Massive C^*-algebras

Ilijas Farah

Winter 2021

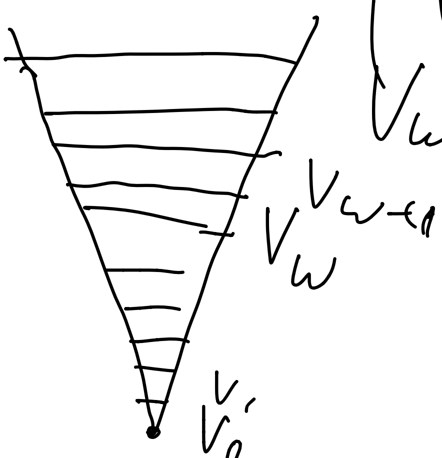
For convenience, references will be given to my 'Combinatorial set theory and C^*-algebras' (Springer Monographs in Mathematics, 2019—pdf ebook available upon request) whenever possible.
Saeed Ghasemi kindly agreed to run tutorials for this course (saeed.ghas@gmail.com).

Class 1, January 11, 2021
The set-theoretic universe

Von Neumann's *cumulative hierarchy* V_α, for $\alpha \in \mathrm{OR}$, is defined by transfinite recursion on ordinals:
$V_0 := \emptyset$, $V_{\alpha+1} := \mathcal{P}(V_\alpha)$, and $V_\beta := \bigcup_{\alpha<\beta} V_\alpha$ if β is a limit ordinal.

$|V_n| = 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$

$V_{\omega+10}$



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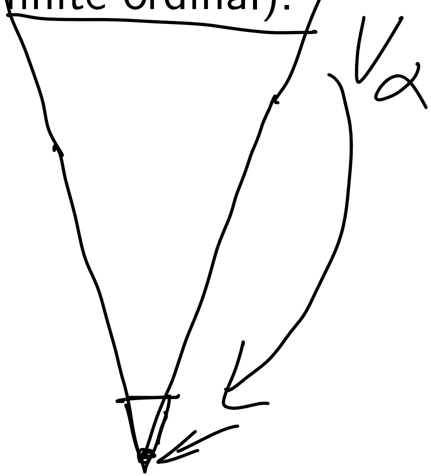

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Virtually all of mathematics takes place in $V_{\omega+10}$ (ω is the least infinite ordinal).
However, the structure of V_α, for some very large α, profoundly affects the structure of $V_{\omega+1}$.
(Think analytic number theory, only a bit more drastic.)

Prerequisites

H: $\ell_2(\mathbb{I})$ for some $\mathbb{I}$

$\mathcal{B}(H)$ — a Banach algebra with involution *.

Abstract C^**-algebra*: complex Banach algebra with an involution that satisfies the C^*-equality, $\|aa^*\| = \|a\|^2$.

Concrete C^*-algebra: norm-closed, self-adjoint subalgebra of $\mathcal{B}(H)$.

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Concrete C^*-algebra: norm-closed, self-adjoint subalgebra of $\mathcal{B}(H)$.

Thm 1.10.1 (GNS) *Every abstract* C^**-algebra* A *is isomorphic to a concrete* C^**-algebra.*

Thm 1.3.1 (Gelfand–Naimark) *Every unital abelian* C^**-algebra is isomorphic to*

$$C(X) = \{f : X \to \mathbb{C} \mid f \text{ is ctns}\} \qquad \|f\|_\infty$$

for some compact Hausdorff space X.

Thm 1.3.2 *The category of unital abelian* C^**-algebras is contravariantly equivalent to the category of compact Hausdorff spaces.*

$$C(X) \twoheadrightarrow C(Y)$$

$$X \leftarrowtail Y$$

Lemma 1.2.10 *Every algebraic *-homomorphism between C^*-algebras is contractive (i.e., 1-Lipshitz).*

$$d(x,y) \leq 1 \Rightarrow d(\phi(x), \phi(x)) \leq 1$$

Lemma 1.2.10 *Every algebraic *-homomorphism between C^*-algebras is contractive (i.e., 1-Lipshitz).*

Coro 1.2.11 *Every injective algebraic *-homomorphism Φ between C^*-algebras is an isometry.*

Convention

$A, B, C, \dots$ - C^*-algebras
$a, b, c, \dots$ - elements of C^*-algebras
$B \leq A$ means 'B is a C^*-subalgebra of A'

Taxonomy of operators (§1.4)

