Massive C^*-algebras

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Winter 2021
Lecture 3, January 18

Tutorials (with Saeed Ghasemi):
Monday, 1-3pm (EST)
Zoom Meeting ID: 940 6387 0029 Passcode: 135882
We now continue the study of coronas using degree-1 conditions and types.

Conditions and types

Def 15.1.1 *A degree-1 condition over a* C^**-algebra* C *is an expression of the form*

$$\|a_0 x a_1 + a_2 x^* a_3 + a_4\| = r \tag{1}$$

with coefficients a_j *in* C *and* $r \in \mathbb{R}_+$.
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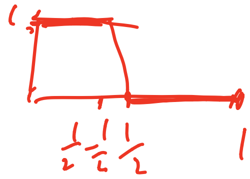
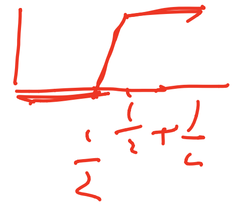
$C([0,1])$

f_n: 

g_n: 

$\|X f_n - f_n\| = 0, \quad \|X g_n\| = 0, \quad n \in \mathbb{N}$

f_n

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(All this can be defined for types in n variables for $n \leq \aleph_0$.)

Def 15.1.4 *A* C^**-algebra* C *is countably degree-1 saturated if every satisfiable countable degree-1 type over* C *in* n *variables, for any* n*, is realized in* C.

$A = \mathcal{K}(\underline{\underline{H}})$

Def 15.1.4 *A* C^**-algebra* C *is countably degree-1 saturated if every satisfiable countable degree-1 type over* C *in* n *variables, for any* n*, is realized in* C*.*

Thm 15.1.5 *The corona of every* σ*-unital, nonunital,* C^**-algebra is countably degree-1 saturated.*

A remark for C^*-algebraists: More is true. Every massive C^*-algebra is countably degree-1 saturated, and ultraproducts associated with free (i.e., nonprincipal) ultrafilters on $\mathbb{N}$ have a stronger property.
We will first prove an easier result, as a warm-up.

Thm *Suppose that B_n, for $n \in \mathbb{N}$, are unital C^*-algebras. The corona of $\bigoplus_n B_n$ is countably degree-1 saturated.*

$$\prod_n B_n = \{ (b_n) \in \bigtimes_n B_n \mid \sup_n \|b_n\| < \infty \}$$

$$\bigoplus_n B_n = \{ (b_n) \in \prod_n B_n \mid \|b_n\| \to 0, \ n \to \infty \}$$

$$e_n = \sum_{j \le n} 1_{B_j} \in \bigoplus_n B_n$$

$$M\left(\bigoplus_n B_n\right) = \prod_n B_n$$

$$\prod_n B_n \Big/ \bigoplus_n B_n$$

Thm *Suppose that B_n, for $n \in \mathbb{N}$, are unital C^*-algebras. The corona of $\bigoplus_n B_n$ is countably degree-1 saturated.*

Proof: This corona is isomorphic to $C := \prod_n B_n / \bigoplus_n B_n$. Let

$$\pi \colon \prod_n B_n \to C$$

be the quotient map.

