Skew Mean Curvature Flow

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Outline

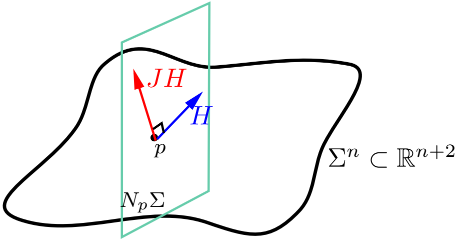
Part I. Introduction

Definition of SMCF

The Skew Mean Curvature Flow (SMCF) or Bi-normal Flow is a family of codimension two immersions $F : [0,T) \times \Sigma^n \to M^{n+2}$ evolving by

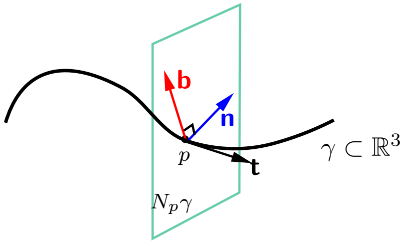
$$\partial_t F = J\mathbf{H}$$

where $\mathbf{H}$ is the mean curvature of Σ_t and J is the complex structure on the normal bundle $\mathcal{N}\Sigma_t$, which rotates a normal vector by $\pi/2$ positively in the normal plane.



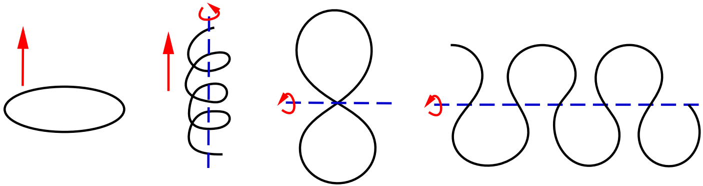
Examples 1: one dimension

- 1-D SMCF in $\mathbb{R}^3$, i.e. Vortex Filament Equation:
$$\gamma_t = \kappa \mathbf{b} = \gamma_s \times \gamma_{ss}$$
- By Hasimoto transformation $\Phi = \kappa e^{i\int \tau}$, equivalent to
$$-i\Phi_t = \Phi_{ss} + \frac{1}{2}|\Phi|^2\Phi,$$
which amounts to rewriting the evolution equation of curvature in a suitable frame (gauge) of the normal bundle.



Examples 1: one dimension

- 1D SMCF in $\mathbb{R}^3$ is completely integrable, has infinitely many conserved quantities, and admits soliton solutions.
- (translating or rotating) soliton curves are related to Euler's elastica and magnetic geodesics.



Circle Helix Figure 8 Wave-like

Examples 2: higher dimension

- Product of spheres $F : S^m(a) \times S^n(b) \to \mathbb{R}^{m+1} \times \mathbb{R}^{n+1}$ satisfies SMCF with $a(0) = b(0) = 1$ if
$$\begin{cases} \partial_t a & = -n/b; \\ \partial_t b & = +m/a. \end{cases}$$
- $m = n$ (eg. Clifford torus): global solution
$$a(t) = e^{-nt}, \quad b(t) = e^{nt}.$$
- $m < n$ (eg. $S^1 \times S^2 \subset \mathbb{R}^5$): finite time solution
$$a(t) = (1-(n-m)t)^{n/(n-m)}, b(t) = (1-(n-m)t)^{m/(m-n)},$$
which blows up at $T = 1/(n-m)$.[Khesin-Yang 2019]

Background 1: Hydrodynamics

SMCF models the locally induced motion of vortex membranes (codim 2 vortex) in a perfect fuild, which is deduced from the Euler equation by applying the Biot-Savart formula.

