k < \infty$, then both $\inf_{\|x\| \leq k} \varphi$ and $\sup_{\|x\| \leq k} \varphi$ are formulas.*

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The space $\mathfrak{F}_A$ of formulas over A has an algebra structure.

If $A \leq B$ and $\varphi(\bar{x}) \in \mathfrak{F}_A$, define the interpretation (i.e., evaluation) φ^B from (an appropriate sort of) B into $\mathbb{R}$.

Examples of formulas over $\mathbb{C}$

Example

1. $\|[x, y]\|$.
2. $\sup_{\|y\| \leq 1} \|[x, y]\|$.
3. $\sup_{\|x\| \leq 1} \sup_{\|y\| \leq 1} \|[x, y]\|$.
4. $\sup_{\|x\| \leq 1} |\|x\|^2 - \|xx^*\||$.

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4. $\sup_{\|x\| \leq 1} |\|x\|^2 - \|xx^*\||$.
5. We can expand the language by continuous functional calculus $\inf_{\|y\| \leq k} \|x - \exp(i\pi y^* y)\|$.

Induction/recursion on complexity of the formula

In order to prove that all formulas in $\mathfrak{F}_A$ have a certain property $\mathbb{P}$, it suffices to prove the following:

1. $\mathbb{P}(\varphi)$ for every atomic φ.
2. If $\mathbb{P}(\varphi_0), \ldots, \mathbb{P}(\varphi_{n-1})$ holds and g is continuous, then $\mathbb{P}(g(\varphi_0, \ldots, \varphi_{n-1}))$ holds.
3. If $\mathbb{P}(\varphi)$ holds and x is a variable, then $\mathbb{P}(\sup_{\|x\| \leq k} \varphi)$ holds and $\mathbb{P}(\inf_{\|x\| \leq k} \varphi)$ holds.

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Def *If $\varphi(\bar{x})$ is in $\mathfrak{F}_A$, $A \leq B$, $\bar{b}$ in B of the same 'sort' as $\bar{x}$, define the interpretation $\varphi^B(\bar{b})$ by recursion on complexity of φ.*

On B^n consider the norm

$$\|\bar{x}\| := \max_{i<n} \|x_j\|.$$

Lemma D.2.3 *To every term $P(\bar{x})$ over A and every formula $\varphi(\bar{x})$ over A one can associate a uniform continuity modulus so that if $A \leq B$ then the interpretations τ^B and φ^B satisfy this uniform continuity modulus, and their ranges are bounded subsets of B and $\mathbb{R}$, respectively.*

The fundamental theorem of ultraproducts

Thm 16.2.8, Łoś's Theorem *If* $A \leq A_j$ *for all* $j \in \mathbb{J}$, $\mathcal{U}$ *is an ultrafilter on* $\mathbb{J}$, $\varphi(\bar{x})$ *is a formula over* A*, and* $C := \prod_{\mathcal{U}} A_j$*, then* $\varphi^C(\bar{a}) = \lim_{j \to \mathcal{U}} \varphi^{A_j}(\bar{a}_j)$ *for every* $\bar{a} = (\bar{a}_j)_{j \in \mathbb{J}}$ *in* $\prod_{\mathcal{U}} A_j$ *of the appropriate sort.*

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Type as a functional

The algebra $\mathfrak{F}_A$ of formulas over A can be endowed with a seminorm,

$$\|\varphi(\bar{x})\| = \sup_{B,\bar{b}} |\varphi^B(\bar{b})|.$$

(The sup is taken over all $A \leq B$ and $\bar{b}$ in B of the appropriate sort.)

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Def 16.1.4, roughly *If $A \leq C$ and $\bar{b} \in C^{\mathbb{N}}$, the type of $\bar{b}$ is the evaluation character (with $\bar{x}$ of the appropriate sort) on $\mathfrak{F}_A^{\bar{x}} := \{\varphi(\bar{x}) | \varphi(\bar{x}) \in \mathfrak{F}_A\}$:*

$$\mathfrak{F}_A^{\bar{x}} \mapsto \mathbb{R} \colon \varphi(\bar{x}) \mapsto \varphi^C(\bar{b}).$$

It is denoted $\mathrm{type}_C(\bar{b}/A)$.

The type of $\bar{b}$ over A codes all first-order properties of $\bar{b}$ with parameters in A.