Def 1.4.1 *Some* $a \in A$ *is (assuming* A *is unital in* (3), (5), (7)*)*

1. *normal if* $aa^* = a^*a$*;*
2. *self-adjoint if* $a = a^*$*;*
3. *projection if* $a^2 = a^* = a$*;*
4. *unitary if* $aa^* = a^*a = 1$*;*
5. *isometry if* $a^*a = 1$*;*
6. *partial isometry if both* aa^* *and* a^*a *are projections, called the range projection and the source projection, respectively, of* a *(see Exercise 1.11.19);*
7. *coisometry if* $aa^* = 1$*;*
8. *contraction if* $\|a\| \leq 1$*.*

Continuous functional calculus

H $\quad\quad \mathcal{K}(H) = \{a \in \mathcal{B}(H) \mid a \text{ is cpct}\}$

$a[H,]$

Def *The unitization of A, $\tilde{A}$, is defined as follows.*
$\tilde{A} = \{a + \lambda | a \in A, \lambda \in \mathbb{C}\}$, *with* $+$, * *defined naturally.*
$(a + \lambda)(b + \mu) = (a + \lambda b + \mu a + \lambda\mu)$

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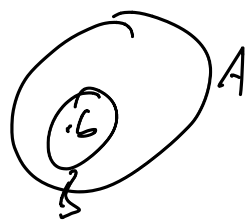
A, if A unital

The spectrum of $a \in A$:

$$\mathrm{sp}(a) = \{\lambda \in \mathbb{C} : a - \lambda I \text{ is not invertible in } \tilde{A}\}.$$

Fact. If $B \leq A$, $1_B = 1_A$, and $b \in B$, then $\mathrm{sp}_B(b) = \mathrm{sp}_A(b)$.



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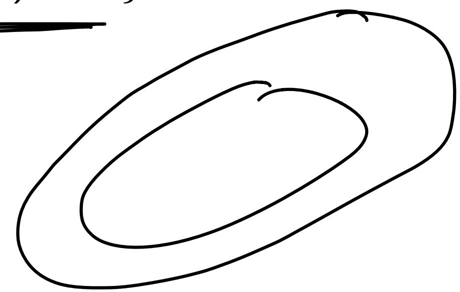
Def $\mathrm{C}^*(S)$: *the* C^**-algebra generated by (a set of operators)* S.
$\mathrm{C}^*(a) = \mathrm{C}^*(\{a\})$, *etc.*

Continuous functional calculus

loc. cpct Hausdff.

$$C_0(X) = \{f \in C(X) \mid \lim_{x\to\infty} f(x) = 0\}.$$



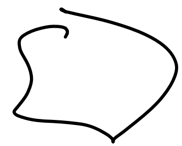
Continuous functional calculus

$$C_0(X) = \{f \in C(X) | \lim_{x\to\infty} f(x) = 0\}.$$

$a^* a = a a^*$

Thm 1.4.2 (Continuous functional calculus) *If $a \in A$ is normal then*

$$\mathrm{C}^*(a) \cong C_0(\mathrm{sp}(a) \setminus \{0\})$$



and the natural isomorphism sends $\mathrm{id}_{\mathrm{sp}(A)}$ to a.
If A is unital, then $\mathrm{C}^*(a,1) \cong C(\mathrm{sp}(a))$.

Coro *If a is normal and $f \in C(\mathrm{sp}(a))$, then we can define $f(a) \in \mathrm{C}^*(a,1)$ (and $f(a) \in \mathrm{C}^*(a)$ if $f \in C_0(\mathrm{sp}(a) \setminus \{0\})$.*

$\exp(a)$ $\qquad$ $\mathrm{sp}(a) \subseteq \mathbb{R}$ $\qquad$ $|a|$

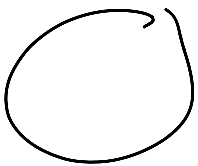
A useful triviality

$\sup_{x\in\cancel{\mathbb{C}}}\{|f(x)|\}$

Lemma *If a is normal and $f \in C(\mathrm{sp}(a))$ then $\|f(a)\| = \|f\|_\infty$, in particular $f(a) = 0$ if and only if $f(\lambda) = 0$ for all $\lambda \in \mathrm{sp}(a)$.*

