Rigidity of Frameworks - Lecture 3
The rank function of the $\mathcal{C}_2^1$-cofactor matroid

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Matroids

A **matroid** $\mathcal{M}$ is a pair $(E, \mathcal{I})$ where E is a finite set and $\mathcal{I}$ is a family of subsets of E satisfying:

- $\emptyset \in \mathcal{I}$;
- if $A \subseteq B \subseteq E$ and $B \in \mathcal{I}$ then $A \in \mathcal{I}$;
- if $A, B \in \mathcal{I}$ and $|A| < |B|$ then there exists $x \in B \setminus A$ such that $A + x \in \mathcal{I}$.

$A \subseteq E$ is **independent** if $A \in \mathcal{I}$ and A is **dependent** if $A \notin \mathcal{I}$.
A is a **circuit** if it is a minimal dependent set.
A set $B \subset A$ is a **base of** A if B is a maximal independent subset of A.
All bases of A have the same cardinality $r(A)$ refered to as the the **rank of** A. We refer to the bases of E as **bases of** $\mathcal{M}$ and to the rank of E as the **rank of** $\mathcal{M}$, and denote it by $r(\mathcal{M})$.
The **closure of** A, $\text{cl}(A)$, is the (unique) maximal superset of A which has the same rank as A.

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The **closure of** A, $\mathrm{cl}(A)$, is the (unique) maximal superset of A which has the same rank as A.
A set $F \subseteq E$ is **spanning** if $r(F) = r(\mathcal{M})$. So F is spanning if and only if $\mathrm{cl}(F) = E$, and all bases of $\mathcal{M}$ are spanning.

A partial order for matroids

Given two matroids $\mathcal{M}_1 = (E, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E, \mathcal{I}_2)$ with the same groundset, we say $\mathcal{M}_1 \preceq M_2$ if $\mathcal{I}_1 \subseteq \mathcal{I}_2$. Equivalently $\mathcal{M}_1 \preceq M_2$ if:

- every base of $\mathcal{M}_1$ is independent in $\mathcal{M}_2$; or
- $r_1(A) \leq r_2(A)$ for all $A \subseteq E$.

The relation '$\preceq$' defines a partial order, the **weak order**, on any set of matroids with the same groundset E.

Truncations and Erections

The **truncation** of a matroid $\mathcal{M}_1 = (E, \mathcal{I}_1)$ of rank k is the matroid $\mathcal{M}_0 = (E, \mathcal{I}_0)$ of rank $k-1$, where $\mathcal{I}_0 = \{I \in \mathcal{I}_1 : |I| \leq k-1\}$.

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Crapo (1970) defined **matroid erection** as the 'inverse operation' to truncation. So M_1 is an **erection** of M_0 if M_0 is the truncation of M_1. (For technical reasons we also consider M_0 to be a **trivial erection** of itself.) Note that, although every matroid has a unique truncation, matroids may have several, or no, non-trivial erections.

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The set of all erections of a matroid M_0 forms a **poset** under the weak order of matroids. Crapo showed that it is actually a **lattice**: it is clear that the trivial erection of M_0 is the unique minimal element in this poset; he showed that there also exists a unique maximal element and called this the **free erection of** $\mathcal{M}_0$.

Truncations and Erections

Suppose $\mathcal{M}_1$ is an erection of $\mathcal{M}_0$ (so $\mathcal{M}_0$ is the truncation of $\mathcal{M}_1$) and we have $r(\mathcal{M}_1) = k$ (so $r(\mathcal{M}_0) = k - 1$).

	$\mathcal{M}_0$	$\mathcal{M}_1$
sets of size $k+1$	junk	spanning circuits | junk
sets of size k	spanning circuits | junk	bases | circuits | junk
sets of size $\leq k-1$	ind | circuits | junk	ind | circuits | junk

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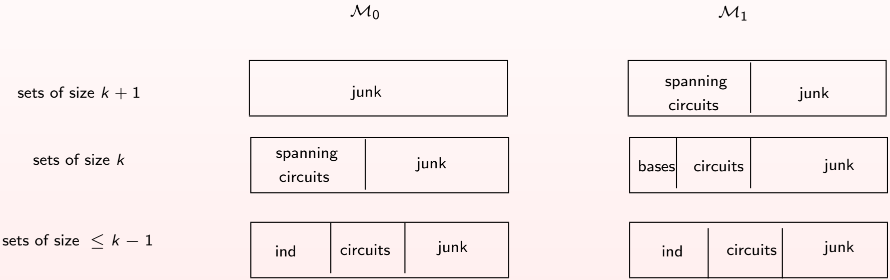

$\mathcal{M}_1$ becomes the free erection of $\mathcal{M}_0$ when we choose the set of circuits of $\mathcal{M}_1$ of size k to be the set of all spanning circuits of $\mathcal{M}_0$ which are implied by its non-spanning circuits.