- [Da Rios 1906] 1-D Vortex filament in $\mathbb{R}^3$

$$\gamma_t = \gamma_s \times \gamma_{ss}$$

- [Shashikanth 2012] 2-D Vortex membrane in $\mathbb{R}^4$
- [Khesin 2012] n-D Vortex membrane in $\mathbb{R}^{n+2}$

Background 2: Superfluid

The Gross-Pitaevskii equation

$$-i\phi_t = \Delta\phi + \frac{1}{\varepsilon}W(|\phi|^2)\phi$$

models the evolution of the wave function $\phi : \mathbb{R}^{n+2} \times [0, \infty) \to \mathbb{C}^1$ associated with a Bose condensate.

Conjecture: Vortices evolve along SMCF.(Physics evidences)

- [Tai-Chia Lin 2000] 1-D vortex filament
- [Jerrard 2002] n-D vortex sphere with multiplicity 1
- Similar structure found in superconductors (parabolic PDEs) and cosmic strings (hyperbolic PDEs)

Background 3: Connection with other flows

- SMCF is the Hamiltonian flow of the volume functional in the (infinite dimensional) symplectic manifold $(\mathcal{I}, \Omega)$. Here $\mathcal{I}$ is the space of immersions moduli diffeomorphisms, Ω is the Marsden-Weinstein symplectic structure

$$\Omega(V, W) = \int_{F(\Sigma)} \iota_V \iota_W d\bar{\mu}$$

- Mean Curvature Flow is the gradient flow of the volume functional.

Background 3: Connection with other flows

Theorem (S., 2017)

The Gauss map $\rho : [0,T] \times \Sigma^n \to G(n,2)$ of SMCF in $\mathbb{R}^{n+2}$ satisfies the Schrödinger map flow

$$\partial_t \rho = J_G \Delta_g \rho.$$

- The Grassmannian manifold $G(n,2)$ is a Kähler manifold
- The underlying metric is evolving by $\partial_t g = -2 \langle J\mathbf{H}, \mathbf{A} \rangle$.
- [Ruh-Vilms, 1970]Gauss map of a minimal submanifold is harmonic.
- [M-T.Wang, 2001]Gauss map of the MCF satisfies the harmonic map heat flow.

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Background 3: Connection with other flows

Complex PDE	Mapping	Sub-manifold
Elliptic	Harmonic map	Minimal sub-manifold
Parabolic	Harmonic heat flow	Mean curvature flow
Hyperbolic	Wave Map	Hyperbolic curvature flow
Schrödinger	Schrödinger map flow	**Skew mean curvature flow**
Ginzburg-Landau	Dirichlet Energy	Volume functional

Part II. Problems and Results

The initial value problem

Consider the initial value problem

$$\begin{cases} \partial_t F = J\mathbf{H} \\ F(0,\cdot) = F_0 \end{cases}$$

In local coordinates $\mathbf{H}$ can be written as

$$\mathbf{H}^\alpha = (\Delta_g F)^\alpha = g^{ij}(\partial_i\partial_j F^\alpha - \Gamma_{ij}^k \partial_k F^\alpha),$$

where

$$g = g(DF), \quad \Gamma = \Gamma(D^2F), \quad J = J(DF).$$

For a graphic solution $F(x) = (x, \phi^1(x), \phi^2(x))$, reduce to

$$\partial_t \phi = i\Delta\phi + O(\partial_x^2\phi|\partial_x\phi|^2).$$

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Global existence of 1-D SMCF

Theorem (H. Gomez, 2004)

Given a smooth initial curve with $\kappa \in L^2$ in a three dimensional Riemannian manifold, the 1-D SMCF admits a unique smooth global solution.

Remark:

- 1D-SMCF is essentially equivalent to a 1-D Schrödinger map arising from ferromagnetism physics.
- The proof used the Hasimoto Transformation and Strichartz-type estimates for Schrödinger equations.
- There exists self-similar solutions which becomes singular in finite time [Gutierrez-Rivas-Vega 2003].