Coro *Assume a is normal.*

1. *a is self-adjoint (i.e., $a = a^*$) iff $\mathrm{sp}(a) \subseteq \mathbb{R}$.*
2. *a is a projection (i.e., $a = a^* = a^2$) iff $\mathrm{sp}(a) \subseteq \{0,1\}$.*
3. *a is a unitary (i.e., $aa^* = a^*a = 1$) iff $\mathrm{sp}(a) \subseteq \mathbb{T}$ (the unit circle in $\mathbb{C}$).*



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Coro *If $\|a - a^*\| < \varepsilon$ then there $b \in \mathrm{C}^*(a)$ such that $b = b^*$ and $\|b - a\| < \varepsilon$.*

$$b = \frac{a + a^*}{2}$$

A useful triviality

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Exercise. $(\forall \varepsilon > 0)(\exists \delta > 0)$ such that for all a, if $\boxed{\max(\|a - a^*\|, \|a - a^2\|) < \delta}$ then there is a projection $p \in \mathrm{C}^*(a)$ with $\|a - p\| < \varepsilon$.

Positivity

Def (see §1.6) *Some $a \in A$ is positive if it satisfies any of the following equivalent conditions.*

1. $a = b^* b$ *for some* $b \in A$.
2. $a = a^*$ *and* $\mathrm{sp}(a) \subseteq [0, \infty)$.

$$(a\xi \mid \xi) \geq 0 \quad \forall \xi$$

H

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Let $A_{\mathrm{sa}} = \{a \in A | a = a^*\}$.

Exercise. $A = A_{\mathrm{sa}} + iA_{\mathrm{sa}}$. $A_{\mathrm{sa}} = A_+ - A_+$.

$$a \quad \frac{a+a^*}{2} + \frac{1}{2i}\left(i(a-a^*)\right)$$

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Lemma *If $a \leq b$ then $cac^* \leq cbc^*$ for all c.*

$b - a = d^* d \qquad c(b-a)c^* = cd^*dc^*$

$= (dc^*)^* dc^* \geq 0$

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Lemma *If $a \leq b$ then $cac^* \leq cbc^*$ for all c.*
If $0 \leq a \leq b$ then $\|a\| \leq \|b\|$ and $\|ac\| \leq \|bc\|$ for all c.

$\|a\| = \max \mathrm{sp}(a)$ (a normal)

Polar decomposition

We define $|a| = (a^* a)^{1/2}$.

Thm 1.1.3 *For every a in $\mathcal{B}(H)$ there exists a partial isometry $v \in \mathcal{B}(H)$ such that $a = v|a| = |a^*|v$.*

$v^* v$ is a proj.

$z = e^{i\theta} \rho$

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Exercise. For every a and every $\underline{\varepsilon > 0}$ there is $\underline{x \in \mathrm{C}^*(a)}$ such that $\|x\| \leq 1$ and $\|\underline{a - x|a|}\| < \varepsilon$.
(Hint: First prove that for every $f \in C_0(\mathrm{sp}(a) \setminus \{0\})$ we have $u\underline{f(|a|)} \in \underline{\mathrm{C}^*(a)}$.)

$u = v$

Some notation

$$A_1 = \{a \in A \,|\, \|a\| \leq 1\}.$$
$$A_+ = \{a \in A_{\mathrm{sa}} | a \geq 0\}.$$
$$A_{+,1} = \{a \in A_+ : \|a\| = 1\}$$

Approximate units

Def 1.6.7 *An approximate unit in* A *is a net* $(e_\lambda : \lambda \in \Lambda)$ *of positive contractions such that* $\lim_\lambda \|a - e_\lambda a\| = 0$ *for all* $a \in A$.

Prop 1.6.8 *Every* C^**-algebra* A *has an approximate unit. If* A *is separable then it has a sequential approximate unit.*

We'll sketch the proof.

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$$(1-a)^{-1} = \sum_{n=0}^{\infty} a^n$$

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(2) $\Lambda = \{a \in A_+ | \|a\| < 1\}$ is directed under $\leq$.

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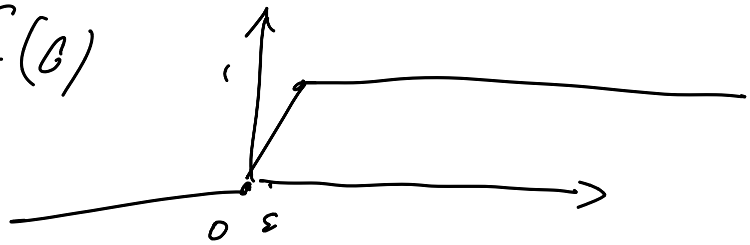
$\Lambda = \{a \in A_+ | \|a\| < 1\}$ is directed under $\leq$.
$\Lambda \to A_+ : a \mapsto (1 - a)^{-1} - 1$ is an order-isomorphism.