Fact. For $(d_n)_{n=0}^{\infty} \in \prod_n B_n$,

$$\|(d_n)\| = \sup_n \|d_n\|$$

$$\|\pi((d_n))\| = \limsup_n \|d_n\|.$$

$$\|\pi((d_n))\| = \lim_{m\to\infty} \|((1-e_m)(d_n))_{n=0}^{\infty}\|$$

e_m

Fact. For $(d_n) \in \prod_n B_n$,

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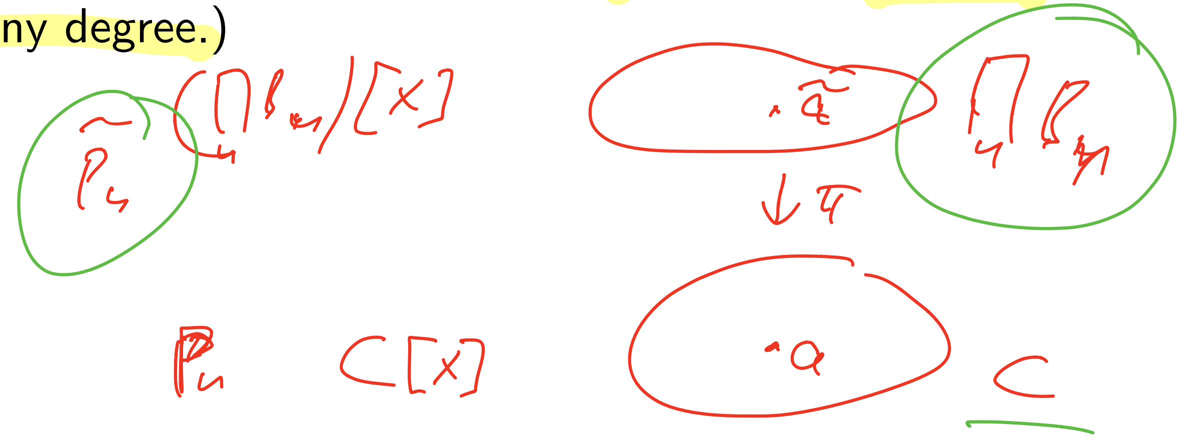
Fix a satisfiable countable degree-1 type $t(x)$, and enumerate it as $\|P_n(x)\| = r_n$, for $n \in \mathbb{N}$. (In this proof, P_n can be a *-polynomial over C of any degree.)



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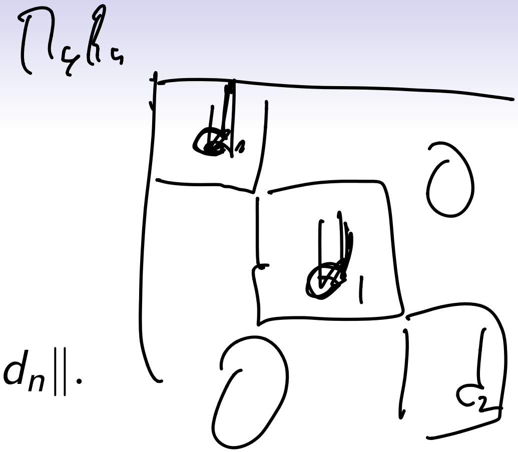
Lift the coefficients of P_n to $\prod_n B_n$, and let $\tilde{P}_n$ be a polynomial over $\prod_n B_n$ that lifts P_n.



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Lift the coefficients of P_n to $\prod_n B_n$, and let $\tilde{P}_n$ be a polynomial over $\prod_n B_n$ that lifts P_n.
We'll need a nice approximate unit for $\bigoplus_n B_n$. Let $e_j = \sum_{n \leq j} 1_{B_n}$, for $j \in \mathbb{N}$.

Fact. Each e_j is a projection, $e_j \leq e_{j+1}$, and e_j is in the center of $\prod_n B_n$.

$e_j e_{j+1} = e_j$

For $n \in \mathbb{N}$ fix $\tilde{b}(n)$ in the unit ball of $\prod_n B_n$ such that

$$\max_{j \leq n} |\|\pi(\tilde{P}_j((\tilde{b}(n)))\| - r_j| < \frac{1}{n}$$

$\|\pi(\tilde{P}_j(\tilde{b}(n)))\| \to r_j$

For $n \in \mathbb{N}$ fix $\tilde{b}(n)$ in the unit ball of $\prod_n B_n$ such that $\quad \limsup = r_j$

$$\max_{j \le n} \big| \|\pi(\tilde{P}_j((\tilde{b}(n))))\| - r_j \big| < \frac{1}{n}$$

Fact. There are $0 < m(0) < m(1) < \dots$ in $\mathbb{N}$ such that for all n and all $k \le n$ we have

$$\big| \|(e_{m(n+1)} - e_{m(n)})\tilde{P}_k(\tilde{b}(n))\| - r_k \big| < \frac{1}{n} \quad \varepsilon,$$

$\tilde{b}(1)$ $\qquad$ $\tilde{b}(2)$

0 $\quad m(1)$ $\quad m(2)$ $\quad m(3)$

$$\hat{P}_1(b(1)) = (d_n)_{n=0}^{\infty}$$

$$\overline{\lim_n} \|d_n\| \leq r_1 + \frac{1}{1}$$

$$\underline{\lim} \|d_n\| \geq r_1 - \frac{1}{1}$$

$n(k)$ $\quad \hat{P}_j(\tilde{b}(k))$ $\quad r_j$

$n(k+1)$ $\quad \hat{P}_j(\tilde{b}(k+1))$

For $n \in \mathbb{N}$ fix $\tilde{b}(n)$ in the unit ball of $\prod_n B_n$ such that

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Let $\tilde{b} := \sum_n (e_{m(n+1)} - e_{m(n)})\tilde{b}_n$.