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Main difficulties

For higher dimensional SMCF ($n \geq 2$):

- Not covered by existing theory on nonlinear Schrödinger equations
- De Turck's trick does not apply
- NO Hasimoto transformation (?)
- Apparently, only preserved quantity is the volume (element), NO conservation laws for curvature
- Even the uniqueness of derivative non-linear Schrödinger equations is difficult

$$u_t = i\Delta u + F(\nabla u).$$

Results 1: Local Existence of 2-D SMCF

Theorem (S.-Sun, 2015)

Given a smooth initial compact surface Σ_0 in $\mathbb{R}^4$,the SMCF admits a smooth local solution, where the existence time depends only on $\|\mathbf{A}_0\|_{H^{2,2}}$ and the volume of Σ_0.

Remark:

- The existence actually holds for $W^{4,2}$-initial data and for more general ambient manifolds.
- The proof relies on a uniform estimate of the second fundamental form which only holds for dimension two.

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Results 2: Existence and Uniqueness of SMCF

Theorem (S. 2019)

Given a n-dimensional smooth initial compact sub-manifold Σ^n in $\mathbb{R}^{n+2}$, the SMCF admits a unique smooth local solution, where the existence time depends only on the $W^{[n/2]+2,2}$-norm of the initial Gauss map.

Remark:

- The existence and uniqueness actually holds for more general initial data and ambient manifolds.
- For $k \geq [n/2]$, the $W^{k+1,2}$-norm of the Gauss map is equivalent to

$$E = vol + \|\mathbf{H}\|_p + \|\mathbf{A}\|_{k,2}.$$

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Open problems

The study of SMCF has just began, lots of open problems.

- Regularity: optimal regularity for existence and uniqueness?
- Local existence: on non-compact manifolds?
- Finite time blow-up: examples in 2d?
- Global existence: small initial data?
- Long time asymptotic behavior: geometric application?
- Solitons: no known non-trivial solitons for dimension ≥ 2, which satisfies, for a Killing vector field K,

$$J\mathbf{H} = K.$$

Recent progress

Long-time existence for graphic submanifold with small data:

Theorem (Ze Li, preprint 2020)

Let $n \geq 3$ and $k \geq n+4$. For a smooth graphic initial submanifold which is a H_k-small transversal perturbation of $\mathbb{R}^n \subset \mathbb{R}^{n+2}$, there exists a global unique smooth solution to the SMCF.

Local well-posedness for non-compact submanifold with small data:

Theorem (Huang-Tataru, preprint 2020)

Let $n \geq 4$ and $k > n/2$. There exists $\varepsilon_0 > 0$ such that any initial submanifold $F_0 : \mathbb{R}^n \to \mathbb{R}^{n+2}$ with $\|\mathbf{H}_0\|_{H^k} \leq \varepsilon_0$, the n-D SMCF is locally well-posed.

Part III. Existence of SMCF

Perturbed SMCF

For $\varepsilon > 0$, consider the perturbed SMCF

$$\begin{cases} \partial_t F = J_\varepsilon \mathbf{H} = \varepsilon \mathbf{H} + J\mathbf{H}; \\ F(0,\cdot) = F_0. \end{cases}$$

- For $\varepsilon > 0$, pSMCF is weakly parabolic and De Turck trick yields a local solution
- Parabolic estimates will blow-up as $\varepsilon \to 0$, need uniform estimates of pSMCF w.r.t. ε
- **Strategy**: energy method, which relies on uniform Sobolev inequalities since the metric is varying along the flow,.

Evolution equations

Along the pSMCF, we have

$$
\begin{aligned}
\partial_t d\mu &= -\varepsilon|\mathbf{H}|^2 d\mu \\
\partial_t g &= -2\left\langle J_\varepsilon \mathbf{H}, \mathbf{A}\right\rangle \\
\partial_t \mathbf{A} &= J_\varepsilon \Delta \mathbf{A} + \mathbf{A} * \mathbf{A} * \mathbf{A}. \\
\partial_t \mathbf{H} &= J_\varepsilon \Delta \mathbf{H} + \mathbf{A} * \mathbf{A} * \mathbf{H}. \\
\partial_t \nabla^l \mathbf{A} &= J_\varepsilon \Delta \nabla^l \mathbf{A} + \sum_{i+j+k=l} \nabla^i \mathbf{A} * \nabla^j \mathbf{A} * \nabla^k \mathbf{A}.
\end{aligned}
$$

