An introduction to tropical geometry: theory and applications
Lecture 1

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Winter School on Geometric Constraint Systems

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Hello from Hong Kong!

Motivation

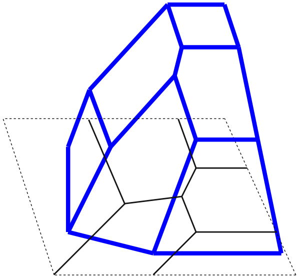
- Much of the study of tropical geometry has been motivated by the **applications** both inside and outside mathematics.
- From theoretical side, it is considered a combinatorial shadow of algebraic geometry with remarkable results in:
  - Enumerative algebraic geometry
  - Toric geometry
- From applied side it is used in:
  - Economics (Bank of England)
  - Phylogenetics
  - Quantum field theory
  - MANY MORE
- **Tropical**: honoring the Brazilian computer scientist Imre Simon.

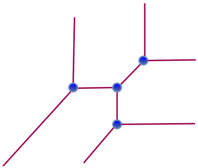
What is tropical geometry?

- A tool for transforming algebraic varieties into polyhedral objects which retain a lot of information about the original variety.

$$\text{polynomials/varieties} \quad \xrightarrow{\text{tropicalization}} \quad \text{polyhedral fans and graphs}$$

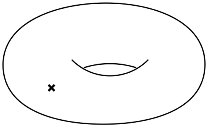
- A piecewise linear shadow of algebraic geometry.

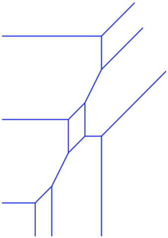


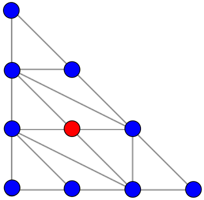


Outline

- Lecture 1: Tropical polynomials

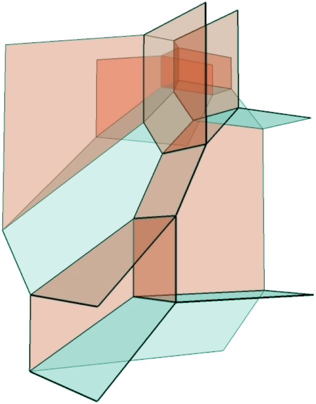






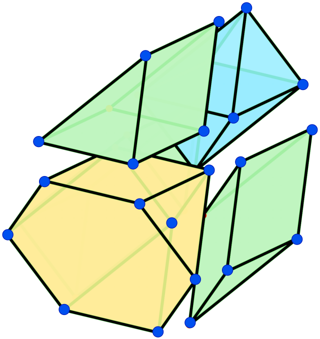
Outline

- Lecture 2: Tropical varieties as polyhedral complexes



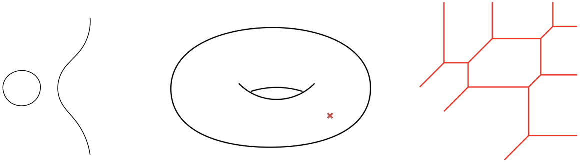
Outline

- Lecture 3: Tropical linear spaces, Grassmannians, and matroids with applications in phylogenetics



What is tropical geometry about?

- We can think of it as a new type of algebraic geometry.
- What is the solution space of polynomials?
- We work over tropical numbers $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$
- $f = y^2 - x^3 + 3x^2 - 2x \subset \mathbb{R}[x, y]$
- Look at $V(f)$:
  - over real numbers $\mathbb{R}$
  - over complex numbers $\mathbb{C}$
  - and the variety of $\operatorname{trop} V(f)$ over tropical numbers $\overline{\mathbb{R}}$



- The degree of $V(f)$ is 3 and its genus is 1.

Tropical Arithmetic

- Tropical geometry is algebraic geometry over the tropical semiring

$$\overline{\mathbb{R}} = (\mathbb{R} \cup \{\infty\}, \oplus, \odot)$$

- Addition and multiplication:

$$x \oplus y = \text{minimum of } x \text{ and } y$$
$$x \odot y = x + y$$

- $3 \odot 4 = 7$ and $3 \oplus 4 = 3$
- $3 \odot (4 \oplus 8) = ?$

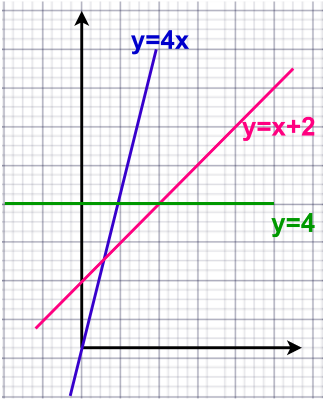
$$\infty \oplus x = x \quad ? \odot x = x \quad \text{and} \quad 2 \oplus x = 8$$

- $\overline{\mathbb{R}}$ is a semiring: commutative, associative with additive and multiplicative identities.

