Introduction to tropical geometry: theory and applications
Lecture 2

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Winter School on Geometric Constraint Systems

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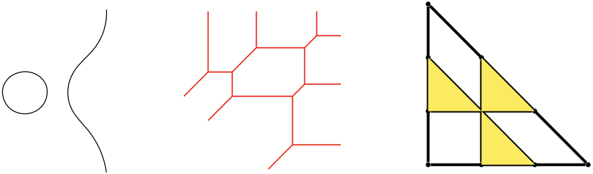
Summary of Lecture 1

- The study of tropical geometry has been motivated by **applications**.
- It is considered a combinatorial shadow of algebraic geometry.
  - A degree 3 curve with genus 1
  - The degree and genus of tropical planar curves



Tropical genus

Let f be a polynomial of degree d and $\mathcal{S}$ the induced dual subdivision of $d\Delta$.

- The genus of $V(f)$ is the number of interior vertices of $\mathcal{S}$.

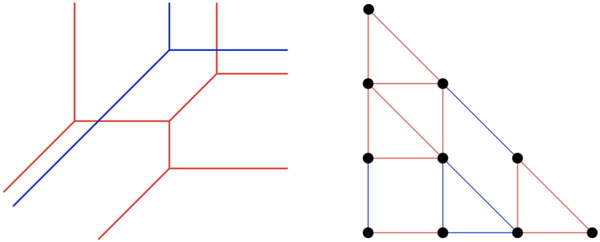
Summary of Lecture 1

- **Goal:** to provide a simple model of algebraic geometry
- Let C and C' be two algebraic planar curves of degree d and d'.
- **Classical Bézout's theorem:** C and C' intersect in dd' points.

Tropical Bézout's theorem

trop(C) and trop(C') intersect in dd' points (up to multiplicity).

- The **multiplicity** of an intersection point p is the area of the parallelogram dual to p in the dual subdivision of $C \cup C'$.



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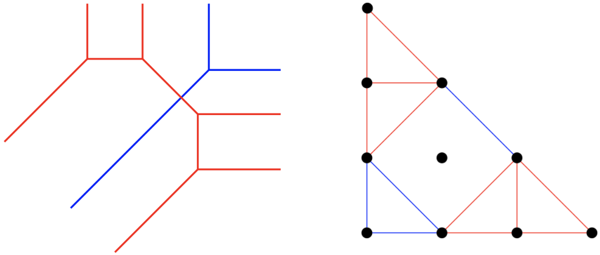


Main references

- **A bit of tropical geometry**

  Erwan Brugallé and Kristin Shaw

- **A First Expedition to tropical geometry**

  Book by Johannes Rau

- **Introduction to Tropical geometry**

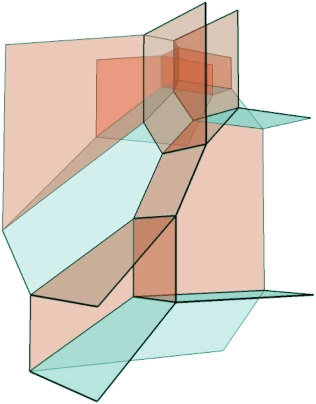
  Book by Diane Maclagan and Bernd Sturmfels

- **Essentials of tropical combinatorics**

  Book by Michael Joswig

Outline

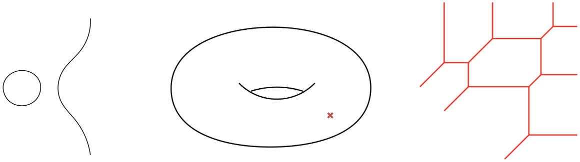
- Lecture 2: Tropical varieties as polyhedral complexes



What is tropical geometry about?