Then $P_k(\tilde{b}) = \sum_n (e_{m(n+1)} - e_{m(n)})\tilde{P}_k(\tilde{b}_n)$.

$e_{m(k)} \hat{P}_j(\tilde{b}(j))$

$= \hat{P}_j(e_{m(k)} \tilde{b}(j))$

$\tilde{b}(2)$

0 $m(1)$ $\tilde{b}(1)$ $m(2)$ $\tilde{b}(2)$ $m(3)$ $m(4)$

For $n \in \mathbb{N}$ fix $\tilde{b}(n)$ in the unit ball of $\prod_n B_n$ such that

$$\max_{j \leq n} \left| \| \pi(\tilde{P}_j((\tilde{b}(n)))\| - r_j \right| < \frac{1}{n}$$

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Then $P_k(\tilde{b}) = \sum_n (e_{m(n+1)} - e_{m(n)}) \tilde{P}_k(\tilde{b}_n)$.
Then $b := \pi(\tilde{b})$ realizes the type t.

$$\| \underline{\underline{P_k(b)}} \| = \limsup_{j \to \infty} \| (P_k(b))_j \| = r_j$$

$$\in \prod B_n$$

Remarks (1) We did not need the assumption that the polynomials P_n were of degree 1.
(2) The proof shows that the corona $\prod_n B_n / \bigoplus_n B_n$ is quantifier-free countably saturated, and it is even countably saturated (in the sense of continuous model theory), but the proof of the latter involves additional ideas.

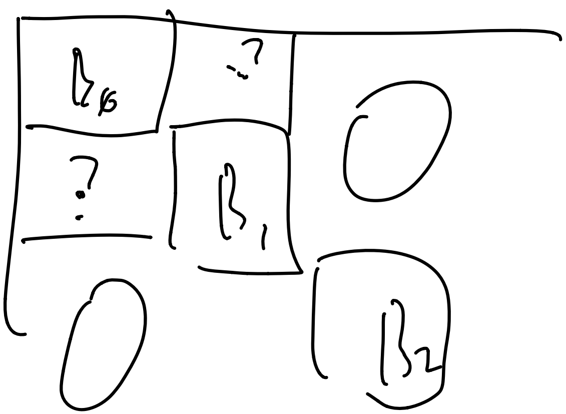
$A = K(H)$

$\bigoplus_n B_n$

Back to the main result: A $M(A)/A$

Thm 15.1.5 *The corona of every σ-unital, nonunital, C^*-algebra is countably degree-1 saturated.*

In the proof we will need a theorem of Arveson.



$e_n P_j(b)$

$= P_j(e_n b)$

Quasi-central approximate units

(1977, Duke)

Def 1.9.1 *Suppose A is an ideal in M and $X \subseteq M$. An approximate unit (e_m) in A is X-quasi-central if $\lim_m \|ae_m - e_m a\| = 0$ for every $a \in X$.*