Def *If* $B \leq C$*, we say that* B *is an elementary submodel of* C*, and write* $B \preceq C$*, if* $\varphi^B(\bar{b}) = \varphi^C(\bar{b})$ *for all* $\varphi \in \mathfrak{F}_B$*. (Equiv., for all* $\varphi \in \mathfrak{F}_A$*, for a fixed* $A \leq B$*.)*

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Thm (Łoś) $A \preceq A_{\mathcal{U}}$.

Type as a set of conditions

Def 16.1.2 *A condition (over A) is an expression of the form $\varphi(x) = r$ for $r \in \mathbb{R}$ and $\varphi(x)$ in $\mathfrak{F}_A$.*
A type (over A) is a set of conditions over A.

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If $A \leq C$, *a type* $\mathrm{t}(\bar{x})$ *is realized in C if there exists* $\bar{b}$ *of the appropriate sort in C such that every condition in* $\mathrm{t}(\bar{x})$ *is satisfied by* $\bar{b}$.

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A type $\mathrm{t}(\bar{x})$ *is approximately realized (or satisfiable) in C if for every finite subset* $\mathrm{t}_0(\bar{x})$ *of* $\mathrm{t}(\bar{x})$ *and every* $\varepsilon > 0$ *there exists* $\bar{b}$ *of the appropriate sort in C such that for every condition* $\varphi(\bar{x}) = r$ *in* $\mathrm{t}_0(\bar{x})$ *we have* $|\varphi^C(\bar{b}) - r| < \varepsilon$. *Such* $\bar{b}$ *is a partial realization of* $\mathrm{t}(\bar{x})$.

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Lemma
If $A \leq C$*,* $b \in C^n$*, then*
$\ker(\mathrm{type}_C(\bar{b}/A)) = \{\varphi(\bar{x}) - r : \varphi^C(\bar{b}) = r\}$*.*

(Full) countable saturation

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Thm 16.4.1 *Suppose that $\mathcal{U}$ is a countably incomplete ultrafilter on $\mathbb{J}$ and that A_j, for $j \in \mathbb{J}$, are C^*-algebras. Then the ultraproduct $C := \prod_{\mathcal{U}} A_j$ is countably saturated.* This is a relative to

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By Łoś, there is $Y_n \in \mathcal{U}$ such that all $j \in Y_n$ and all $k \leq n$ satisfy

$$|\varphi_k^{A_j}(\bar{b}(n)_j) - r_k| < \frac{1}{n}.$$

Coro *If $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$ and A_n, for $n \in \mathbb{N}$, are C^*-algebras, $C := \prod_{\mathcal{U}} A_n$, and $B \leq C$ is separable, then each one of C and $B' \cap C$ is countably degree-1 saturated. It is therefore SAW*, CRISP, every uniformly bounded reporesentation of an amenable group into it is uniformizable, it allows 'discontinuous functional calculus', it is essentially non-factorizable, satisfies the conclusion of Kasparov's Technical Theorem, etc.*

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Coro (Effros–Rosenberg, Kirchberg) *For every separable B the following are equivalent*

1. *For all $F \Subset B$, for all $\varepsilon > 0$, $M_2(\mathbb{C}) \hookrightarrow B$ so that it ε-commutes with all $b \in F$.*
2. $M_2(\mathbb{C}) \hookrightarrow B_{\mathcal{U}} \cap B'$.
3. $M_2(\mathbb{C}) \hookrightarrow B_{\mathcal{U}} \cap C'$ *for every separable* $C \leq B_{\mathcal{U}}$.
4. $\bigotimes_{\mathbb{N}} M_2(\mathbb{C}) \hookrightarrow B_{\mathcal{U}} \cap B'$
5. $\bigotimes_{\aleph_1} M_2(\mathbb{C}) \hookrightarrow B_{\mathcal{U}} \cap B'$.

A formula φ is *quantifier-free* if it does not involve quantifiers sup or inf; that is, $\varphi = f(\psi_0, \ldots, \psi_{n-1})$ for atomic formulas ψ_j, $j < n$. Quantifier-free formulas over A form an algebra.

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Quantifier-free conditions, quantifier-free types, and quantifier-free saturation are defined in a natural way.