- We can think of it as a new type of algebraic geometry.
- **Goal:** To understand the solution space of a system of polynomials?
- We work over tropical semifield $\overline{\mathbb{R}} = (\mathbb{R} \cup \{\infty\}, \oplus, \odot)$

$$x \oplus y = \text{minimum of } x \text{ and } y \quad \text{and} \quad x \odot y = x + y$$



- Given a degree 3 polynomial f in 2 variables we compute $V(f)$:
  - over real numbers $\mathbb{R}$
  - over complex numbers $\mathbb{C}$
  - and the variety of trop(f) over tropical numbers $\overline{\mathbb{R}}$

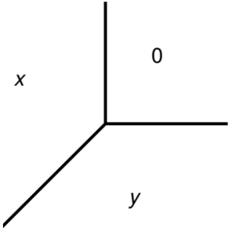
Tropical hypersurfaces

- **Tropical polynomials** with coefficients in $\overline{\mathbb{R}}$.

$$f = \oplus a_u \odot x^u = \oplus a_u \odot x^{u_1} \odot \cdots \odot x^{u_n} = \min\{a_u + u_1 x_1 + \cdots + u_n x_n\}$$

$V(f) = \{\mathbf{w} \in \overline{\mathbb{R}}^n : \ f(\mathbf{w}) = \infty$ or the min in $f(\mathbf{w})$ is achieved at least twice$\}$.

- $f = x \oplus y \oplus 0 = \min\{x, y, 0\}$



$$x = y < 0, \quad x = 0 < y, \quad y = 0 < x$$

Also, $V(f)$ contains $(0, \infty)$ and $(\infty, 0)$

Fundamental Theorem

- What we have seen so far:
  - **tropical polynomials** whose coefficients live in $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$.
- More generally:
  - **tropicalization of arbitrary polynomials**
  - coefficients live in an arbitrary field K with a **valuation** $val : K \to \overline{\mathbb{R}}$
  - For every $f = \sum a_u x^u = \sum a_u x^{u_1} \cdots x^{u_n}$:

We define: $\quad \text{trop}(f) = \oplus val(a_u) \odot x^u = \min\{val(a_u) + x \cdot u\}$.

- **Example:** Trivial valuation: $val(a_u) = 0$
- $V(\text{trop}(f)) = \{\mathbf{w} \in \overline{\mathbb{R}}^n : \text{trop}(f)(\mathbf{w}) = \infty$ or the min in $\text{trop}(f)(\mathbf{w})$ is achieved at least twice$\}$.

- **Fundamental Theorem** (under mild conditions on K)
  - solutions of tropical equations = tropicalization of the solutions
  - $V(\text{trop}(f)) = \text{trop}(V(f))$

Tropicalizations of a variety

- Let $f_1, \ldots, f_s \in K[x_1, \ldots, x_n]$ and I their generating ideal
- The **variety of** $I = \langle f_1, \ldots, f_s \rangle$ is the set of common solutions of $f_1, \ldots, f_s$.

$$V(I) = \bigcap_{f \in I} V(f) = V(f_1) \cap \cdots \cap V(f_s)$$

Question

How to compute the tropicalization of $V(I)$?

- The **tropicalization of the ideal** I is $\mathrm{trop}(I) = \langle \mathrm{trop}(g) : \ g \in I \rangle$.
- The **tropicalization of the variety** $V(I)$

$$\mathrm{trop}(V(I)) = \bigcap_{f \in \mathrm{trop}(I)} V(f) \subset \overline{\mathbb{R}}^n$$

Example: Tropicalization of a variety

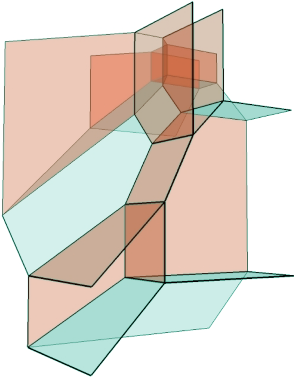
- $\text{trop}(I) = \langle \text{trop}(g) : \ g \in I \rangle$ and $\text{trop}(V(I)) = \bigcap_{f \in \text{trop}(I)} V(f)$
- **Example:** Let $g = x - 3y + 5$ and $I = \langle g \rangle$. Then $\text{trop}(g) = x \oplus y \oplus 0$.
- $\text{trop}\,V(I) = V(\text{trop}(g))$ is a tropical line.
- **Note:** $\text{trop}(I) = \langle \text{trop}(f) : \ f \in I \rangle$ is not generated by $\text{trop}(g)$.
- $xg = x^2 - 3xy + 5x$ and $3yg = 3xy - 9y^2 + 15y$.
- $\text{trop}(xg) = x^2 \oplus (x \odot y) \oplus x$ and $\text{trop}(3yg) = (x \odot y) \oplus y^2 \oplus y$.
- $h := xg + 3yg = x^2 + 5x - 9y^2 + 15y$ and $\text{trop}(h) = x^2 \oplus x \oplus y^2 \oplus y$
- $\text{trop}(h)$ cannot be written as any combination of $\text{trop}(g)$.
- Tropical varieties have nicer combinatorial structures than ideals.