$\psi : A_+ \to \Lambda \qquad \psi(b) = (b+1)^{-1} + 1$

$A_+ \qquad$ check: $a \in A \Rightarrow (\forall \varepsilon > 0)\ \exists e \in \Lambda\ \|a - ea\| < \varepsilon$

$\underline{a > 0}$ $\quad e = f(0)$

$\| f(\varepsilon) a - a \| < \varepsilon$



$\underline{\underline{A}}$

An *ideal* in a C^*-algebra will be a two-sided, norm-closed, self-adjoint ideal unless otherwise specified.

$J = J^*$

Lemma 2.5.2 *Every quotient of a C^*-algebra is a C^*-algebra.*

(It is true, but not obvious, that the C^*-equality holds in the quotient.)

$\boxed{(\|aa^*\| = \|a\|^2}$

$J \triangleleft A$

$B(H)/K(H)$ Calkin 1942

Topologies on $\mathcal{B}(H)$. von Neumann algebras

Out of the uncountably many important topologies on $\mathcal{B}(H)$, we'll need the following two.

Strong operator topology (SOT) in $\mathcal{B}(H)$: induced by the family of seminorms $a \mapsto \|a\xi\|$, for ξ in H. This is the topology of pointwise convergence on H.

Weak operator topology (WOT): induced by the family of seminorms $a \mapsto |(a\xi|\eta)|$, for ξ and η in H.

(Recall that $\|\xi\|_2 = (\xi|\xi)^{1/2}$ and $(\xi|\eta) = \frac{1}{4}\sum_{j=0}^{3} i^j \|x + i^j\eta\|_2$.)

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vNA $\Rightarrow C^*$

$\not\Leftarrow$

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Def 3.1.2 *A von Neumann algebra is a strongly closed, unital C^*-subalgebra of $\mathcal{B}(H)$.*

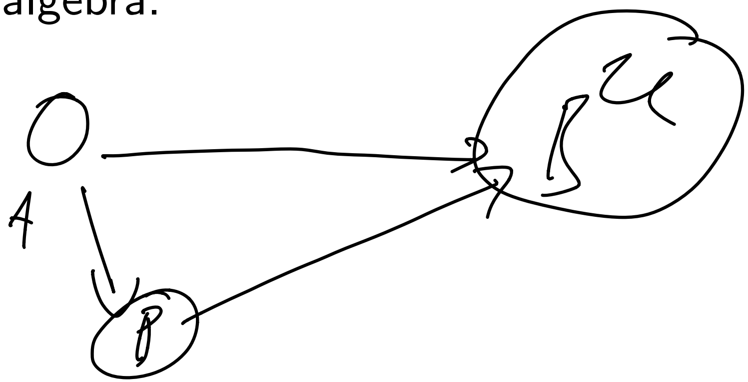
Lemma 3.1.3 *Suppose M is a von Neumann algebra and a_λ, for $\lambda \in \Lambda$, is an increasing net in M_+ which is bounded above by some $b \in M_+$. Then there exists $a \in M_+$ such that $\text{SOT-lim}_\lambda a_\lambda = \sup_\lambda a_\lambda = a$.*

$C([0,1])$

Massive C^*-algebras (ultraproducts, asymptotic sequence algebras, ultrapr~~oducts~~, coronas. . .)

owers

The structure of separable C^*-algebras and *-homomorphisms between them is often better understood when they are embedded into a massive C^*-algebra.



Massive C^*-algebras (ultraproducts, asymptotic sequence algebras, ultraproducts, coronas. . .)

The structure of separable C^*-algebras and *-homomorphisms between them is often better understood when they are embedded into a massive C^*-algebra.
There is no formal definition of a 'massive C^*-algebra' (but we know one when we see it). Massive C^*-algebras are constructed from (a sequence of) separable C^*-algebras (and possibly ultrafilters on $\mathbb{N}$) in a canonical way. Some of their basic properties are sensitive to the choice of the axioms of set theory.

