Introduction to tropical geometry: theory and applications
Lecture 3 (Tropicalization of Linear Spaces)

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Linear spaces

- For this talk assume that $K = \mathbb{C}$.
- A d-dim **linear vector subspace** $L \subset K^n$ can be given as:
  - A span of d vectors $v_1, \ldots, v_d$ in K^n.
  - A row space of a $d \times n$ matrix.
  - The solution set of $n - d$ linear equations of form $a_1X_1 + \cdots + a_nX_n = 0$
  - **Plücker coordinates**

Goal

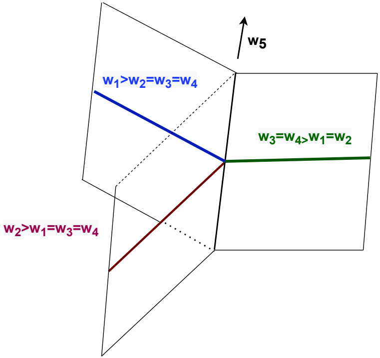
- To describe the tropicalized variety trop(L).
- To find a minimal set of equations needed to generate trop(L).
- To describe the moduli space of ordinary and tropical linear spaces.
- To describe an abstract notion of tropical linear varieties.

Tropical linear space (Sturmfels, Ardila-Klivans)

Tropical linear space: obtained by tropicalizing the equations of circuits of L.

Example: Tropical linear subspace

- $L =$ row space $\begin{pmatrix} 0 & 0 & 0 & 0 & 2 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 & 0 \end{pmatrix}$ with columns $X_1, \ldots, X_5$
- $X_1 - 2X_2 + X_3 = 0$, $X_4 - X_3 = 0$ and $X_1 - 2X_2 + X_4 = 0$
- $\text{trop}(L) = V(X_1 \oplus X_2 \oplus X_3) \cap V(X_1 \oplus X_2 \oplus X_4) \cap V(X_3 \oplus X_4) \bigcap_{\substack{\text{infinitely many} \\ f \in \text{trop}(I_L)}} V(f)$



- $\text{trop}(L) = \{w :\ \min\{w_1, w_2, w_3\}, \ldots, \min\{w_3, w_4\}$ is achieved at least twice$\}$

Plücker coordinates of matrices

- Consider a $d \times n$ matrix A with entries in a field K.
- **Plücker coordinate** $p_I(A)$ for $I \in \binom{[n]}{d}$: The minor of A on the columns I.
- The **Plücker vector** of A in K^n is

$$p(A) = (p_{I_1}, \ldots, p_{I_{\binom{n}{d}}})$$

- $A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ -1 & 0 & 1 & 1 \end{bmatrix}$ with $p(A) = (1, 3, 1, 1, 1, 2) = (p_{12}, p_{13}, \ldots, p_{34})$
- The associated linear space of a Plücker vector $p \in \mathbb{P}^{\binom{n}{d}-1}$ is:

$$L_p = \{X \in K^n : \sum_{i \in T} (-1)^i p_{T \setminus i} X_i = 0 \text{ for all } T \subset [n],\ |T| = d+1\}$$

  - $\{1,2,3\} : p_{12}X_3 - p_{13}X_2 + p_{23}X_1 = 0 \Longrightarrow X_3 - 3X_2 + X_1 = 0$
  - $\{1,2,4\} : p_{12}X_4 - p_{14}X_2 + p_{24}X_1 = 0 \Longrightarrow X_4 - 1X_2 + X_1 = 0$
  - $\{1,3,4\} : p_{13}X_4 - p_{14}X_3 + p_{34}X_1 = 0 \Longrightarrow 3X_4 - X_3 + 2X_1 = 0$
  - $\{2,3,4\} : p_{23}X_4 - p_{24}X_3 + p_{34}X_2 = 0 \Longrightarrow X_4 - X_3 + 2X_2 = 0$

Plücker coordinates of linear subspaces

Parameterization of Plücker vectors

How to check whether a given vector is a Plücker vector of a linear subspace?

- Consider the projective space $\mathbb{P}^{\binom{n}{d}-1} = K^n \backslash \{(0, \ldots, 0)\}/\sim$

$$(p_1, \ldots, p_{\binom{n}{d}}) \sim (\lambda p_1, \ldots, \lambda p_{\binom{n}{d}}) \text{ for all } \lambda \in K\backslash\{0\}$$

- A vector $p \in \mathbb{P}^{\binom{n}{d}-1}$ is the Plücker vector of a matrix if and only if for all $S, T \subset [n]$ with $|S| = d-1$ and $|T| = d+1$:
- **Quadratic Plücker relations:** $\sum_{i \in T\backslash S} (-1)^{\alpha_i} p_{T\backslash i} p_{S\cup i} = 0$
- For a 2×4 matrix: $p_{12}p_{34} - p_{13}p_{24} + p_{23}p_{14} = 0$
- A Plücker vector of a matrix determines the corresponding linear space which is independent of the choice of the matrix A representing it.