Tropical polynomials

A tropical polynomial is a **piecewise linear function** with integer slopes, and a finite number of linear pieces.

$$f(x) = x^4 \oplus 2 \odot x \oplus 4 = \min\{4x, 2 + x, 4\}$$



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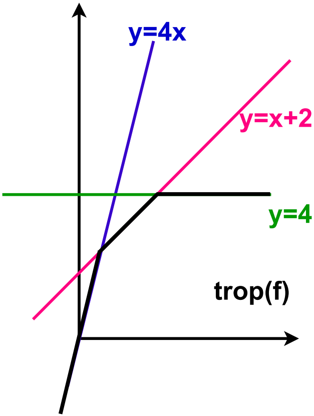
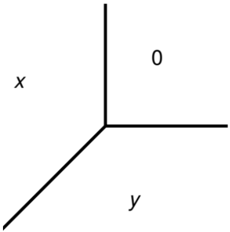


Tropical hypersurfaces

- The tropical hypersurface of $f \in \overline{\mathbb{R}}[x_1, \dots, x_n]$ is

  $$V(f) = \{\mathbf{w} \in \mathbb{R}^n : \; f(\mathbf{w}) = \infty \text{ or the min in } f(\mathbf{w}) \text{ is achieved at least twice}\}.$$

- $f = x \oplus y \oplus 0 = \min\{x, y, 0\}$

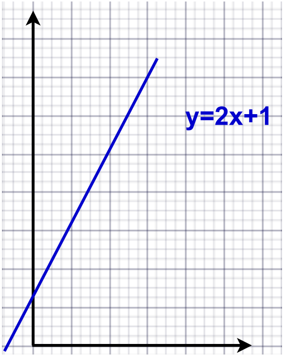


$$x = y < 0, \quad x = 0 < y, \quad y = 0 < x$$

Also, $V(f)$ contains $(0, \infty)$ and $(\infty, 0)$

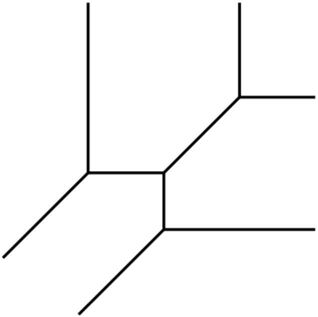
Tropical hypersurfaces

- Draw the tropical hypersurface $V(3 \odot x^3 + 2 \odot x \odot y)$.
- When $\min\{3 + 3x, 2 + x + y\}$ is attained twice?
- This is the case iff $3 + 3x = 2 + x + y \iff y = 2x + 1$.



Tropical hypersurfaces and dual subdivisions

- $f = 1 \oplus (0 \odot x) \oplus (0 \odot y) \oplus (0 \odot xy) \oplus (1 \odot x^2) \oplus (1 \odot y^2)$



- $V(f)$ has 4 vertices, 9 edges and 6 connected cells in $\mathbb{R}^2 \setminus V(f)$

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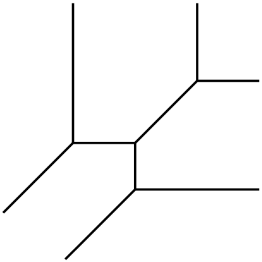

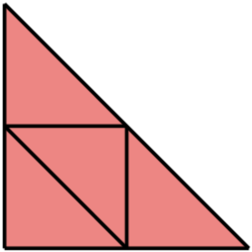

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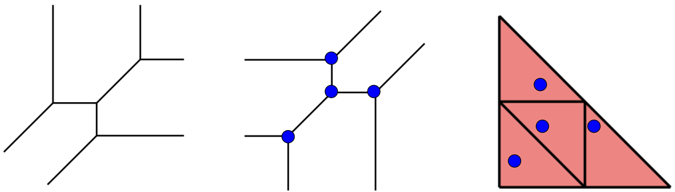
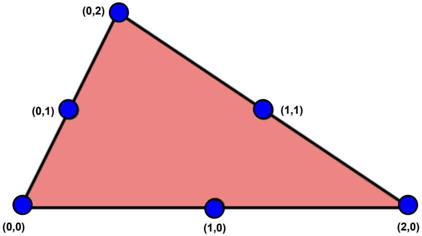