Multiplier algebras

Def 2.5.5 *An ideal J in a C^*-algebra A is essential if for every $a \in A \setminus \{0\}$ we have $aJ \neq \{0\}$.*

$Y = [0,1]$ $f(0)=0$

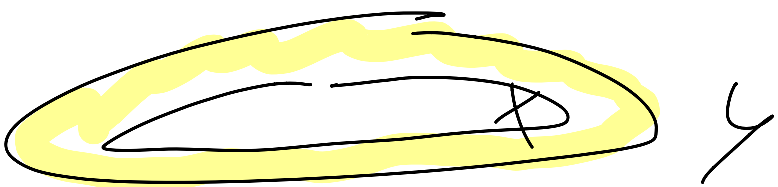
$X = (0,1]$

Example

If Y is a compact Hausdorff space and $X \subseteq Y$ is dense and locally compact, then $J = \{f \in C(Y) | f(y) = 0 \text{ for all } y \in Y \setminus X\}$ is an essential ideal of $C(Y)$.

Note that $J \cong C_0(X)$, where

$$C_0(X) = \{f \in C(X) | \lim_{x \to \infty} f(x) = 0\}.$$



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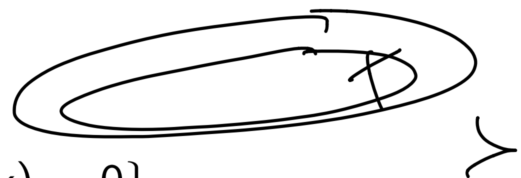


$$C_0(X) = \{f \in C(X) | \lim_{x \to \infty} f(x) = 0\}.$$

Here, Y is a compactification of X.
We will define the non-commutative analog of the Čech–Stone compactification, βX.

(βX is the compact Hausdorff space that contains X as a dense subspace and has the property that every bounded continuous $f\colon X \to [0, 1]$ has a continuous extension $\tilde{f}\colon Y \to \mathbb{C}$.)

(I'll write $B \leq C$ for 'B is a C^*-subalgebra of C'.)
Suppose $A \leq \mathcal{B}(H)$. The *idealizer* of A is

$$M = \{b \in \mathcal{B}(H) : bA \subseteq A, Ab \subseteq A\}.$$

" $\{ba \mid a \in A\}$

Fact. This implies M is a C^*-algebra and A is an ideal in M. It is essential if A is *nondegenerate*, i.e., if
$A^{\perp} = \{b \in \mathcal{B}(H) | bA = Ab = \{0\}\}$ is trivial.

annihilator

Exercise. Prove that if $A \cong C_0(X)$ then $M \cong C(\beta X)$.

$\hookrightarrow \mathcal{B}(H)$

(I'll write $B \leq C$ for 'B is a C^*-subalgebra of C'.)
Suppose $A \leq \mathcal{B}(H)$. The *idealizer* of A is

$$M = \{b \in \mathcal{B}(H) : bA \subseteq A, Ab \subseteq A\}.$$

Fact. This implies M is a C^*-algebra and A is an ideal in M. It is essential if A is *nondegenerate*, i.e., if
$A^{\perp} = \{b \in \mathcal{B}(H) | bA = Ab = \{0\}\}$ is trivial.

Exercise. Prove that if $A \cong C_0(X)$ then $M \cong C(\beta X)$.

It is not obvious that M depends only on A, and not on the way A is embedded into $\mathcal{B}(H)$.
There are (at least) three routes towards proving this, and constructing the multiplier algebra of A: strict completion, multipliers, and Hilbert modules.

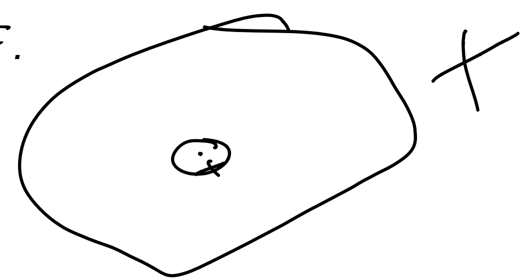
Weak topology induced by a family of seminorms; filters

In non-metrizable topological spaces, one can define convergence in terms of nets or in terms of filters. Following the tradition in operator algebras, my book uses nets, but in one respect the filters are more convenient.

