F = number field

F_∞/F = pro-p extn (often $\mathbb{Z}_p$, in particular the cyclotomic one!)

Thm (Iwasawa) Let F_∞/F be a $\mathbb{Z}_p$-extn & $p^{e_n} \| h(F_n)$

$$e_n = \mu p^n + \lambda n + \nu \qquad (n \gg 0)$$

where $\mu, \lambda \in \mathbb{Z}_{\geq 0}$, $\nu \in \mathbb{Z}$

Conj (Iwasawa) For the cyclotomic $\mathbb{Z}_p$-extn $\mu = 0$.

[Ferrero-Washington, 1979] $F/\mathbb{Q}$ = Abelian

Let $\Gamma = Gal(F_\infty/F) \simeq \mathbb{Z}_p$, $\Lambda(\Gamma) = \mathbb{Z}_p[[\Gamma]] \simeq \mathbb{Z}_p[[T]]$

(Structure Thm) Let M be a fin gen $\Lambda(\Gamma)$-module

$$M \sim \Lambda^r \oplus \bigoplus_{i=1}^{s} \frac{\Lambda}{p^{m_i}} \oplus \bigoplus_{j=1}^{t} \frac{\Lambda}{f_j^{n_j}} \qquad (r=0)$$

$$f_M(T) := p^\mu \prod_j f_j^{n_j} \qquad \mu = \sum m_i$$

Let E/F be an elliptic curve

$Sel(E/F)$ = p^∞-Selmer group

$= \ker\left(H^1(G_S(F), E[p^\infty]) \to \bigoplus_{v \in S} H^1(F_v, E)[p^\infty]\right)$

$R(E/F)$ = p^∞-fine Selmer group

$$0 \to R(E/F) \to Sel(E/F) \to \bigoplus_{v|p} E(F_v) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

$Sel(E/F_\infty) = \varinjlim_n Sel(E/F_n)$, $R(E/F_\infty) = \varinjlim_n R(E/F_n)$

Conj A (Coates-Sujatha): $R(E/F_{cyc})^\vee$ is $\Lambda(\Gamma)$-torsion & $\mu = 0$.

§ Evidence.

Thm (K.-Sujatha) Let F be a number fld and E/F be a rk 0 elliptic curve. Suppose that $Ш(E/F)$ is finite*. Then, as we vary over all primes of good ordinary reduction, $\mathrm{Sel}(E/F_{cyc})$ is finite for density one primes.

In particular, given E/F of rk 0, Conjecture A holds for density one good ordinary primes.

Sketch:

$$f_E(0) \sim \underbrace{\prod_{v \text{ bad}} c_v^{(p)}}\cdot \underbrace{\prod_{v|p} |\tilde{E}_v(f_v)_p|^2} \cdot |\mathrm{Sel}(E/F)|^2 \Big/ |E(F)_p|$$

$E(F)[p]$ = triv for all but fin many primes.

$|\mathrm{Sel}(E/F)| = |Ш(E/F)|$ = triv for all but fin many.

[p is non-anomalous for density one primes.
$(p \nmid |\tilde{E}_v(f_v)|)$

//

* $\mathrm{Sel}(E/F_{cyc})$ is in fact trivial

p-rational number field

F_{S_p} = max p-ramified extn of F

$F_{S_p}(p)$ = max pro-p quotient

F_{cyc}

F

Defn: F is p-rational if $\mathcal{G}_{S_p}(F) = \mathrm{Gal}(F_{S_p}(p)/F)$ is a free pro-p group

ex: $\mathbb{Q}$, K = Quad field if $p \nmid h(K)$, p = regular, $\mathbb{Q}(\mu_p)$

Thm (K.) Let F be a p-rational number field. The Classical Conjecture holds for F_{cyc}/F.

examples of non-Abelian number fields where Iwasawa $\mu = 0$ holds.

Cor (K.) Let F be p-rational and $F \supseteq \mu_p$.
Let $E/_F$ st $E(F)[p] \neq 0$. Then
Conjecture A holds for $\mathcal{Y}(E/F_{cyc}) = R(E/F_{cyc})^\vee$

$F(E[p]/F)$ = deg p extn or triv.

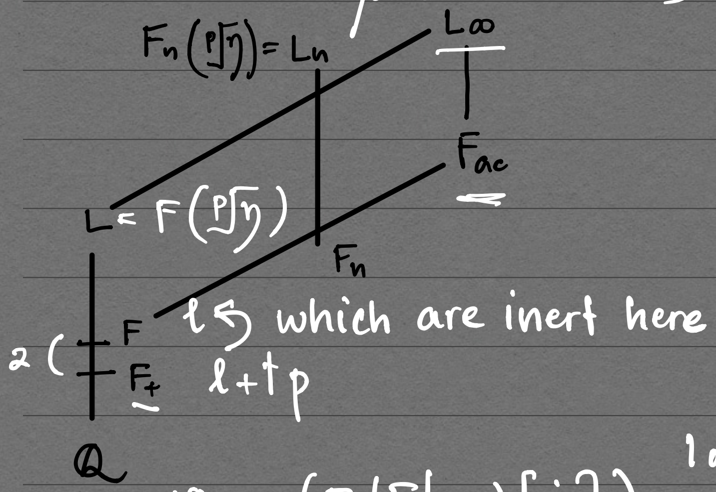
Thm (Coates-Sujatha '05, Lim-Murty '16): Let E/F be an elliptic curve with $E[p] = E(F)[p]$. Then
Conj A $\Leftrightarrow$ Classical Iwasawa $\mu = 0$ Conj