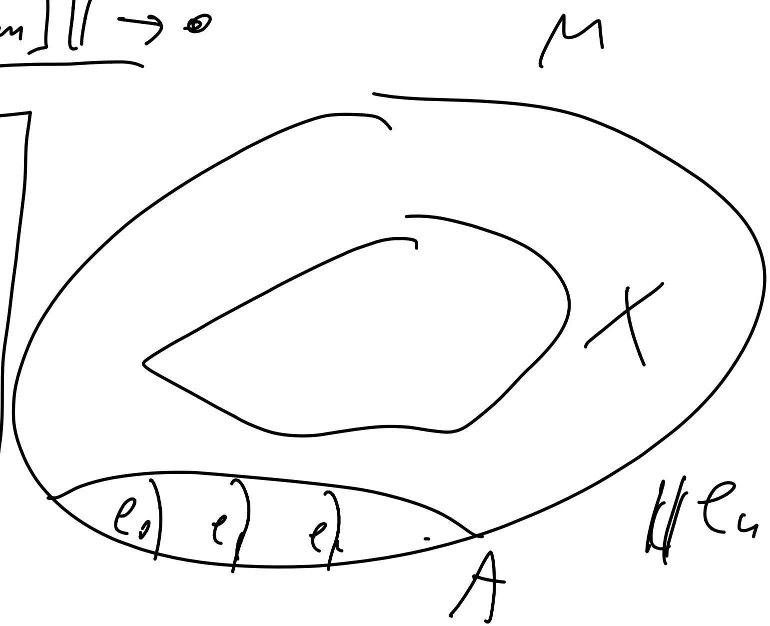
$\|[a, e_m]\| \to 0$

$A = K(H)$

$M = B(H)$

(e_n) - o.n. basis for H

$S\xi_n = \xi_{n+1}$



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Prop 1.9.3 *Suppose A is σ-unital ideal in a C^*-algebra M and $X \subseteq M$ is separable. Then there exists an X-quasi-central approximate unit e_n, for $n \in \mathbb{N}$, in A such that $e_{n+1}e_n = e_n$.*

The proof of this fact (due to Arveson) uses GNS representations in a clever way; since I promised that I'll not go into the representation theory, and since the proof is presented in the text, I'll skip it.

$e_{n+1} e_n = e_n e_{n+1}$ $\qquad$ $1_A = 1$

One more fact about commutation

$f(t) = \sqrt{t}$

Lemma

$\mathbb{D}$

Suppose that $S \subseteq \mathbb{C}$ is compact and $f \in C(S)$. Then for all $\varepsilon > 0$ there is $g_f(\varepsilon) > 0$ such that for all a and b with a normal and sp$(a) \subseteq S$ *we have*

$$\|[a,b]\| < g_f(\varepsilon) \Rightarrow \|[f(a),b]\| < \varepsilon.$$

δ

Pf ① If $f(x) = x^n$

$\|[a^n, b]\| \leq (n+1)\|[a,b]\|$

(2) If f is a $*$-poly.
applies (1)

(3) Use Stone-Weierstrass.

$$\| f - \underline{\underline{p}} \| < \varepsilon/4 .$$

Proof that every corona $\mathcal{M}(A)/A$ of a σ-unital C^*-algebra is countably degree-1 saturated

Fix a satisfiable countable degree-1 type $\mathrm{t}(x)$, and enumerate it as $\|P_n(x)\| = r_n$, for $n \in \mathbb{N}$.

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Fix a satisfiable countable degree-1 type $\mathrm{t}(x)$, and enumerate it as $\|P_n(x)\| = r_n$, for $n \in \mathbb{N}$.
Lift the coefficients of P_n to $\mathcal{M}(A)$, and let $\tilde{P}_n$ be a polynomial over ~~$\mathbb{M}_n/B_n$~~ that lifts P_n.
For $n \in \mathbb{N}$ fix $\tilde{b}(n)$ in the unit ball of $\mathcal{M}(A)$ such that

$\mathcal{M}(A)$

$$\max_{j \leq n} \left| \|\pi(\tilde{P}_j((\tilde{b}(n)))\| - r_j \right| < \frac{1}{n}$$

Let X_j be the set of all coefficients of $\tilde{P}_n$ and all $b(n)$, for $n \leq j$.
Let $X := \bigcup_j X_j$.
Fix an X-quasicentral approximate unit (e_n) for A such that $e_{n+1}e_n = e_n$ for all n.

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$$\|[e_j, c]\| < g_{\sqrt{\cdot}\restriction[0,1]}(2^{-j})/2$$

for all $c \in X_j$ and all j.

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$e_{j+1} - e_j \geq 0$

Let $f_j := (e_{j+1} - e_j)^{1/2}$ (with $e_{-1} := 0$).

1. For all $n \leq j$, $\tilde{P}_n(f_j \tilde{b}_j f_j) \approx_{2^{-j}} \tilde{f}_j P_n(\tilde{b}_j) f_j \approx_{2^{-j}} \tilde{P}_n(\tilde{b}_j) f_j^2$.

$b(j)$ $\qquad$ $b(j)$

$P_u(\cancel{t}X) = a_1 X a_2 + a_3 X^* a_4 + a_5$

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2. $\sum_j f_j^2 = 1$ (the infinite sum strictly converges).