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Fact

Saturation $\Rightarrow$ *quantifier-free saturation* $\Rightarrow \ldots \Rightarrow$ *degree-*$n+1$ *saturation* $\Rightarrow$ *degree-n saturation* $\Rightarrow \ldots \Rightarrow$ *degree-2 saturation* $\Rightarrow$ *degree-1 saturation*

Q: Which, if any, of these arrows are reversible?

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Prop *If C is countably saturated and $A \leq C$ is separable, then $A' \cap C$ is countably quantifier-free saturated but not necessarily countably saturated.*

A proof can be found in today's lecture.

Notably, the proofs of Łoś's Theorem and countable saturation of ultraproducts have nothing to do with C^*-algebras. They are general theorems of model theory, applicable to arbitrary (appropriately defined) metric structures.
Let's take a look at a relevant example.

Tracial ultraproducts

Def *A state* τ *on a* C^**-algebra is a positive functional of norm 1. It is tracial if* $\tau(ab) = \tau(ba)$ *for all a and b in A.*
$T(A) := \{\tau | \tau$ *is a tracial state on* $A\}$.

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Def *A unital* C^**-algebra* A *is finite if there is no* $v \in A$ *such that* $v^*v = 1_A$ *and* $vv^* < 1_A$.
It is stably finite if $M_n(A)$ *is finite for all n.*

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if $T(A) \neq \emptyset$ *then A is stably finite.*

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if $T(A) \neq \emptyset$ then A is stably finite.

The converse is an open problem (deep partial results by Haagerup, Kirchberg, Haagrup–Thornbjørsen.)
(Note that 'A is not finite' is equivalent to $\psi^A = 0$, with ψ defined as

$$\inf_{\|x\| \leq 1} \|1 - x^* x\| + \|x^* x x x^* - x^* x\| + |1 - \|x^* x - x x^*\||.$$

Lemma *If* $\tau \in T(A)$ *then*

$$\|a\|_{2,\tau} := \tau(a^*a)^{1/2}$$

is a seminorm on A *and* $J_\tau := \{a | \|a\|_{2,\tau} = 0\}$ *is an ideal of* A.
If $T(A) \neq \emptyset$, *then*

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Exercise. If A is abelian, then $\|\cdot\|$ and $\|\cdot\|_{2,u}$ agree on A.

Caveat: $\|\cdot\|_{2,u}$ is uniformly continuous with respect to $\|\cdot\|$, but not vice versa, except in very specific situations.

Tracial ultraproduct

Def D.2.14, C.7.1 *Suppose* $\mathcal{U}$ *is an ultrafilter on an index set* $\mathbb{J}$, A_j, *for* $j \in \mathbb{J}$, *are unital* C^**-algebras with* $T(A_j) \neq \emptyset$. *Then*

$$J_{\mathcal{U}} := \{a \in \prod_j A_j : \lim_{j \to \mathcal{U}} \|a_j\|_{2,u} = 0\}$$

is a two-sided, self-adjoint, norm-closed ideal of $\prod_j A_j$, *and the quotient*

$$\prod^{\mathcal{U}} A_j := \prod_j A_j / c_{\mathcal{U}}$$

is the (tracial) ultraproduct associated to $\mathcal{U}$. *If all* A_j *are equal to some* A, *the tracial ultraproduct is denoted* $A^{\mathcal{U}}$ *and called tracial ultrapower.*

(See e.g., C. Schafhauser *A new proof of the Tikuisis–White–Winter theorem*, Crelle, 2020 or Castillejos et. al., *Nuclear dimension of simple* C^**-algbras*, Inv. Math. 2020)

Formulas, revisited

Recall that $A[\bar{x}]$ is the algebra of *-polynomials in $\bar{x}$ with coefficients in A, called *terms*. Suppose $T(A) \neq \emptyset$ and A is unital.

Def D.2.2 *Formulas over A are defined recursively:*

1. *The atomic formulas are expressions of the form* $\|P(\bar{x})\|_2$ *for* $P(\bar{x})$ *a term over A.*
2. *If* $n \geq 1$, $f : \mathbb{R}^n \to \mathbb{R}$ *is a continuous function, and* $\varphi_0, \ldots, \varphi_{n-1}$ *are formulas then* $f(\varphi_0, \ldots, \varphi_{n-1})$ *is a formula.*
3. *If* φ *is a formula, x is a variable, and* $k < \infty$, *then both* $\inf_{\|x\| \leq k} \varphi$ *and* $\sup_{\|x\| \leq k} \varphi$ *are formulas.*

The space $\mathfrak{F}_{A,t}$ *of formulas over A has an algebra structure.*

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Def *If* $\varphi(\bar{x})$ *is in* $\mathfrak{F}_{A,t}$, $A \leq B$, $T(B) \neq \emptyset$, $\bar{b}$ *in* B *of the same 'sort' as* $\bar{x}$, *define the* ***interpretation*** $\varphi^{B}(\bar{b})$ *by recursion on complexity of* φ.