- $V(f)$ has 4 vertices, 9 edges and 6 connected cells in $\mathbb{R}^2 \setminus V(f)$
- The dual subdivision of 2Δ encodes the combinatorial structure of $V(f)$
- $\{\text{vertices of } V(f)\} \iff \{\text{maximal cells of } 2\Delta\}$
- $\{\text{edges of } V(f)\} \iff \{\text{edges of } 2\Delta\}$
- $\{\text{connected components of } \mathbb{R}^2 \setminus V(f)\} \iff \{\text{vertices of } 2\Delta\}$

Newton polytope

- $f = \oplus a_u \odot x^u = \oplus a_u \odot x^{u_1} \odot \cdots \odot x^{u_n} = \min\{a_u + u_1 x_1 + \cdots + u_n x_n\}$
- **supp**$(f) := \{$exponents u for which $a_u \neq \infty\}$
- **Newton polytope** of $f :=$ The convex hull of supp(f).



- Example: $f = 1 \oplus x \oplus y \oplus xy \oplus x^2 \oplus y^2$

Lifted Newton polytope

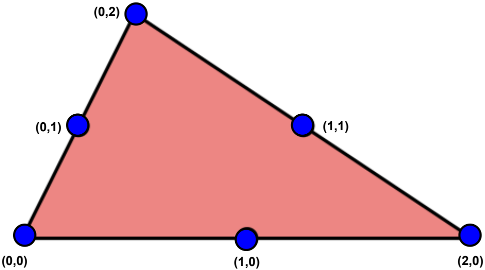
- $f = \oplus a_u \odot x^u = \oplus a_u \odot x^{u_1} \odot \cdots \odot x^{u_n} = \min\{a_u + u_1 x_1 + \cdots + u_n x_n\}$
- The **lifted Newton polytope** is the convex hull of

$$\{(u, a_u) : \ u \in \text{supp}(f)\} \subset \mathbb{Z}^n \times \mathbb{R}$$

- Project the lower facets into $\mathbb{R}^n$ (facets visible from below).
- This provides a regular subdivision of the Newton polytope of f.

Example: dual subdivision

- $f = 1 \oplus (0 \odot x) \oplus (0 \odot y) \oplus (0 \odot xy) \oplus (1 \odot x^2) \oplus (1 \odot y^2)$
- Goal: To show that the **dual subdivision** of the lifted Newton polytope of f encodes the **combinatorial structure** of $V(f)$.



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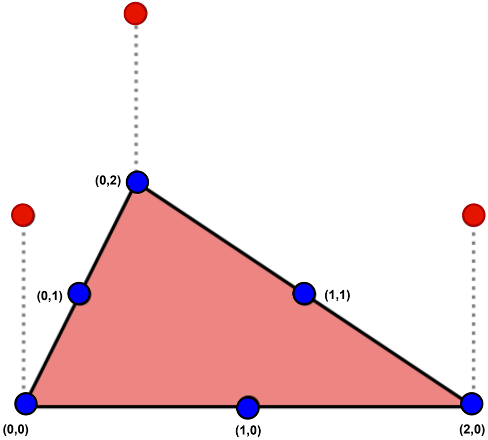

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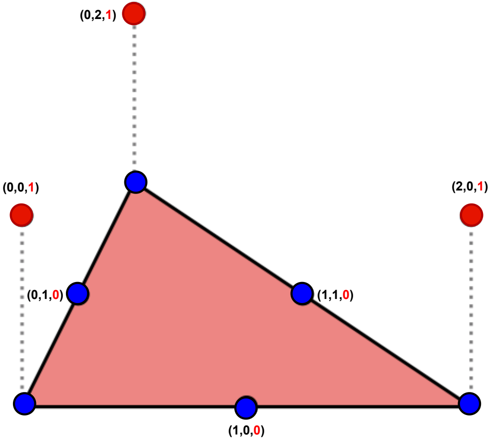