Uniform Sobolev inequality

Theorem (Mantegazza, GAFA, 2002)

Suppose M^n is a compact submanifold of Euclidean space. If $Vol + \|H\|_{n+\delta} \leq B$ for some $\delta > 0$, then there exists $C = C(B, n)$ such that

$$\|D^j T\|_p \leq C \|T\|_{W^{k,q}}^a \|T\|_r^{1-a},$$

where $j \in [0, k], p, q, r \in [1, \infty]$ and $a \in [j/k, 1]$ satisfies

$$\frac{1}{p} = \frac{j}{m} + a\left(\frac{1}{q} - \frac{k}{m}\right) + \frac{1-a}{r} > 0.$$

In particular, when $kq > m$, we have

$$\|T\|_\infty \leq C \|T\|_{W^{k,q}}.$$

Proof of existence I

Along pSMCF, since

$$\partial_t \nabla^l \mathbf{A} = J_\varepsilon \Delta \nabla^l \mathbf{A} + \sum_{i+j+k=l} \nabla^i \mathbf{A} * \nabla^j \mathbf{A} * \nabla^k \mathbf{A},$$

it follows

$$\partial_t \|\nabla^l A\|_2^2 \leq C \sum_{i+j+k=l} \int_M |\nabla^i A| \cdot |\nabla^j A| \cdot |\nabla^k A| \cdot |\nabla^l A| d\mu.$$

Assume $V + \|H\|_p \leq B$ for some $p > n$, then by uniform Sobolev, we have for $k > n/2$

$$\partial_t \|A\|_p^2 \leq C(B) \|A\|_{k,2}^2 (1 + \|A\|_p^2)$$
$$\partial_t \|A\|_{k,2}^2 \leq C(B) \|A\|_{k,2}^2 \cdot \|A\|_{k,2}^2$$

Proof of existence II

By setting the energy $E = V + \|A\|_p^2 + \|A\|_{k,2}^2$, we conclude

$$\partial_t E \leq C(B)E \cdot (1 + E).$$

Lemma

For pSMCF with $\varepsilon > 0$, there exists a uniform time $T > 0$ only depending on E_0 such that $E(t) \leq 2E_0$ for all $t \in [0, T_0]$.

Once we have uniform time T and estimates of A, the convergence of pSMCF and existence of SMCF follow by standard arguments. □

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□

Part IV. Uniqueness of SMCF

Uniqueness of SMCF

Consider the initial value problem of SMCF

$$\begin{cases} \partial_t F = J\mathbf{H}, \\ F(0,\cdot) = F_0. \end{cases}$$

For two solutions F and $\tilde{F}$, show that $F = \tilde{F}$.

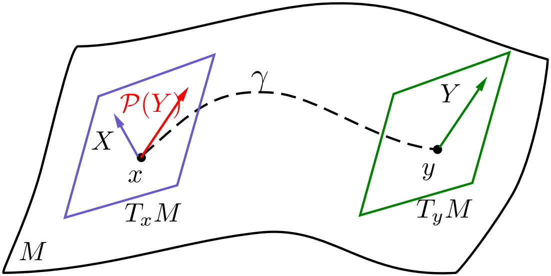
- Since no maximal principal, again we will use energy methods, which is also useful in parabolic flows,
  e.g. Ricci flow by [Kotchwar], and MCF by [Lee & Ma].
- **Key idea**: Measure the difference/distance of two solutions intrinsically by Parallel transportation/Relative gauge.

Distance of vector fields

Suppose $x, y \in (M, g)$ is connected by a unique geodesic γ, then for any $X \in T_xM, Y \in T_yM$, define

$$d_1(X, Y) = |X - \mathcal{P}(Y)|$$

where $\mathcal{P} : T_yM \to T_xM$ is the parallel transportation along γ.