Structure theorem

Theorem (Bieri-Groves, Cartwright-Payne)

Let $X = V(I)$ be an irreducible variety with $\dim(X) = d$. Then $\mathrm{trop}(X)$ is the support of an **$\mathbb{R}$-rational polyhedral complex** of dimension d which is:

- **pure**, **balanced** and **connected through codimension one**.



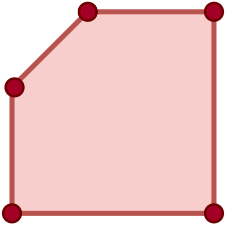
A crash course on polyhedral geometry

- A **polyhedron** in $\mathbb{R}^n$ is the intersection of finitely many half-spaces.
- Given the rational vector $a \in \mathbb{Q}^n$ and $b \in \mathbb{R}$, the hyperplane

$$\mathcal{H} = \{x : \ x \cdot a = b\} \subset \mathbb{R}^n$$

divides $\mathbb{R}^n$ in two half-spaces

$$\mathcal{H}^+ = \{x : \ x \cdot a \geq b\} \quad \text{and} \quad \mathcal{H}^- = \{x : \ x \cdot a \leq b\}.$$



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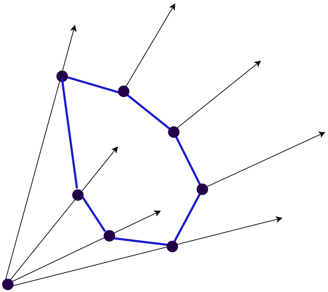
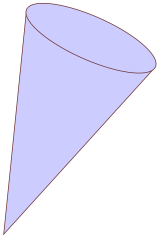


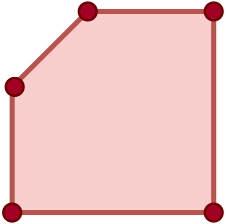
Polyhedral Cone



Cone

Faces of polyhedra

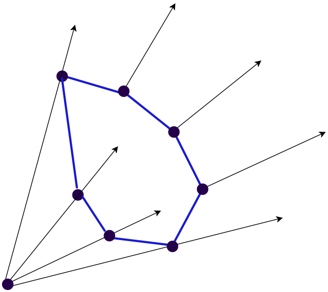
- Let $P \subseteq \mathbb{R}^n$ be the intersection of half-spaces $\mathcal{H}_1^+, \ldots, \mathcal{H}_k^+$.
- For any hyperplane $\mathcal{H}$ with $P \subseteq \mathcal{H}^+$, $P \cap \mathcal{H}^+$ is called a **face** of P.
- How many faces does this polytope have?
- What is the maximum dimension of its faces?



Polyhedral cones and fans

- **Polyhedral cone**: the positive hull of a finite subset of $\mathbb{R}^n$ as

$$C = \text{pos}(v_1, \ldots, v_s) = \{\sum_{i=1}^{s} t_i v_i : \; t_i \geq 0\}$$



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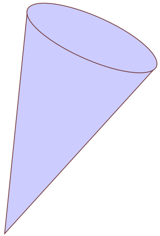

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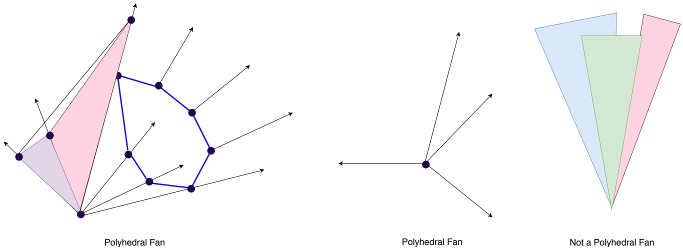
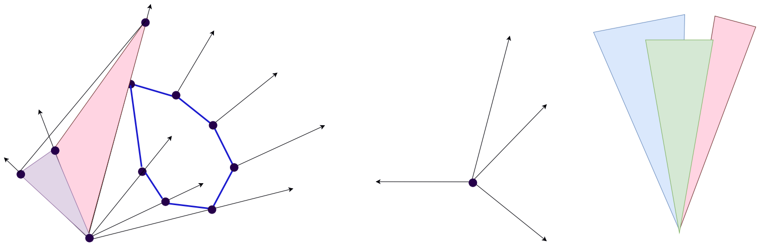


- **Polyhedral fan**: A collection $\mathcal{F}$ of polyhedral cones in $\mathbb{R}^n$ such that the intersection of any two cones is a face of each.