Łoś's Theorem and countable saturation hold for tracial ultraproducts

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Thm 16.2.8, Łoś's Theorem *If* $A \leq A_j$ *are unital,* $T(A_j) \neq \emptyset$ *for all* $j \in \mathbb{J}$, $\mathcal{U}$ *is an ultrafilter on* $\mathbb{J}$, $\varphi(\bar{x}) \in \mathfrak{F}_{A,t}$, *and* $C := \prod^{\mathcal{U}} A_j$, *then* $\varphi^C(\bar{a}) = \lim_{j \to \mathcal{U}} \varphi^{A_j}(\bar{a}_j)$ *for all* $\bar{a}$ *in* $\prod^{\mathcal{U}} A_j$ *of the appropriate sort.*

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Łoś's Theorem and countable saturation hold for tracial ultraproducts

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Coro *If $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$ and A_n, for $n \in \mathbb{N}$, are unital C^*-algebras, $T(A_n) \neq \emptyset$, $C := \prod^{\mathcal{U}} A_n$, then C is countably saturated with respect to $\mathfrak{F}_{C,t}$. It is therefore SAW*, CRISP,...*

Q: If $a \in C$, $0 \leq a \leq 1$ and $0 \in \mathrm{sp}(a)$, is $a^{\perp} \cap C \neq \{0\}$?

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A: Not necessarily! Let's see why.

Example

Let A be the CAR algebra M_{2^∞}. It has a unique tracial state τ. let $C := A^{\mathcal{U}}$. Choose $a \in A_+$ such that $\mathrm{sp}(a) = [0, 1]$ and $\tau^{\mathcal{U}} \restriction \mathrm{C}^*(a) \cong C([0, 1])$ is the Lebesgue measure. (I.e., $\tau(f(a)) = \int f \, d\lambda$ for all $f \in C([0, 1])$.)

Formulas, re-revisited

Recall that $A[\bar{x}]$ is the algebra of *-polynomials in $\bar{x}$ with coefficients in A, called *terms*.

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This language describes pairs $(C, C/J)$, where $J = \{a \mid \|a\|_{2,u} = 0\}$ (the quotient map $\pi\colon C \to C/J$ is definable in this language).

Łoś's Theorem and countable saturation hold for tracial ultraproducts

$$
\begin{array}{ccc}
A & \xrightarrow{\ \iota\ } & A_{\mathcal{U}} \\
{\scriptstyle \mathrm{id}}\downarrow & \searrow^{\iota} & \downarrow{\scriptstyle \pi} \\
\overline{A}^{t} & \xrightarrow[\ \pi\ ]{} & A^{\mathcal{U}}
\end{array}
$$

$$
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Suppose that A is a separable C^*-algebra, $T(A) \neq \emptyset$. If $D \leq A_{\mathcal{U}}$ is separable and $a \in \pi[D]' \cap A^{\mathcal{U}}$, consider the type with conditions

$$\|a - x\|_2 = 0, \|[d, x]\| = 0, \; d \in D.$$

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This type is consistent and "countable".
So there is $\tilde{a} \in A_{\mathcal{U}} \cap D'$ such that $\pi(\tilde{a}) = a$.

Prop (Sato, Kirchberg–Rørdam) *If $T(A) \neq \emptyset$ and $D \leq A_{\mathcal{U}}$ is separable, then $\pi[D' \cap A_{\mathcal{U}}] = \pi[D]' \cap A^{\mathcal{U}}$.*

Kirchberg's invariant:
$F(A) = (A_{\mathcal{U}} \cap A')/(A^{\perp} \cap A_{\mathcal{U}})$.

Fact

If A is unital, then $F(A) = A_{\mathcal{U}} \cap A'$.

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Thm (F.–Hart–Sherman) *The answer to either question cannot be decided in ZFC.*