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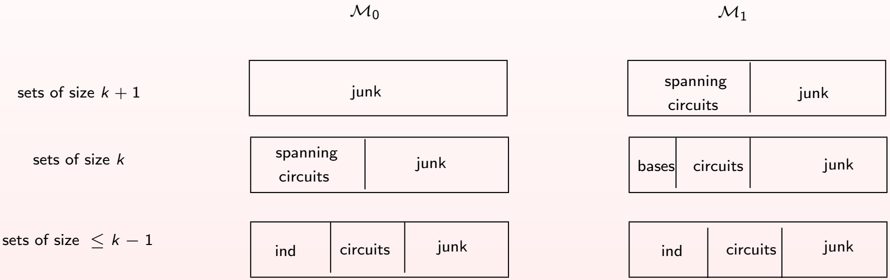

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$\mathcal{M}_0$ has no non-trivial erection if every spanning circuit of $\mathcal{M}_0$ is implied by its set of non-spanning circuits.

Matroid Elevations

A **partial elevation** of a matroid $\mathcal{M}_0$ is any matroid $\mathcal{M}$ which can be constructed from $\mathcal{M}_0$ by taking a sequence of erections. A **(full) elevation** of $\mathcal{M}_0$ is a partial elevation $\mathcal{M}$ which has no non-trivial erection.

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The set of all partial elevations of $\mathcal{M}_0$ forms a poset $P(\mathcal{M}_0)$ under the weak order and $\mathcal{M}_0$ is its unique minimal element. Every maximal element of $P(\mathcal{M}_0)$ will have no non-trivial erection so will be a (full) elevation of $\mathcal{M}_0$.

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Given Crapo's result that the poset of all erections of $\mathcal{M}_0$ is a lattice, it is tempting to conjecture that $P(\mathcal{M}_0)$ will also be a lattice and that the free elevation of $\mathcal{M}_0$ will be its unique maximal element.

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Free elevations

Lemma [Clinch, BJ, Tanigawa, 2019+]

Let $\mathcal{M}_0$ be a matroid on E and $\mathcal{M}$ be its free elevation. Then $\mathcal{M}$ is a maximal element in the poset of all partial elevations of $\mathcal{M}_0$.

Proof Let $\mathcal{M}'$ be a partial elevation of $\mathcal{M}_0$. We show $\mathcal{M}' \not\succ \mathcal{M}$. Let $\mathcal{M} = \mathcal{M}_t, \mathcal{M}_{t-1}, \ldots, \mathcal{M}_0$ and $\mathcal{M}' = \mathcal{M}'_s, \mathcal{M}'_{s-1}, \ldots, \mathcal{M}_0$ be the sequences of truncations which construct $\mathcal{M}_0$ from $\mathcal{M}$ and $\mathcal{M}'$, respectively.

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Let i be the largest integer such that $\mathcal{M}_i = \mathcal{M}'_i$. Then $\mathcal{M}_{i+1}$ and $\mathcal{M}'_{i+1}$ are both erections of $\mathcal{M}_i$. Since $\mathcal{M}_{i+1}$ is the free erection of $\mathcal{M}_i$, Crapo's result gives $\mathcal{M}'_{i+1} \prec \mathcal{M}_{i+1}$. Hence there exists $F \subseteq E$ such that F is dependent in $\mathcal{M}'_{i+1}$ and independent in $\mathcal{M}_{i+1}$.

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Then F is a spanning circuit of $\mathcal{M}_i$ which becomes a base of $\mathcal{M}_{i+1}$ and a non-spanning circuit of $\mathcal{M}'_{i+1}$. This implies that F is a circuit in $\mathcal{M}'_s$ and is independent in $\mathcal{M}_t$. Hence
$\mathcal{M}' = \mathcal{M}'_s \not\succ \mathcal{M}_t = \mathcal{M}$.