Polyhedral complex

- A **polyhedral complex** is a collection of polyhedra in $\mathbb{R}^n$ such that the intersection of any two polyhedra is a face of both.

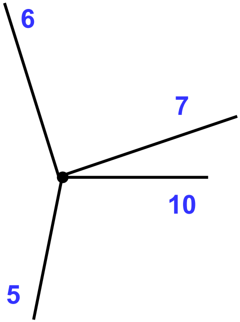


Polyhedral Fan   Polyhedral Fan   Not a Polyhedral Fan

- **Support set:** The set of points of $\mathbb{R}^n$ contained in a polyhedral complex.
- **Pure complex:** all its maximal polyhedra have the same dimension.

Balanced polyhedral fan

- Consider a weighted 1-dim rational polyhedral fan $\mathcal{F} \subseteq \mathbb{R}^n$ with rays $\rho_1, \ldots \rho_k$ weighted by $m_1, \ldots, m_k$.



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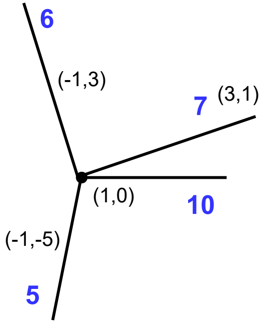

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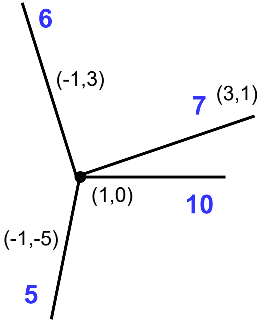

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- Then $\mathcal{F}$ is **balanced** if $m_1 p_1 + \cdots + m_k p_k$ is the zero vector in $\mathbb{R}^n$.

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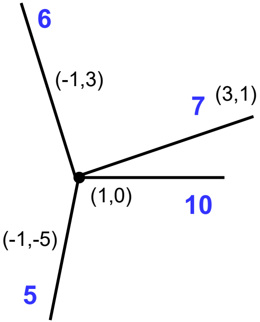

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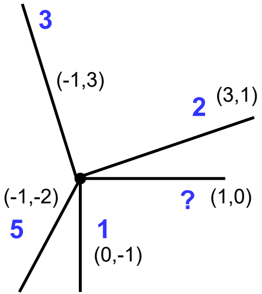

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- What multiplicity makes this fan balanced?

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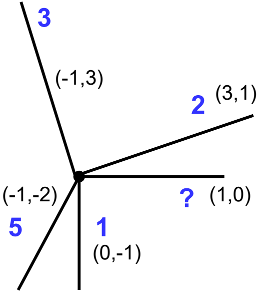

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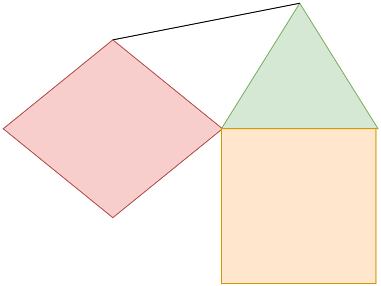
- $6(-1,3) + 7(3,1) + 10(-1,0) + 5(-1,-5) = (0,0)$
- What multiplicity makes this fan balanced? $3.(-1) + 2.3 + 5.(-1) + 2.1 = 0$.

Connected through codimension 1

- A polyhedral complex P of dim d is **connected through codimension 1** if for any two d-dim polyhedra C and D in P there is a sequence of d-dim polyhedra $C = P_0, P_1, \ldots, P_k = D$ such that

$$\dim(P_i \cap P_{i+1}) = d - 1 \quad \text{for all } i$$



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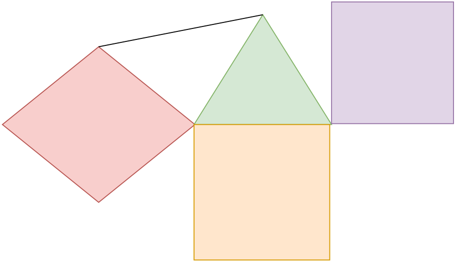

- This property is frequently used in tropical algorithms.