$\{Y \mid Y \subseteq X\}$

Def *Given a set X, some $\mathcal{F} \subseteq \mathcal{P}(X)$ is a filter on X if the following holds.*

1. *$Y \in \mathcal{F}$ and $Z \supseteq Y$ implies $Z \in \mathcal{F}$.*
2. *$Y \in \mathcal{F}$ and $Z \in \mathcal{F}$ implies $Y \cap Z \in \mathcal{F}$.*
3. *If $\emptyset \notin \mathcal{F}$, then $\mathcal{F}$ is a proper filter.*



Weak topology induced by a family of seminorms

Suppose that X is a topological vector space, $\mathcal{N}$ is a family of seminorms on X, and $\mathcal{F}$ is a filter on X.

Def

$\mathcal{F} \to x$

1. *$\mathcal{F}$ converges to $x \in X$ if for all $\rho \in \mathcal{N}$ and all $\varepsilon > 0$ we have $\{y \in X | \rho(x-y) < \varepsilon\} \in \mathcal{F}$.*
2. *$\mathcal{F}$ is Cauchy if for all $\rho \in \mathcal{N}$ and all $\varepsilon > 0$ we have $Y \in \mathcal{F}$ such that $\rho(x-y) < \varepsilon$ for all x and y in Y.*
3. *X is complete (with respect to the topology induced by $\mathcal{N}$) if every Cauchy filter on X converges.*

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The completion of X with respect to $\mathcal{N}$ is defined in a natural way—see e.g., Gabriel Nagy's lecture notes (https://www.math.ksu.edu/ nagy/func-an-F07-S08.html, lecture TVS IV.).

Strict topology

Def 13.1.1 *Suppose* $A \leq M$. *To every* $h \in A$ *we associate two seminorms on* M, $\lambda_h(b) := \|hb\|$ *and* $\rho_h(b) := \|bh\|$. *The weak topology induced by these seminorms is called the* A*-strict topology, or just the strict topology if* A *is clear from the context.*

A unital => norm

$h = 1$

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Proof: The algebraic operations on $\mathcal{M}(A)$ are defined in a natural way.
$+, \cdot, *$

To define the norm, let $\mathcal{E}$ be an approximate unit of A. If $\mathcal{F}$ is a bounded Cauchy filter in A, let $\|\mathcal{F}\| \neq \sup_{e\in\mathcal{E}} \sup_{Y\in\mathcal{F}} \inf_{b\in Y} \|eb\|$.

$A \hookrightarrow M(A)$ $\qquad$ $a \longrightarrow$

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Def 13.1.6 *$\mathcal{M}(A)$ is the multiplier algebra of A.*

$\{a \in A \mid \|a\| \leq K\}$

$\mathcal{F}_1 \sim \mathcal{F}_2 \quad \forall \rho \quad \forall \varepsilon$

$$\exists X_1 \in \mathcal{F}_1 \qquad \exists X_2 \in \mathcal{F}_2$$
$$\forall a_1 \in X_1 \qquad \forall a_2 \in X_2$$
$$\{\rho(a_1, a_2) < \varepsilon\}$$

Example 13.2.4

1. *If X is a locally compact Hausdorff space then* $\mathcal{M}(C_0(X)) \cong C(\beta X)$.
2. $\mathcal{M}(\mathcal{K}(H)) \cong \mathcal{B}(H)$.
3. *If B_n, for $n \in \mathbb{N}$, are unital* C^**-algebras, then* $\mathcal{M}(\bigoplus_n B_n) \cong \prod_n B_n$.

$M_n(\mathbb{C})$

$\bigoplus_n B_n = \{ (a_n) \mid \|a_n\| \to 0 \}$

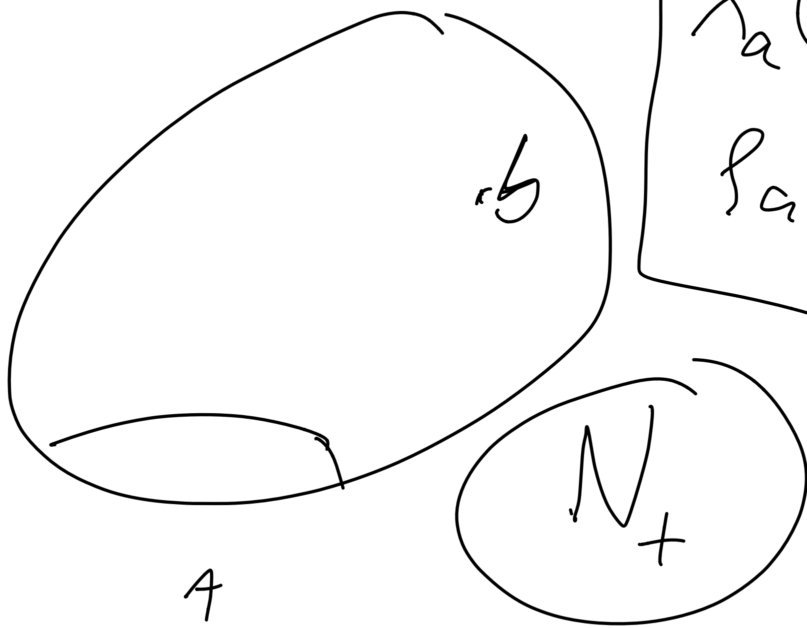
$\prod_n B_n = \{ (a_n) \mid \sup_n \|a_n\| < \infty \}$