Implication for $\mathcal{C}_1^2$

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Corollary [Clinch, BJ, Tanigawa, 2019+]

Let $\mathcal{M}_0$ be the rank 10 matroid on $E(K_n)$ in which the set of non-spanning circuits is the set of all copies of K_5 in K_n. Then $\mathcal{C}_1^2(K_n)$ is the free elevation of M_0.

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Proof We have seen that $\mathcal{C}_1^2(K_n)$ is the unique maximal element in the poset of all abstract rigidity matroids on $E(K_n)$. The same proof gives the stronger result that $\mathcal{C}_1^2(K_n)$ is the unique maximal element in the poset of all partial elevations of $\mathcal{M}_0$. Since the lemma implies that the free elevation of $\mathcal{M}_0$ is a maximal element in this poset, it must be equal to $\mathcal{C}_1^2(K_n)$. □

A bound on the rank function of a matroid

Let $\mathcal{M}$ be a matroid. A sequence of circuits $(C_1, C_2, \ldots, C_t)$ of M is **proper** if $C_i \not\subseteq \bigcup_{j=1}^{i-1} C_j$ for all $2 \leq i \leq t$.

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Lemma [Clinch, BJ, Tanigawa, 2019+]

Let $(C_1, C_2, \ldots, C_m)$ be a proper sequence of circuits in $\mathcal{M}$. Then

$$r(\bigcup_{i=1}^m C_i) \leq |\bigcup_{i=1}^m C_i| - m.$$

Proof We use induction on m. The lemma holds when $m = 1$ since C_1 is a circuit so $r(C_1) = |C_1| - 1$. When $m \geq 2$,

$$\begin{aligned} r(\bigcup_{i=1}^m C_i) &\leq r(\bigcup_{i=1}^{m-1} C_i) + r(C_m \setminus \bigcup_{i=1}^{m-1} C_i) \\ &\leq |\bigcup_{i=1}^{m-1} C_i| - m + 1 + |C_m \setminus \bigcup_{i=1}^{m-1} C_i| \\ &\leq |\bigcup_{i=1}^{m} C_i| - m. \end{aligned}$$

since r is submodular and $(C_1, C_2, \ldots, C_m)$ is proper.

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Corollary [Clinch, BJ, Tanigawa, 2019+]

Suppose $\mathcal{M} = (E, r)$ is a matroid and $F \subseteq E$. Then

$$r(F) \leq \min\{|F_0| + |\bigcup_{i=1}^m C_i| - m\}$$

where the minimum is taken over all $F_0 \subseteq F$ and all proper sequences of circuits $(C_1, C_2, \ldots, C_m)$ which cover $F \backslash F_0$.

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Proof This follows using $r(F) \leq r(F_0) + r(F \backslash F_0) \leq |F_0| + r(F \backslash F_0)$ and applying the lemma to bound $r(F \backslash F_0)$.

A bound on the rank function of a partial elevation

Corollary [Clinch, BJ, Tanigawa, 2019+]

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Theorem [Clinch, BJ, Tanigawa, 2019+]

Suppose $\mathcal{M}_0$ is a matroid on E, $\mathcal{M} = (E, r)$ is a partial elevation of M_0 and $F \subseteq E$. Then

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Proof This follows from the corollary since every non-spanning circuit of M_0 is a circuit of M.

The rank function of the free elevation?

Theorem [Clinch, BJ, Tanigawa, 2019+]

Suppose $\mathcal{M}_0$ is a matroid on E. For $F \subseteq E$ let

$$\rho(F) = \min\{|F_0| + |\bigcup_{i=1}^m C_i| - m\}$$

where the minimum is taken over all $F_0 \subseteq F$ and all proper sequences of non-spanning circuits $(C_1, C_2, \ldots, C_m)$ of $\mathcal{M}_0$ which cover $F \setminus F_0$. Suppose ρ is submodular on 2^E. Then the free elevation of $\mathcal{M}_0$ is the unique maximal element in the poset of all partial elevations of $\mathcal{M}_0$ and its rank function is ρ.