Thm (Iwasawa 1973) Let F be the cyclotomic field of p-th roots of unity $(p \neq 2)$ [or the 4-th roots of unity $(p=2)$]. Given $N \geq 1$, $\exists\, L/F$ which is cyclic of deg p and a $\mathbb{Z}_p$-extn L_∞/L st $\mu(L_\infty) \geq N$

$\mathbb{Q}(\mu_p)$

Thm (K. 2020) Let $p \neq 2$ and F be the cyclotomic field of p-th roots of unity. Let E/F be an elliptic curve st $E(F)[p] \neq 0$. Given $N \geq 1$, there exists a cyclic Galois extn L/F of degree p and a $\mathbb{Z}_p$-extn L_∞/L st

$$\mu(\mathcal{Y}(E/L_\infty)) \geq N.$$



$\eta \in F^\times$ divisible exactly by the first power of ℓ_i where $1 \leq i \leq m$

$$\underbrace{\dim_{\mathbb{F}_p}\left(R(E/L_n)[p]\right)}_{\mu(\mathcal{Y}) \geq} \overset{1 \text{ or } 2}{\geq} * \underbrace{\dim_{\mathbb{F}_p}\left(Cl(L_n)[p]\right)}_{\geq\, mp^n} - c_{bad}\, p^n$$

Thm (K. 2020) Let $N \in \mathbb{Z}_{\geq 0}$. Let F be a number field and G be any uniform pro-p group st an Iwasawa-type result holds.*
Let E/F be an elliptic curve with $E(F)[p] \neq 0$ then $\exists\, L/F$ = deg p extn and a G-extn L_∞/L st $\mu(\mathcal{Y}(E/L_\infty)) \geq N$

(Hajir-Maire) $\exists$ large class of pro-p groups G st you can construct a G-extn $/F$ st $\exists$ ∞ly many primes in F that split completely in F_∞/F.

→ Iwasawa-type result holds (μ becomes arb large in such a G-extn)

Control Theorems :

$$s_n : \mathrm{Sel}(E/F_n) \longrightarrow \mathrm{Sel}(E/F_\infty)^{\mathrm{Gal}(F_\infty/F_n)}$$

$E/_F$ = good ord redn at p $(p \neq 2)$

Thm (Mazur's Control Thm, 1972) Let F_∞/F be a $\mathbb{Z}_p$-extn ; $\ker(s_n)$ and $\mathrm{coker}(s_n)$ are finite and bounded independent of n

Thm (Greenberg, 2003) Extended this to general p-adic Lie extensions

In his theorem(s), the ker and coker are finite but not always bounded.

(E has pot good ord redn at p)

- Work with Meng Fai Lim

$$r_n : R(E|F_n) \longrightarrow R(E|F_\infty)^{Gal(F_\infty|F_n)}$$

(1) Rubin (some $\mathbb{Z}_p^d$ - extns) fin

(2) Wuthrich (all $\mathbb{Z}_p$- extns)
ker and coker are fin and bounded

Control Diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & & \arrow[d] & & 0 \arrow[d] \\
0 \arrow[r] & R(E/F_n) \arrow[r] \arrow[d, "r_n"] & & H^1(G_S(F_n), E[p^\infty]) \arrow[r] \arrow[d, "g_n"] & \bigoplus_{v \in S} H^1(F_{n,v}, E[p^\infty]) \arrow[d, "h_n"] \\
0 \arrow[r] & R(E/F_\infty)^{G_n} \arrow[r] \arrow[d] & & H^1(G_S(F_\infty), E[p^\infty])^{G_n} \arrow[r] \arrow[d] & \bigoplus_{v \in S} H^1(F_{\infty,v}, E[p^\infty])^{G_n} \\
 & & & &
\end{tikzcd}$$

(1) E/F be any elliptic curve

$\mathrm{corank}_{\mathbb{Z}_p}$ $\ker(r_n) = O(1)$

$\mathrm{corank}_{\mathbb{Z}_p}$ $\mathrm{coker}(r_n) = O(p^{(d-1)n})$

(2) Specialize to extensions

(i) $\mathbb{Z}_p^d$ extensions:

$$\mathrm{ord}_p |\ker(r_n)| = O(n) \; *$$
$$\mathrm{ord}_p |\mathrm{Coker}(r_n)| = O(p^{(d-1)n})$$

→ If Conjecture A is true, then

$$\mathrm{ord}_p (R(E/F_n)^\vee [p^\infty]) = O(np^{(d-1)n})$$

(ii) Trivializing extns $F_\infty = F(E[p^\infty])$

with CM:

$$\mathrm{ord}_p |\ker(r_n)| = O(n) \; *$$
$$\mathrm{ord}_p |\mathrm{Coker}(r_n)| = O(n)$$

without CM:

$$\mathrm{ord}_p |\ker(r_n)| = O(n)$$
$$\mathrm{ord}_p |\mathrm{coker}(r_n)| = O(np^{2n})$$

(iii) Kummer Extensions (good reduction)