Distance of second fundamental forms

Question: For two submanifolds $F, \tilde{F} : \Sigma^n \to \mathbb{R}^m$, how to compare their second fundamental forms A and $\tilde{A}$ intrinsically?

- If Gauss maps $\rho, \tilde{\rho}$ lie close enough, define

$$d_1(A, \tilde{A}) = d_1(d\rho, d\tilde{\rho}).$$

  by using parallel transportation $\mathcal{P} : \tilde{\rho}^* TG \to \rho^* TG$
- Actually we can do better! Observe

$$\rho^* TG = \rho^*(\mathcal{G}^\top \otimes \mathcal{G}^\perp) = F^*(T\bar{\Sigma} \otimes N\bar{\Sigma}) =: \mathcal{H} \otimes \mathcal{N}$$

  the parallel transportation $\mathcal{P}$ actually splits

$$\mathcal{P}^\top : \tilde{\mathcal{H}} \to \mathcal{H}, \quad \mathcal{P}^\perp : \tilde{\mathcal{N}} \to \mathcal{N}.$$

Distance of arbitrary tensors

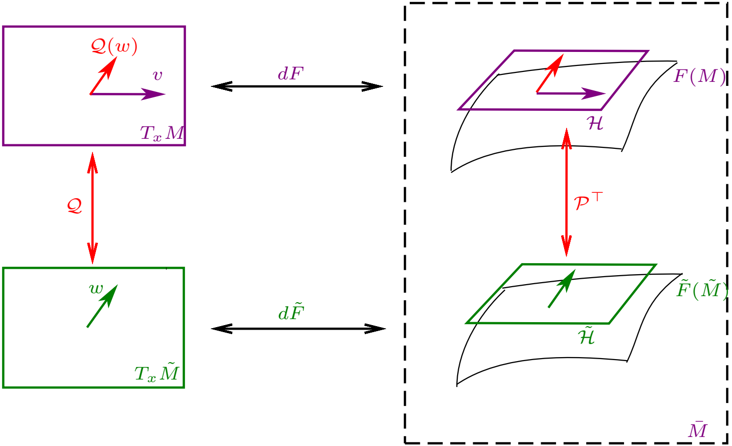
- $\mathcal{P}$ in turn gives a "parallel transportation" $\mathcal{Q} : T\tilde{\Sigma} \to T\Sigma$ by

$$\begin{array}{ccc} T\tilde{\Sigma} & \xrightarrow{d\tilde{F}} & \tilde{\mathcal{H}} \\ {\scriptstyle \mathcal{Q}}\downarrow & & \downarrow{\scriptstyle \mathcal{P}^{\top}} \\ T\Sigma & \xrightarrow[dF]{} & \mathcal{H} \end{array}$$

- Now for any tensor $\Phi \in \Gamma(\mathcal{N} \otimes (T\Sigma)^p), \tilde{\Phi} \in \Gamma(\tilde{\mathcal{N}} \otimes (T\tilde{\Sigma})^p)$, we can define their "intrinsic distance" by

$$d(\Phi, \tilde{\Phi}) = |\Phi - \mathcal{P}^{\perp} \otimes \mathcal{Q}^p(\tilde{\Phi})|.$$

Parallel transport for tensors



Idea of proof

Step 1: For a sufficiently small time, we can define the parallel transportation $\mathcal{P}$ and $\mathcal{Q}$ and derive estimates of their derivatives.

Step 2: Define the energy functional

$$\mathcal{L} = \int_\Sigma \Big(|d(\rho, \tilde{\rho})|^2 + |d(A, \tilde{A})|^2 + |d(\nabla A, \tilde{\nabla}\tilde{A})|^2 + |g - \tilde{g}|^2 + |\Gamma - \tilde{\Gamma}|^2 + |\mathbf{I} - \mathcal{Q}|^{\mathbf{2}} \Big) dv.$$

Step 3: By the evolution equations of SMCF, we can derive a Gronwall inequality for the energy $\mathcal{L}$, which implies uniqueness. □

Thank you for your attention!