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Question For which matroids $\mathcal{M}_0$ is ρ a submodular function?

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Question For which matroids $\mathcal{M}_0$ is ρ a submodular function?
Note If we can show that ρ is submodular when $\mathcal{M}_0$ is the rank $\binom{d+2}{2}$ matroid on $E(K_n)$ in which the set of non-spanning circuits is the set of all copies of K_{d+2} in K_n then this will verify part (a) of Graver's conjecture (that there is a unique maximal abstract d-rigidity matroid).

A characterisation of the rank function of $\mathcal{C}_2^1(K_n)$

A K_5-**sequence in** K_n is a sequence of subgraphs $(K_5^1, K_5^2, \ldots, K_5^m)$ each of which is isomorphic to K_5. It is **proper** if $K_5^i \not\subseteq \bigcup_{j=1}^{i-1} K_5^j$ for all $2 \leq i \leq t$.

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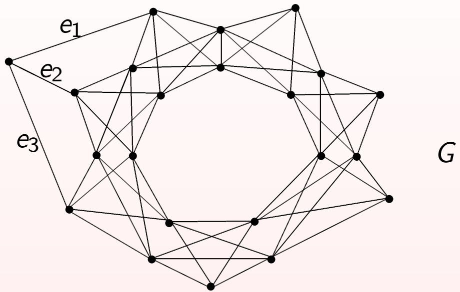
Theorem [Clinch, Tanigawa, BJ, 2019+]

Let $F \subseteq E(K_n)$. Then the rank of F in $\mathcal{C}_1^2(K_n)$ is

$$r(F) = \min\{|F_0| + |\bigcup_{i=1}^{t} E(K_5^i)| - m\}$$

where the minimum is taken over all $F_0 \subseteq F$ and all proper K_5-sequences $(K_5^1, K_5^2, \ldots, K_5^m)$ in K_n which cover $F \backslash F_0$.

Example



Let $F = E(G)$, $F_0 = \{e_1, e_2.e_3\}$ and $(K_5^1, K_5^2, \ldots, K_5^7)$ be the 'obvious' proper K_5-sequence which covers $F \backslash F_0$. We have

$$r(F) \leq |F_0| + \left|\bigcup_{i=1}^{7} E(K_5^i)\right| - 7 = 59 < 60 = |F|$$

so F is not $\mathcal{C}_2^1$-independent.

Shellable covers

Let $\mathcal{X}$ be a family of subsets of $V(K_n)$ of size at least five. A **hinge** of $\mathcal{X}$ is a pair of vertices $\{x, y\}$ with $X_i \cap X_j = \{x, y\}$ for some $i \neq j$. Let $H(\mathcal{X})$ be the set of all hinges of $\mathcal{X}$. The **degree** $\deg_{\mathcal{X}}(h)$ of a hinge $h = \{x, y\}$ of $\mathcal{X}$ is the number of sets in $\mathcal{X}$ which contain h. Following Dress (1983), we say $\mathcal{X}$ is **2-thin** if $|X_i \cap X_j| \leq 2$ for all distinct $X_i, X_j \in \mathcal{X}$ and define the **value** of $\mathcal{X}$ to be

$$\text{val}(\mathcal{X}) = \sum_{X \in \mathcal{X}} (3|X| - 6) - \sum_{h \in H(\mathcal{X})} (\deg_{\mathcal{X}}(h) - 1).$$

Shellable covers

Let $\mathcal{X}$ be a family of subsets of $V(K_n)$ of size at least five. A **hinge** of $\mathcal{X}$ is a pair of vertices $\{x, y\}$ with $X_i \cap X_j = \{x, y\}$ for some $i \neq j$. Let $H(\mathcal{X})$ be the set of all hinges of $\mathcal{X}$. The **degree** $\deg_{\mathcal{X}}(h)$ of a hinge $h = \{x, y\}$ of $\mathcal{X}$ is the number of sets in $\mathcal{X}$ which contain h. Following Dress (1983), we say $\mathcal{X}$ is **2-thin** if $|X_i \cap X_j| \leq 2$ for all distinct $X_i, X_j \in \mathcal{X}$ and define the **value** of $\mathcal{X}$ to be

$$\text{val}(\mathcal{X}) = \sum_{X \in \mathcal{X}} (3|X| - 6) - \sum_{h \in H(\mathcal{X})} (\deg_{\mathcal{X}}(h) - 1).$$

The family $\mathcal{X}$ is k-**shellable** if its elements can be ordered as a sequence $(X_1, X_2, \ldots, X_m)$ so that $|X_i \cap \bigcup_{j=1}^{i-1} X_j| \leq k$ for all $2 \leq i \leq m$. The concept of a k-shellable cover is related to that of an 'iterated cover' (Jackson and Jordán, 2006) and a 'generalized partial k-tree (Chen and Sitharam, 2014).