Tropical linear space

- Tropical projective space $\mathbb{TP}^{\binom{n}{d}-1} = (\overline{\mathbb{R}}^{\binom{n}{d}} \setminus \{(\infty, \ldots, \infty)\})/\sim$

$$(p_1, \ldots, p_{\binom{n}{d}}) \sim (\lambda \odot p_1, \ldots, \lambda \odot p_{\binom{n}{d}}) \text{ for all } \lambda \in \mathbb{R}$$

- $L_p = \{X \in K^n : \sum_{i \in T} (-1)^i p_{T \setminus i} X_i = 0 \text{ for all } T \subset [n], \; |T| = d+1\}$
- The tropical linear space associated to $p \in \mathbb{TP}^{\binom{n}{d}-1}$ is obtained by tropicalizing all polynomials above.

Question

What is the minimal generating set for such tropical equations? How to describe them combinatorially?

Tropical linear space (Sturmfels, Ardila-Klivans)

Tropical linear space: obtained by tropicalizing the equations of circuits of L.

A crash course in matroid theory

- Let A be a matrix with columns $X_1, \ldots, X_n$. The minimal subsets of $[n]$ whose corresponding columns are dependent are called circuits.
- A **matroid** M on $[n]$ is a collection $\mathcal{C}$ of circuits (subsets of $[n]$) such that:
  (1) $\emptyset$ is not a circuit.
  (2) No circuit contains another circuit.
  (3) For every pair of circuits C, C' and $a \in C \cap C'$ the set $C \cup C' \setminus \{a\}$ contains a circuit.
- A basis of M is a maximal independent set. The **matroid polytope** of M is

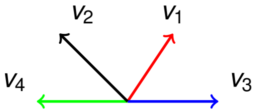
$$P_M = \text{convex hull}\{e_{b_1} + \cdots + e_{b_r} : \{b_1, \ldots, b_r\} \text{ is a basis of } M\}$$

- Edmonds 1970, Gelfand-Goresky-MacPherson-Serganova 1987

Theorem

A $0/1$ polytope is a matroid polytope $\Longleftrightarrow$ all its edges are of the form $e_i - e_j$.

Example 1: Matroid polytope



$$A = [v_1|v_2|v_3|v_4]$$

$$\text{rank}(12) = \text{rank}(13) = \text{rank}(123) = 2$$

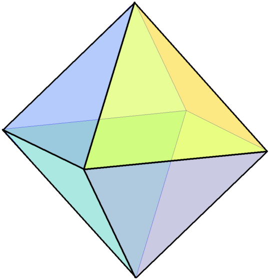
- The circuits of the associated matroid M are $34, 124, 123$
- The basis elements of M are $12, 13, 14, 23, 24$.
- The matroid polytope is

$$P_M = \text{conv}\{(1,1,0,0),(1,0,1,0),(1,0,0,1),(0,1,1,0),(0,1,0,1)\}.$$

Example 2: Uniform matroid $U(d, n)$

- The circuits of $U(2, 4)$ are all 3-subsets.
- The basis elements of $U(2, 4)$ are all 2-subsets.
- The hypersimplex $\Delta(d, n)$: the matroid polytope of $U(d, n)$.
- For the uniform matroid $M = U(2, 4)$ we have:

$$P_M = \operatorname{conv}\{(1, 1, 0, 0), (1, 0, 1, 0), \dots, (0, 0, 1, 1)\}.$$



Tropical linear subspace

- Let L be a linear subspace and M be its associated matroid on $[n]$.
- Let P_M be the matroid polytope of M.
- A matroid is **loopless** if every element $i \in [n]$ is in some basis of M.

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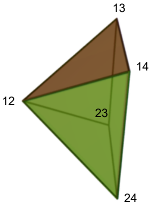
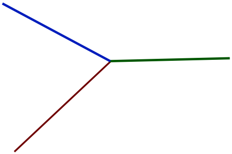
Theorem (Sturmfels)

Tropical L is the dual fan of the loopless faces of the matroid polytope P_M.