$M(A)$

$sp(f_i) \subseteq [0,1]$

$0 \leq t \leq 1$

Let X_j be the set of all coefficients of $\tilde{P}_n$ and all $b(n)$, for $n \leq j$.
Let $X := \bigcup_j X_j$.
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By going to a subsequence, assure that $0 \leq e_j \leq 1$

$$\|[e_j, c]\| < g_{\sqrt{\cdot}\restriction[0,1]}(2^{-j})/2$$

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2. $\sum_j f_j^2 = 1$ (the infinite sum strictly converges).
3. For $(a_j) \in \ell_\infty(A)$ the sum $\sum_j f_j a_j f_j$ strictly converges and satisfies $\|\sum_j f_j a_j f_j\| \leq \sup_j \|a_j\|$.
4. $\|\pi(\sum_j f_j a_j f_j)\| \leq \lim_{n\to\infty} \|\sum_{j\geq n} f_j a_j f_j\|$.
5. If moreover $|\|a_j\| - \|a_j f_j^2\|| \to 0$, then

$$\|\pi(\textstyle\sum_j a_j f_j^2)\| \geq \limsup_j \|a_j f_j^2\|. = \limsup_j \|a_j\|$$

Pf (5) Assume $A \subseteq B(H)$

Fix $\xi_j \in H$, $\|\xi_j\| = 1$

$$\|a_j f_j^2 \xi_j\| \simeq \|a_j\| \Rightarrow f_j^2 \xi_j \simeq \xi_j$$

$\|f_j^2 \xi_j\| \simeq 1$

$\sum f_k = 1$

$f_k \xi_j = 0$

want:

$$\Big\|\sum_k a_k f_k^2 \xi_j\Big\| \geq \Big|\|a_j f_j^2 \xi_j\| - \Big\|\sum_{k \neq j} a_k f_k^2 \xi_j\Big\|\Big|$$

$\simeq 0$

$e_{k+1} e_k = e_k$

$f_k f_\ell = 0 \quad \text{if } |k - \ell| > 1$

$\|a + b\| \geq \|a\| - \|b\|$

$\|a_j \zeta\| \leq \|a_j\| \, \|\zeta\|$

$\|a_j f_j^2 \xi\| \simeq \|a_j\|$

$\|f_j^2 \xi\| < 1$

$\|a_j f_j^2 \xi_j\| \leq \|a_j\| \, \|f_j^2 \xi_j\|$

(3) w.l.o.g. $K = \sup_j \|a_j\| < \infty$

Prove:

$$\forall n \; \left\| \sum_{j \le n} f_j a_j f_j \right\| \le K$$

$$A = \begin{bmatrix} f_0 & f_1 & f_2 & \cdots & f_n \\ & & 0 & & \end{bmatrix} \cdot \begin{bmatrix} a_0 & & & 0 \\ & a_1 & & \\ & & \ddots & \\ 0 & & & a_n \end{bmatrix} = B$$

$$C = \begin{bmatrix} f_0 & \\ f_1 & \\ \vdots & 0 \\ f_n & \end{bmatrix} = \begin{bmatrix} \sum_{j \le n} f_j a_j f_j & 0 \\ 0 & 0 \end{bmatrix}$$

So, $\left\| \sum_{j \le n} f_j a_j f_j \right\| \le \|A\| \cdot \underbrace{\|B\|}_{\max_j \|a_j\|} \|C\| \le \max_i \|a_i\|$

$$\|A\|^2 = \|AA^*\| = \left\| \begin{bmatrix} \sum_j f_j^2 & 0 \\ 0 & 0 \end{bmatrix} \right\| \le 1$$

$$\|C\|^2 = \|C^* C\| = \ldots \le 1$$

$f_j = (e_{j+1} - e_j)^{1/2}$

Finally, replace (e_j) with a subsequence such that

$$\left|\|f_j\tilde{P}_n(\tilde{b}_j)f_j\| - r_n\right| < \frac{1}{j}$$

for all $n \leq j$.

$\|\pi(\tilde{P}_n(b_j))\| \approx r_n$

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for all $n \leq j$.
Then $b := \pi(\sum_j f_j \tilde{b}_j f_j)$ is in $(\mathcal{M}(A)/A)_1$.

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$$|\|f_j \tilde{P}_n(\tilde{b}_j) f_j\| - r_n| < \frac{1}{j}$$

for all $n \leq j$.
Then $b := \pi(\sum_j f_j \tilde{b}_j f_j)$ is in $(\mathcal{M}(A)/A)_1$and satisfies

$$\|P_n(b)\| = \|P_n(\sum_j f_j b_j f_j)\| = \|\sum_j f_j P_n(b_j) f_j\| = r_j.$$

Therefore b realizes the type t.