A second characterisation of independence in $\mathcal{C}_2^1(K_n)$

Theorem [Clinch, Tanigawa, BJ, 2019+]

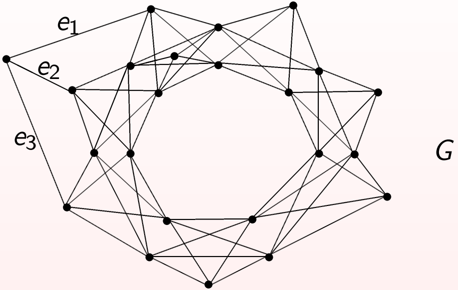
Let $F \subseteq E(K_n)$. Then the rank of F in $\mathcal{C}_1^2(K_n)$ is given by

$$r(F) = \min\{|F_0| + \mathrm{val}(\mathcal{X})\}$$

where the minimum is taken over all $F_0 \subseteq F$ and all 2-thin, 4-shellable covers of $F \setminus F_0$. Furthermore, we can construct a cover which minimises the RHS by choosing $\mathcal{X}$ to be the set of all maximal complete graphs of size at least 5 in $\mathrm{cl}(F)$ and taking F_0 to be the set of all edges in F which are not covered by $\mathcal{X}$.

This result confirms that conjectured expressions for the rank function of $\mathcal{R}_3(K_n)$ due to Jackson and Jordán (2006), and Dress (1987) are valid for $\mathcal{C}_2^1(K_n)$.

Example



Let $F = E(G)$, $F_{=}\{e_1, e_2.e_3\}$ and $\{X_1, X_2, \ldots, X_7\}$ be the 'obvious' 2-thin, 4-shellable cover of $F \backslash F_0$. We have $\text{val}(\mathcal{X}) = 6 \times 9 + 12 - 7 = 59$ and

$$r(F) \leq |F_0| + \text{val}(\mathcal{X}) = 3 + 59 < 63 = |F|$$

so F is not $\mathcal{C}_2^1$-independent.

Application

Theorem [Clinch, BJ, Tanigawa 2019+]

Every 12-connected graph is C_2^1-rigid.

Lovász and Yemini (1982) conjectured that the analogous result holds for $\mathcal{R}_3$-rigidity. Examples constructed by Lovász and Yemini show that the hypothesis of 12-connectivity is best possible for both $\mathcal{C}_2^1$ and $\mathcal{R}_3$.

Open Problems

Problem 1 Determine whether the X-replacement operation preserves independence in the generic 3-dimensional rigidity matroid (Tay and Whiteley, 1985).

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Problem 3 Determine whether the following function $\rho_d : 2^{E(K_n)} \to \mathbb{Z}$ is submodular.

$$\rho_d(F) = \min \left\{ |F_0| + \left| \bigcup_{i=1}^{m} E(K^i_{d+2}) \right| - m \right\}$$

where the minimum is taken over all $F_0 \subseteq F$ and all proper K_{d+2}-sequences $(K^1_{d+2}, K^2_{d+2}, \ldots, K^m_{d+2})$ in K_n which cover $F \setminus F_0$.
An affirmative answer would tell us that there is a unique maximal abstract d-rigidity matroid and ρ_d is its rank function.

Preprints

K. Clinch, B. Jackson and S. Tanigawa, Abstract 3-rigidity and bivariate C_2^1-splines I: Whiteley's maximality conjecture, preprint available at https://arxiv.org/abs/1911.00205.

K. Clinch, B. Jackson and S. Tanigawa, Abstract 3-rigidity and bivariate C_2^1-splines II: Combinatorial Characterization, preprint available at https://arxiv.org/abs/1911.00207.