- Every face is a matroid. Loopless faces: $\{12, 23, 24\}$ and $\{13, 14, 23\}$

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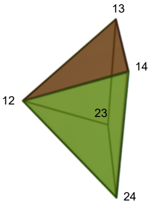

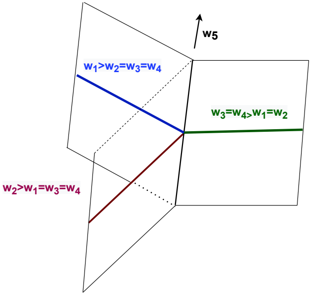

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Grassmannian Gr(d, n)

- **Grassmannian** Gr(d, n): The set of d-dim subspaces in K^n.
- Each element of Gr(d, n) is represented by a full-rank $d \times n$ matrix X.

$$X = \begin{bmatrix} 1 & 1 & 2 & 3 \\ -1 & 0 & 1 & 2 \end{bmatrix} \in \mathrm{Gr}(2,4) \quad \text{with} \quad p_{12} = 1, p_{13} = 3, p_{23} = 1, \ldots$$

- **Plücker coordinate** p_I for $I \in \binom{[n]}{d}$: The minor of X on the columns I.
- **The Plücker ideal** is generated by quadratic relations:

$$p_I p_J = \sum_{\lambda=1}^{k} p_{i_1 \cdots i_{r-1} j_\lambda i_{r+1} \cdots i_k} p_{j_1 \cdots j_{\lambda-1} i_r j_{\lambda+1} \cdots j_k} \quad \text{for all } I, J.$$

- Tropicalization of Gr(d, n)
- **Tropicalized Plücker vectors**

Tropicalized Plücker vectors

- Let A be a $d \times n$ matrix with entries in $\overline{\mathbb{R}}$.
- For every d-subset $I = \{i_1, \ldots, i_d\}$ we define

$$\mathrm{trop}(p_I) = \mathrm{trop}(\det(A_I)) = \min_\sigma\{a_{1,\sigma(i_1)} + \cdots + a_{d,\sigma(i_d)}\}$$

- The **tropicalized Plücker vector** of A in $\overline{\mathbb{R}}^{\binom{n}{d}}$ as

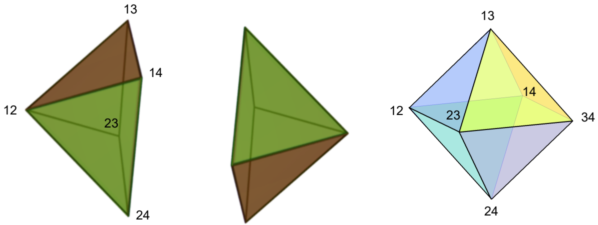
$$\mathrm{trop}(p(A)) = (\mathrm{trop}(p_{I_1}), \ldots, \mathrm{trop}(p_{I_{\binom{n}{d}}})) \text{ with } I_1 < \cdots < I_{\binom{n}{d}} \text{ in lex order.}$$

- Tropical linear subspaces are corresponding to matroid polytopes
- What are the combinatorial properties of tropical Plücker vectors?
- Tropical Plücker vectors are corresponding to subdivisions of $\Delta(d, n)$

Tropical Plücker vectors

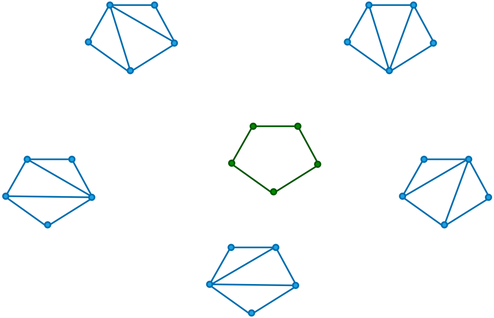
Definition. A vector $p \in \overline{\mathbb{R}}^{\binom{n}{d}}$ is a **tropical Plücker vector** if every cell of the corresponding **regular subdivision** of $\Delta(d, n)$ is a matroid polytope.

- Tropicalized Plücker vectors are realizable tropcial Plücker vectors.
- The dual polyhedral complex of the corresponding subdivision of $\Delta(d, n)$ is a tropical linear space.



Subdivisions of polytopes

- A **subdivision** of a polytope is a finite union of polytopes s.t. every two polytopes are either disjoint or intersect by a common proper face.