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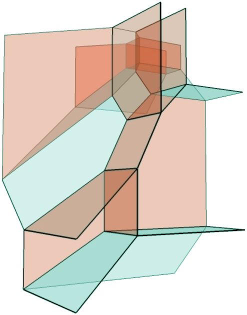


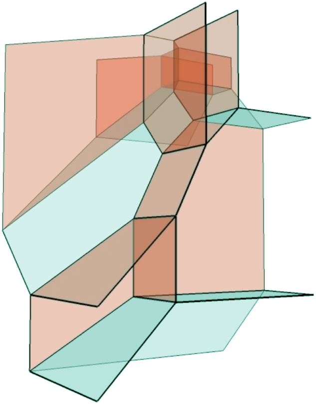
- This property is frequently used in tropical algorithms.

Structure theorem

Theorem

For any irreducible variety $X = V(I)$ of dimension d, its tropicalization $\text{trop}(X)$ is the support of an $\mathbb{R}$**-rational polyhedral complex** of dimension d which is:

- **pure**, **balanced** and **connected through codimension one**.

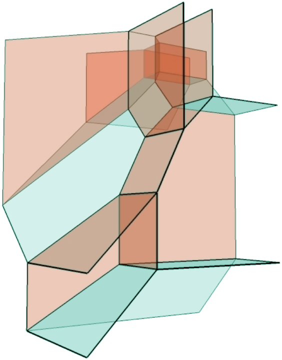


Realizability question

Question

Given a polyhedral complex Σ satisfying the conclusion of the structure theorem, is there an irreducible variety (over some field) s.t. $\Sigma = \text{trop}(X)$?

- Relates to realizability of matroids
- Next lecture: Tropical linear spaces and tropicalized linear spaces



Fundamental theorem

- $\text{trop}(I) = \langle \text{trop}(g) : \; g \in I \rangle$ and $\text{trop}(V(I)) = \bigcap_{f \in \text{trop}(I)} V(f)$

Theorem (Kapranov, Speyer-Sturmfels, Payne, Driasma)

Let $I \subset K[x_1, \ldots, x_n]$. Under some mild conditions on K we have:

$$\text{trop}(V(I)) = \text{closure}\{(val(a_1), \ldots, val(a_n)) : \; a = (a_1, \ldots, a_n) \in V(I)\}$$

- Given $f \in K[x_1, \ldots, x_n]$, the solution space of the tropical polynomial $\text{trop}(f)$ **is equal to** the tropicalization of the solution space of f.
- solutions of tropical equations = tropicalization of the solutions

Algebraic Geometry over which fields?

- Given a field K, a **valuation** $val : K \to \overline{\mathbb{R}}$ is a map s.t. for all $a, b \in K$:
  - $val(ab) = val(a) + val(b)$
  - $val(a+b) \geq \min(val(a), val(b))$
  - $val(a+b) = \min(val(a), val(b))$ if $val(a) \neq val(b)$.
  - $val(a) = \infty$ if and only if $a = 0$.
- **Trivial valuation** over any field:

$$val(a) = 0 \text{ for all } a \neq 0.$$

- **Puiseux series**: $K = \mathbb{C}\{\{t\}\} = \cup_{n \geq 0} \mathbb{C}((t^{\frac{1}{n}}))$.

$$val(a) = \text{the minimum exponent of } t \text{ in } a.$$

  - $val(-t^{3/5} + 2t^2 + 8t^{9/2} + \cdots) = 3/5$.
  - $val(4 + t^{1/5} + 6t^3 + \cdots) =?$

Tropicalization of polynomials

- Let K be a field with a valuation *val*.
- Let $f = \sum a_u x^u$ be a polynomial in $K[x_1, \ldots, x_n]$ where $x^u = x_1^{u_1} \cdots x_n^{u_n}$

We define: $\text{trop}(f) = \oplus val(a_u)_{\odot} x^u = \min\{val(a_u) + x \cdot u\}$.

