Rigidity of Frameworks - Lecture 1
Bar-Joint Rigidity in $\mathbb{R}^d$

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- It is **rigid** if every continuous motion of the vertices of (G, p) in $\mathbb{R}^d$, which preserves the lengths of all edges of (G, p), also preserves the distances between all pairs of vertices of (G, p).)

Example

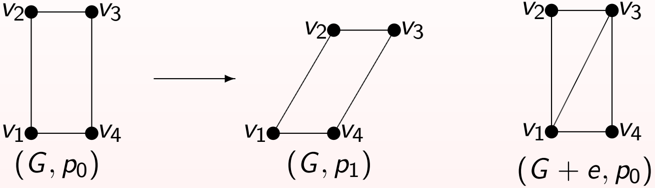

Figure: The 2-dimensional frameworks (G, p_0) and (G, p_1) are not rigid since (G, p_1) can be obtained from (G, p_0) by a continuous motion in $\mathbb{R}^2$ which preserves all edge lengths, but changes the distance between v_1 and v_3. The 2-dimensional framework $(G + e, p_0)$ obtained by adding a 'brace' $e = v_1 v_3$' to (G, p_0) is rigid. However, if we consider $(G + e, p_0)$ to be a 3-dimensional framework in which the vertices all lie in the same plane, then it will not be rigid since we can rotate the vertex v_2 about the 'hinge' $v_1 v_3$ and change the distance between v_2 and v_4.

Algorithmic Complexity

- A 1-dimensional framework (G, p) is rigid if and only if G is connected, but it is NP-hard to determine whether a given d-dimensional framework is rigid for any $d \geq 2$ (Abbot 2008).

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- The algorithmic problem of determining when a graph is generically rigid in $\mathbb{R}^d$ is solved for $d = 1$ (easy) and $d = 2$ (using a result of Pollaczek-Geiringer). It is an important open problem for $d \geq 3$.

The Rigidity Matrix

The rigidity of a given framework (G, p) is determined by the solution space of the system of quadratic equations

$$\|p_t(u) - p_t(v)\|^2 = \|p(u) - p(v)\|^2 \text{ for all } uv \in E \tag{1}$$

where $p_t(u)$ is the position of u at time t.

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Differentiating (1) wrt t and putting $t = 0$, we obtain the following system of linear equations for the **instantaneous velocities** $\dot{p}(u)$ at time $t = 0$.

$$(p(u) - p(v)) \cdot (\dot{p}(u) - \dot{p}(v)) = 0 \text{ for all } uv \in E \qquad (2)$$

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The **rigidity matrix** $R(G, p)$ of (G, p) is the matrix of coefficients of (2). It is an $|E| \times d|V|$ matrix with rows indexed by E and sequences of d consecutive columns indexed by V, in which the row indexed by $e = uv \in E$ is given by

$$_{e=uv}\begin{bmatrix} 0\dots 0 & \overset{u}{p(u)-p(v)} & 0\dots 0 & \overset{v}{p(v)-p(u)} & 0\dots 0 \end{bmatrix}.$$

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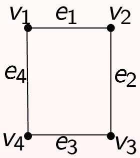

$$\begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} p(v_1)-p(v_2) & p(v_2)-p(v_1) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & p(v_2)-p(v_3) & p(v_3)-p(v_2) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & p(v_3)-p(v_4) & p(v_4)-p(v_3) \\ p(v_1)-p(v_4) & \mathbf{0} & \mathbf{0} & p(v_4)-p(v_1) \end{pmatrix} \end{array}$$

Assuming each edge has positive length, we have rank $R(G,p)=3$ when v_1, v_2, v_3, v_4 are collinear and otherwise rank $R(G,p)=4$.

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We say that (G,p) is **infinitesimally rigid** if

$$\text{rank } R(G,p) = \min\{d|V| - \tbinom{d+1}{2}, \tbinom{|V|}{2}\}.$$

Generic Rigidity and Gluck's Theorem

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Suppose (G, p) is a generic d-dimensional framework. Then (G, p) is rigid if and only if it is infinitesimally rigid.

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- The rigidity of a generic framework (G, p) depends only on the graph G and the dimension d. We say that G is **rigid in** $\mathbb{R}^d$ if some (or equivalently every) generic realisation of G in $\mathbb{R}^d$ is rigid.

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- We can determine whether G is rigid in $\mathbb{R}^d$ if we can determine when a given set of rows of $R(G, p)$ is linearly independent when p is generic.

Matroids

A **matroid** $\mathcal{M}$ is a pair $(E, \mathcal{I})$ where E is a finite set and $\mathcal{I}$ is a family of subsets of E satisfying:

- $\emptyset \in \mathcal{I}$;
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A set $B \subset A$ is a **basis of** A if B is a maximal independent subset of A. All bases of A have the same cardinality $r(A)$ refered to as the the **rank of** A. We refer to the bases of E as **bases of** $\mathcal{M}$ and to the rank of E as the **rank of** $\mathcal{M}$, and denote it by $r(\mathcal{M})$.