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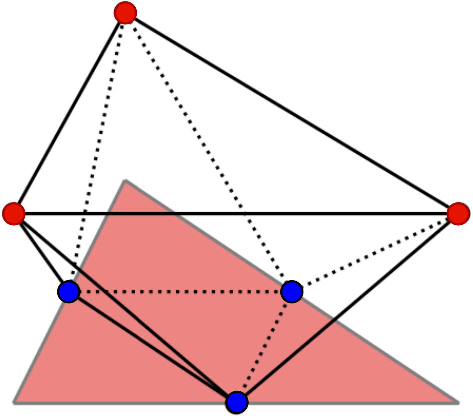

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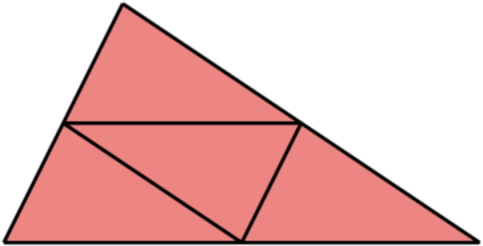

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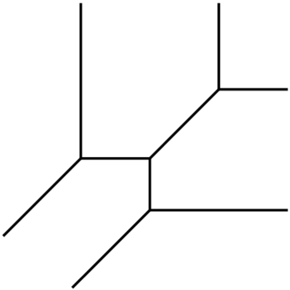
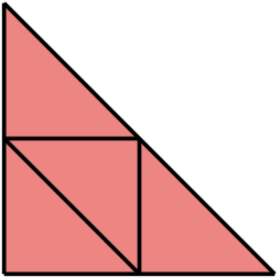


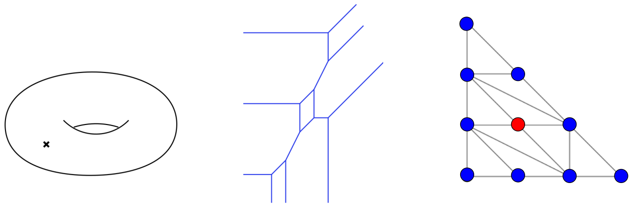


Genus formula for planar curves

Classical version

The genus of a smooth planar curve of degree d is equal to $g = \frac{(d-1)(d-2)}{2}$.

- Let $V(f)$ be a degree 3 curve with genus 1



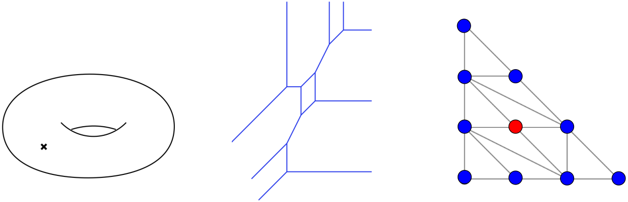
Combinatorial analogue

What is the genus of a tropical planar curve?

Genus formula for tropical planar curves

- Let $f \in \overline{\mathbb{R}}[x, y]$ be a tropical polynomial of degree d s.t.

$$\text{Newton polytope of } f = d\Delta := \text{convex hull}\{(0,0), (d,0), (0,d)\}.$$



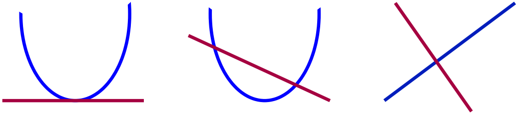
Tropical version

The genus of $V(f)$ is the number of vertices of the dual subdivision in the interior of $d\Delta$. If each integer point of $d\Delta \cap \mathbb{Z}^2$ occurs as a vertex in the subdivision, then we obtain the classical formula.

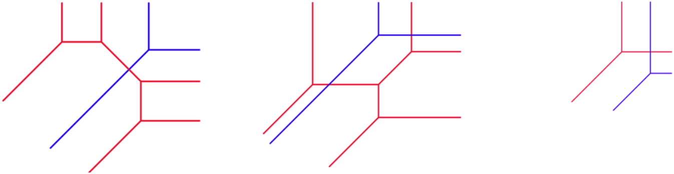
Bézout's theorem

Classical version

Two algebraic planar curves of degree d and d' intersect in dd' points.

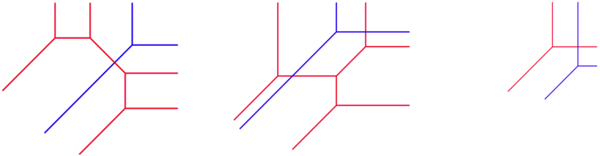


How about the tropical version?