- Trivial valuation: $\text{trop}(x^3 + xy^4 + 3y^6) =min\{3x, x + 4y, 6y\}$
- **Puiseux series**: $f = (-3 + t + t^{5/2})x^3 + (t^{3/2} + t^4)y - 5$.

  $\text{trop}(f) = 0_{\odot}x^3 + 3/2_{\odot}y + 0 = \min\{3x, 3/2 + y, 0\}$
- Example: $f = -t^2x + (1 + t)y + 3xy + t^{1/2}$

  $\text{trop}(f) = 2_{\odot}x \oplus 0_{\odot}y + 0_{\odot}x_{\odot}y + 1/2 = \min\{2 + x, y, x + y, 1/2\}$

Fundamental theorem of tropical algebraic geometry

- $\text{trop}(I) = \langle \text{trop}(g) : \; g \in I \rangle$ and $\text{trop}(V(I)) = \bigcap_{f \in \text{trop}(I)} V(f)$

Theorem (Kapranov, Speyer-Sturmfels, Payne, Driasma)

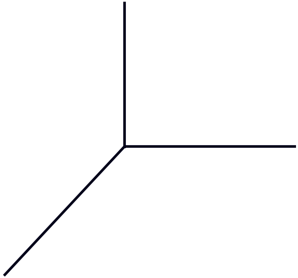
Let $I \subset K[x_1, \ldots, x_n]$. Under some mild conditions on K we have:

$$\text{trop}(V(I)) = \text{closure}\{(val(a_1), \ldots, val(a_n)) : \; a = (a_1, \ldots, a_n) \in V(I)\}$$

- Condition: K is algebraically closed with a non-trivial valuation.
- If not, then take a field extension of K with a non-trivial valuation.
- Consider the valuations induced by **Puiseux series**.
- Let $I = \langle x - t, y - t^3 \rangle$.
- The variety of I is the single point $V(I) = \{(t, t^3)\}$.
- $\{(val(t), val(t^3))\} = \{(1, 3)\}$ which is equal to its closure.
- $\text{trop}(V(I)) = V(x + 1) \cap V(y + 3) = \{(1, 3)\}$

Example of the fundamental theorem

- Let $I = \langle x + y - 1 \rangle \subset \mathbb{C}[x, y]$. Then $V(I) = \{(a, 1 - a) : a \in \mathbb{C}\}$.

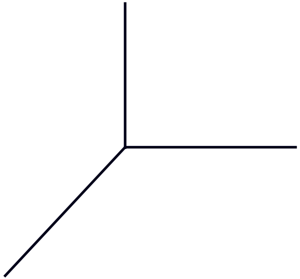


- To compute the tropicalization of $V(I)$ over $\mathbb{C}$ using valuations:
  - We first consider the valued field extension $\mathbb{C}\{\{t\}\}$.

$$val(a) > 0 \qquad val(a) < 0 \qquad val(a) = 0$$

Example of the fundamental theorem

- $\text{trop}(V(I)) = \text{closure}\{(val(a), val(1-a)) : \; a \in \mathbb{C}\{\{t\}\}\}$.



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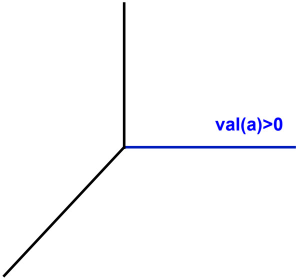

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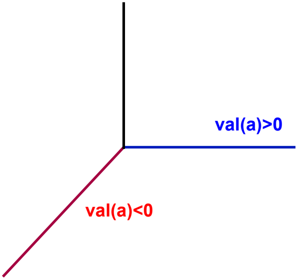

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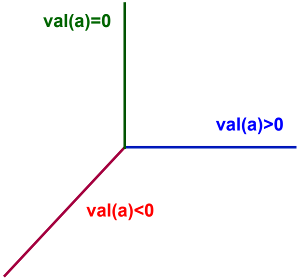

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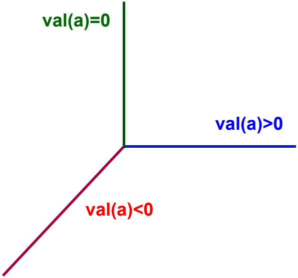

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