$N = B(H)$

Lemma 13.1.9 *If N is a von Neumann algebra and $A \subseteq N$ is a C^*-algebra such that $A^\perp \cap N = \{0\}$ then N is A-strictly complete.*

$\lambda_a(b) = \|ab\|$

$\rho_a(b) = \|ba\|$



$\mathcal{E} \subseteq A$

$e \in \mathcal{E}$

$eNe = \underline{\underline{eAe}}$

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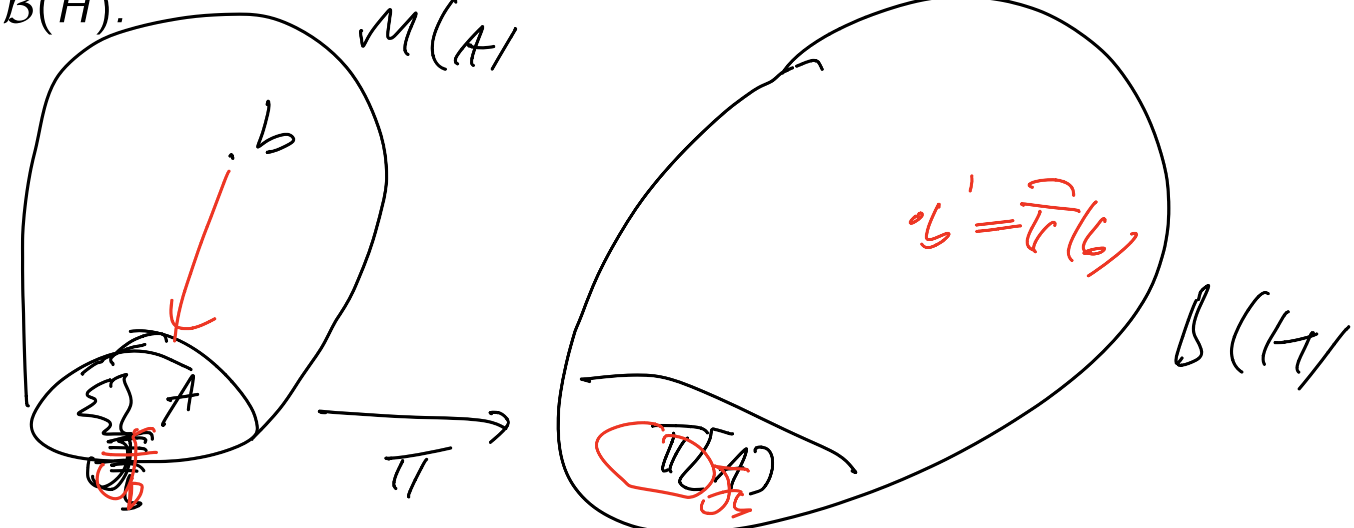
Prop 13.2.1 *Suppose $\pi \colon A \to \mathcal{B}(H)$ is a nondegenerate faithful representation. Then π has a unique extension to a representation $\tilde{\pi} \colon \mathcal{M}(A) \to \mathcal{B}(H)$, and $\tilde{\pi}[\mathcal{M}(A)]$ is equal to the idealizer of $\pi[A]$ in $\mathcal{B}(H)$.*



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Exercise. How many nonisomorphic algebras as in (**??**) can you find?

Def $\mathrm{Proj}(A)$ *is the poset of projections in* A

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Fact. $\mathrm{Proj}(\mathcal{B}(H))$ is a lattice.

Prop (Weaver) *The poset* $\mathrm{Proj}(\mathcal{Q}(H))$ *is not a lattice.*

(For a proof see Proposition 13.3.3.)