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Given a matrix R, the **row matroid** of R is the matroid with groundset E given by the rows of R in which a set $F \subseteq E$ is **independent** if the rows of $R(G, p)$ indexed by F are linearly independent.

The d-**dimensional rigidity matroid** of a graph $G = (V, E)$ is the row matroid $\mathcal{R}_d(G)$ of the rigidity matrix $R(G, p)$ for any generic $p : V \rightarrow \mathbb{R}^d$. We say that G is $\mathcal{R}_d$-**independent** if E is independent in $\mathcal{R}_d(G)$.

The d-dimensional rigidity matroid and Maxwell's theorem

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We can use Maxwell's observation on rank $R(G, p)$ to obtain a necessary condition for $\mathcal{R}_d$-independence. Given $X \subseteq V$ let $i(X)$ denote the number of edges of G induced by X.

Theorem [Maxwell, 1864]

Suppose $G = (V, E)$ is $\mathcal{R}_d$-independent. Then

$$i(X) \leq d|X| - \binom{d+1}{2}$$

for all $X \subseteq V$ with $|X| \geq d + 1$.

The case $d = 2$: Pollaczek-Geiringer's theorem

Maxwell's necessary condition for independence is also sufficient when $d = 1$ (since it implies that G is a forest). Hence $\mathcal{R}_1(G)$ is the well known *cycle matroid of* G.

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Theorem [Pollaczek-Geiringer, 1927]

A graph $G = (V, E)$ is $\mathcal{R}_2$-independent if and only if

$$i(X) \leq 2|X| - 3$$

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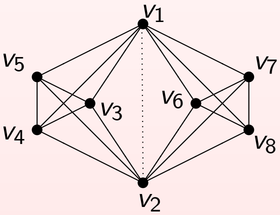
Jacobs and Hendrickson (1997) used this characterisation of independence to give an efficient pebble game algorithm for testing generic rigidity in $\mathbb{R}^2$.

Maxwell's condition is not sufficient when $d \geq 3$

The graph G given below shows that Maxwell's necessary condition,

$$i(X) \leq 3|X| - 6 \text{ for all } X \subseteq V \text{ with } |X| \geq 3,$$

does not imply $\mathcal{R}_3$-independence.



$$|E| = 18 = 3|V| - 6$$

Bases of $\mathcal{R}_d(K_n)$ and the Henneberg construction

Let K_n be the complete graph on $n \geq d+1$ vertices. Maxwell's theorem implies that $\mathcal{R}_d(K_n)$ has rank $dn - \binom{d+1}{2}$ and that the bases of $\mathcal{R}_d(K_n)$ are the (edge sets of the) **minimally rigid spanning subgraphs** of K_n i.e. subgraphs which are rigid in $\mathbb{R}^2$ and have n vertices and $dn - \binom{d+1}{2}$ edges.

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Pollaczek-Geiringer's characterisation of independence in $\mathcal{R}_2(K_n)$ is inductive and gives a recursive contruction for all graphs which are minimally rigid in $\mathbb{R}^2$ using the following operations first suggested by Henneberg in 1911. The **0-extension operation** adds a vertex of degree two to a graph. The **1-extension operation** deletes an edge e and then adds a vertex of degree three which includes both end-vertices of e in its neighbour set.

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Theorem

A graph is minimally rigid in $\mathbb{R}^2$ if and only if it can be constructed for K_3 by recursively applying the 0- and 1-extension operations.

Lemma

Suppose G is obtained from H by a 0-extension which adds a new vertex v and two new edges vw, vx. Let p be a realisation of G in $\mathbb{R}^2$ such that $p(v), p(w), p(x)$ are not collinear and $R(H, p|_H)$ has independent rows. Then $R(G, p)$ has independent rows.

Proof We have

$$R(G,p) = \begin{pmatrix} p(v) - p(w) & * \\ p(v) - p(x) & * \\ \mathbf{0} & R(H, p|_H) \end{pmatrix}$$

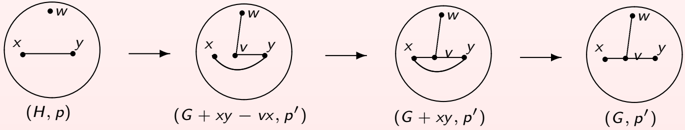
Since $R(H, p|_H)$ has independent rows and $p(v), p(w), p(x)$ are not collinear, $R(G, p)$ has independent rows. □

1-extensions preserve independence in $\mathcal{R}_2(K_n)$

Lemma

Suppose H is $\mathcal{R}_2$-independent and G is obtained from H by a 1-extension which adds a new vertex v, three new edges vw, vx, vy, and deletes the edge xy. Then G is $\mathcal{R}_2$-independent.

Proof Choose a generic realisation p of H and extend it to a realisation p' of G by putting $p'(v)$ on the line through $p(x)$ and $p(y)$. Then rank $R(G + xy - vx, p') =$ rank $R(H, p) + 2 = |E|$ since $(G + xy - vx, p')$ can be obtained from (H, p) by a 0-extension.