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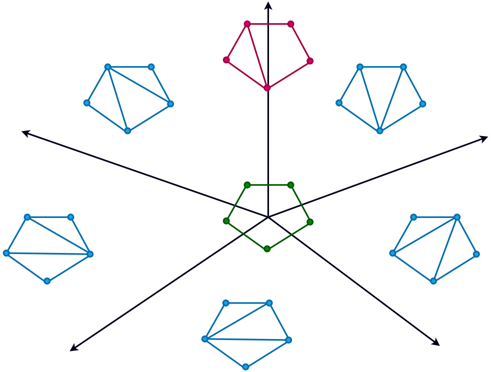
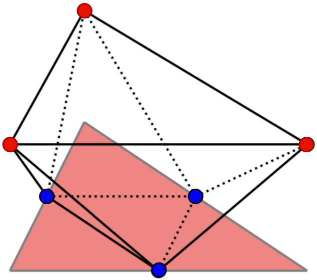


- **The secondary fan of** P**:** A fan whose faces correspond to the regular subdivisions of P. Gelfand-Kapranov-Zelevinski, 1990

Regular subdivisions of polytopes

- A **regular** subdivision: is obtained by lifting some points of P.



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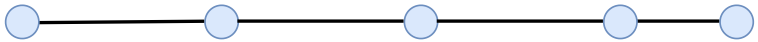

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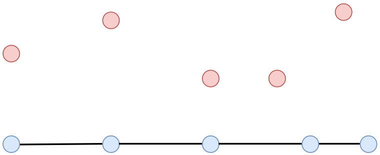

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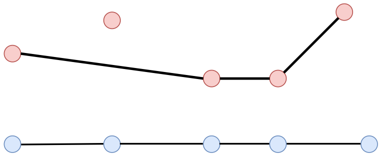

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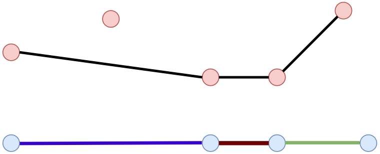

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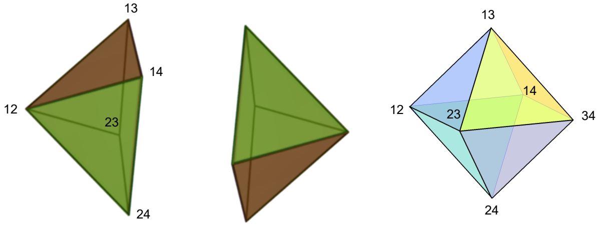

Definition. A vector $p \in \overline{\mathbb{R}}^{\binom{n}{d}}$ is a **tropical Plücker vector** if every cell of the corresponding **regular subdivision** of $\Delta(d, n)$ is a matroid polytope.

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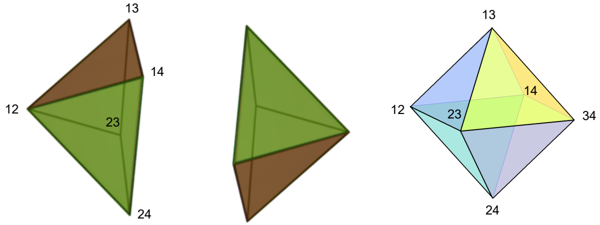

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Theorem (Kapranov 1992, Speyer-Sturmfels 2004)

The polyhedral dual fan of the subdivision of $\Delta(d, n)$ is a tropical linear space.

Dressians and tropical Grassmannians

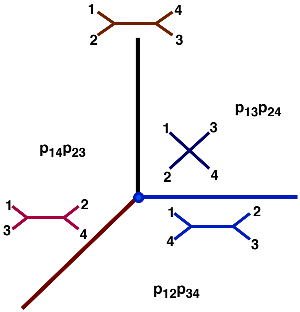
- **Dressian** $\mathrm{Dr}(d, n)$ (Kapranov, Speyer, Joswig, ...)
  - parameterizes abstract tropical linear spaces
  - the moduli space of tropical Plücker vectors
  - has a natural fan structure
  - subfan of the secondary fan of $\Delta(d, n)$ corr. to matroid subdivisions

- **Tropical Grassmannian** (Hacking-Keel-Tevelev, Speyer-Sturmfels, ...)
  - the tropicalization variety of the Plücker ideal.
  - The (closure) of the images of classical Plücker vectors under the valuation map are tropicalized Plücker vectors.

- $\mathrm{Trop}(\mathrm{Gr}(d, n)) \subseteq \mathrm{Dr}(d, n)$

Tropical Grassmannian trop(Gr(2, n))

- Trop Gr(2, n): tropicalized lines in tropical projective ($n-1$)-space
- The space of phylogenetic trees with n leaves (Speyer-Sturmfels 2003).
- Trop Gr(2, n)=Dr(2, n)



- Trop Gr(3, n): metric tree arrangements, but computationally challenging.