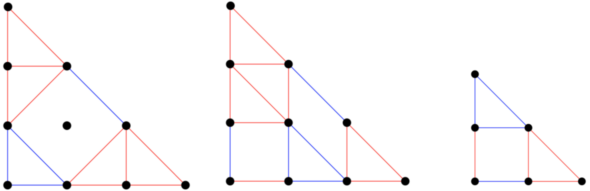


Tropical Bézout's theorem

- **Goal:** to provide a simple model of algebraic geometry.
- Why some of the intersection points are counted twice?

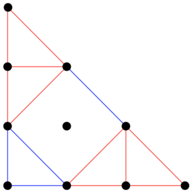


- Generic case: the curves intersect in finitely many points.
- The union of the curves of f and g is the curve of trop(fg).



Tropical Bézout's theorem

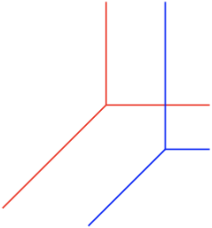
- Each **intersection point** is contained in an edge of both curves.
- The polygon dual to such a vertex of curve is a parallelogram.

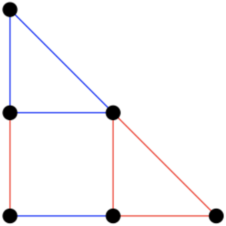


- **Observation:** The area of the parallelogram dual to an intersection point is related to its multiplicity.

Tropical Bézout's theorem

- Let $V(f)$ and $V(g)$ be two tropical curves of degree d and d', intersecting in a finite number of points and away from the vertices of the two curves.
- The **tropical multiplicity** of an intersection point p is the area of the parallelogram dual to p in the dual subdivision of $V(f) \cup V(g)$.





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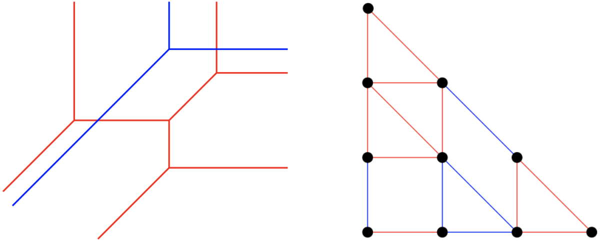

Tropical Bézout's theorem (Sturmfels)

The sum of the tropical multiplicities of all intersection points of $V(f)$ and $V(g)$ is equal to dd'.

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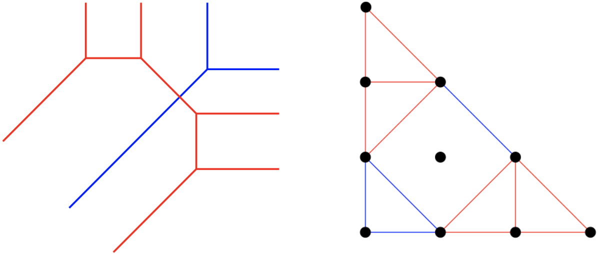

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Proof of tropical Bézout's theorem

Tropical Bézout's theorem (Sturmfels)

Let $V(f)$ and $V(g)$ be two tropical curves of degree d and d', intersecting in a finite number of points. The sum of the tropical multiplicities of all intersection points of $V(f)$ and $V(g)$ is equal to dd'.

- The curve $V(f) \cup V(g)$ is of degree $d + d'$. Hence, the sum of the areas of all polygons is equal to the area of $(d + d')\Delta$ that is $(d + d')^2/2$.
- There are 3 types of polygons in the dual subdivision of $V(f) \cup V(g)$:
  (1) Red polygons: Those which are dual to a vertex of $V(f)$. The sum of their areas is equal to the area of $d\Delta$ that is $d^2/2$.
  (2) Blue polygons: Those which are dual to a vertex of $V(g)$. The sum of their areas is equal to the area of $d'\Delta$ that is $d'^2/2$.
  (3) bicolored polygons: Those dual to an intersection point. Their areas sum up to

$$(d + d')^2/2 - d^2/2 - d'^2/2 = dd'.$$

Main references

- **A bit of tropical geometry**

  Erwan Brugallé and Kristin Shaw

- **A First Expedition to tropical geometry**

  Book by Johannes Rau

- **Introduction to Tropical geometry**

  Book by Bernd Sturmfels and Diane Maclagan

- **Essentials of tropical combinatorics**

  Book by Michael Joswig