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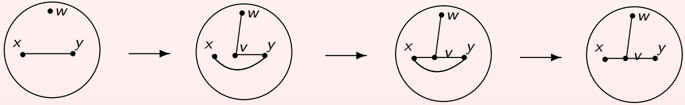
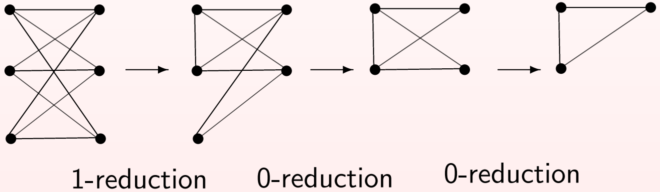


Since $p'(x), p'(y), p'(v)$ are collinear, $\{xy, yv, vx\}$ is a ciruit in the row matroid of $R(G + xy, p')$. Hence rank $R(G, p') =$ rank $R(G + xy, p') =$ rank $R(G + xy - vx, p') = |E|$ so $R(G, p')$ has linearly independent rows. □

Example

We can use the (inverse) Henneberg operations to construct a combinatorial certificate that $K_{3,3}$ is minimally rigid.



Since $K_{3,3}$ can be obtained from K_3 by a sequence of 0- and 1-extensions, it is $\mathcal{R}_2$-independent. Since each operation preserves the condition $|E| = 2|V| - 3$, $K_{3,3}$ is minimally rigid.

Rank fuction of $\mathcal{R}_2(K_n)$ and the Lovász-Yemini Theorem

Lovász and Yemini used Pollaczek-Geiringer's Theorem and a result of Edmunds from matroid theory to characterise the rank function $r_2(.)$ of $\mathcal{R}_2(K_n)$. We need the following concept.

A 1-**thin cover** of a graph G is a family $\mathcal{X}$ of subsets of V of size at least two such that each edge of G is induced by at least one set in $\mathcal{X}$ and $|X_i \cap X_j| \leq 1$ for all distinct $X_i, X_j \in \mathcal{X}$.

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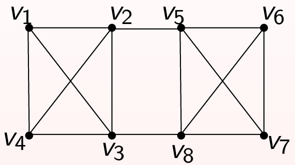
Theorem [Lovász and Yemini, 1982]

Suppose $G = (V, E)$ is a graph and $F \subseteq E$. Then

$$r_2(F) = \min \left\{ \sum_{X \in \mathcal{X}} (2|X| - 3) \right\}$$

where the minimum is taken over all 1-thin covers $\mathcal{X}$ of (V, F).

Example



Let $\mathcal{X} = \{X_1, X_2, X_3, X_4\}$ where $X_1 = \{v_1, v_2, v_3, v_4\}$, $X_2 = \{v_5, v_6, v_7, v_8\}$, $X_3 = \{v_2, v_5\}$ and $X_4 = \{v_3, v_8\}$.
Then

$$r_2(G) \leq \sum_{X \in \mathcal{X}} (2|X| - 3) = 5 + 5 + 1 + 1 = 12 < 13 = 2|V| - 3$$

so G is not rigid in $\mathbb{R}^2$.

Submodular functions and Edmund's Theorem

A set function $f : 2^E \to \mathbb{Z}$ is **submodular** if

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for all $A, B \subseteq E$.

Theorem [Edmonds, 1982]

Suppose $f : 2^E \to \mathbb{Z}$ is submodular and non-decreasing, and is non-negative on $2^E \backslash \{\emptyset\}$. Then

$$\mathcal{I} = \{I \subseteq E : |J| \leq f(J) \text{ for all } \emptyset \neq J \subseteq I\}$$

is the set of independent sets in a matroid $M_f = (E, \mathcal{I})$ on E. In addition, if $f(e) \leq 1$ for all $e \in E$, then the rank of any $F \subseteq E$ in M_f is given by

$$r(F) = \min \left\{ \sum_{F_i \in \mathcal{F}} f(F_i) \right\}$$

where the minimum is taken over all partitions $\mathcal{F}$ of F.

Application of Edmund's Theorem to $\mathcal{R}_2(K_n)$

Let $E = E(K_n)$ and $f : 2^E \to \mathbb{Z}$ be defined by putting $f(F) = 2|V(F)| - 3$. Then f is submodular and non-decreasing and is non-negative on $2^E \backslash \{\emptyset\}$. In addition, Pollaczek-Geiringer's Theorem gives $\mathcal{R}_2(K_n) = M_f$. So Edmond's theorem immediately implies that, for all $F \subseteq E$,

$$r_2(F) = \min \left\{ \sum_{F_i \in \mathcal{F}} (2|V(F_i)| - 3) \right\} \qquad (3)$$

where the minimum is taken over all partitions $\mathcal{F}$ of F.

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The expression for $r_2(G)$ given in the Lovász-Yemini theorem can be obtained by choosing a partition $\mathcal{F}$ for which equality holds in (3) and taking $\mathcal{X} = \{V(F_i) : F_i \in \mathcal{F}\}$. Then showing that $|X_i \cap X_j| \leq 1$ for all $X_i, X_j \in X$.