On the properties of Lambda quantiles and financial applications

Ilaria Peri
Birkbeck, University of London

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Outline

1. **Theory of Lambda quantiles**
2. **Computing and backtesting of Lambda quantiles**
   - Lambda estimation: benchmark approach
   - Lambda quantile regression
3. **Extensions and future research**

Idea of Λ-quantiles

Λ-quantiles are quantiles that instead of a fixed level λ are based on variable Λ function.

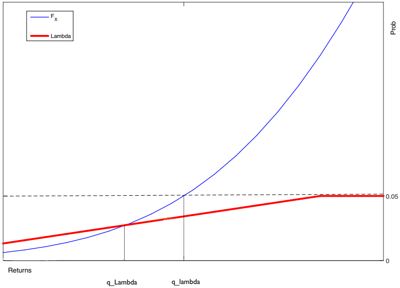
From λ-**quantile**, with $\lambda \in [0, 1]$:

$$q_\lambda(X) = \inf\{x : F(x) > \lambda\}$$

To Λ-**quantiles**, with $\Lambda \colon \mathbb{R} \to [0, 1]$:

$$q_\Lambda(X) = \inf\{x : F(x) > \Lambda(x)\}$$

If $\Lambda \equiv \lambda$ then $q_\Lambda = q_\lambda$.



Motivation in risk management

Notations:

- X random variable of profits and losses, F its distribution function.
- $\mathcal{M}$ set of all distribution functions.

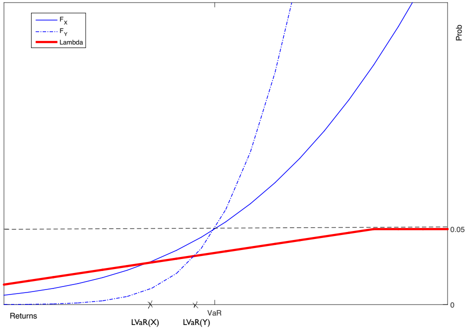
Risk measures represent capital requirements, hence:

$$VaR_\lambda := -q_\lambda(X)$$

$$\Lambda VaR := -q_\Lambda(X)$$

Desiderable features:

- **flexibility** of the confidence level
- ability to better **capture "tail risk"**



Original definition

Definition (Frittelli, Maggis and P., 2014)

Given a monotone and right continuous function

$$\Lambda \colon \mathbb{R} \to [\lambda^m, \lambda^M]$$

with $0 < \lambda^m \leq \lambda^M < 1$, the Lambda VaR is the map $\Lambda VaR : \mathcal{M} \to \mathbb{R} \cup \{+\infty\}$ defined as

$$\Lambda VaR(F) := -\inf\{x \in \mathbb{R} : F(x) > \Lambda(x)\}.$$

Remark: If $\Lambda \equiv \lambda$ then $\Lambda VaR(F) = VaR(F)$.

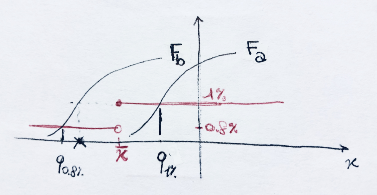
Example: Λ step function

Nondecreasing case:

$$\Lambda(x) = 0.008 \cdot \mathbf{1}_{(-\infty,\bar{x})}(x) + 0.01 \cdot \mathbf{1}_{[\bar{x},+\infty)}(x)$$

In risk management, $\bar{x}$ is the highest loss we are expecting to lose with probability 1%.

$$q_\Lambda^+ = \begin{cases} q_{1\%}^+ & \text{if } q_{1\%} \geq \bar{x} \\ q_{0.8\%}^+ & \text{if } q_{1\%} < \bar{x} \end{cases}$$

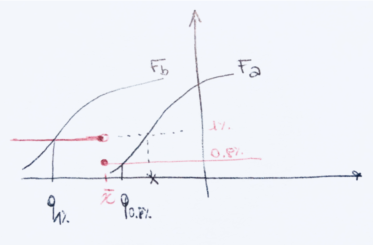


More sensitive than $q_{1\%}$ to left-tail events.

Nonincreasing case:

$$\Lambda(x) = 0.01 \cdot \mathbf{1}_{(-\infty,\bar{x})}(x) + 0.008 \cdot \mathbf{1}_{[\bar{x},+\infty)}(x)$$

$$q_\Lambda^+ = \begin{cases} q_{1\%}^+ & \text{if } q_{1\%} \geq \bar{x} \\ q_{0.8\%}^+ & \text{if } q_{1\%} < \bar{x} \end{cases}$$



Less sensitive than $q_{1\%}$ to right-tail events.

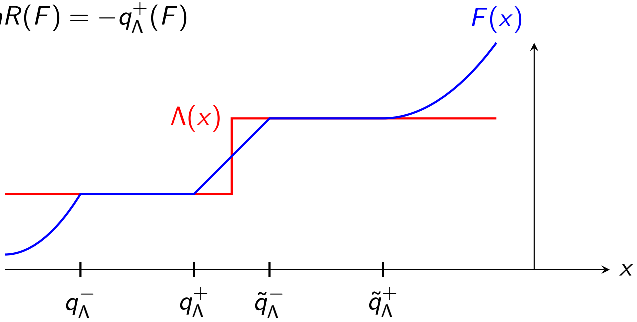
General definition of Lambda quantiles

Four possible Λ-quantiles:

Definition (Bellini and P., 2020)

$$q_\Lambda^-(F) := \inf\{x \in \mathbb{R} | F(x) \geq \Lambda(x)\}, \quad q_\Lambda^+(F) := \inf\{x \in \mathbb{R} | F(x) > \Lambda(x)\},$$
$$\tilde{q}_\Lambda^-(F) := \sup\{x \in \mathbb{R} | F(x) < \Lambda(x)\}, \quad \tilde{q}_\Lambda^+(F) := \sup\{x \in \mathbb{R} | F(x) \leq \Lambda(x)\}.$$

Notice: $\Lambda VaR(F) = -q_\Lambda^+(F)$



Elementary properties

Proposition (Bellini and P., 2020)

a) $q_\Lambda^-(F) \leq q_\Lambda^+(F)$, $\tilde{q}_\Lambda^-(F) \leq \tilde{q}_\Lambda^+(F)$

b) q_Λ^-, q_Λ^+, $\tilde{q}_\Lambda^-$, $\tilde{q}_\Lambda^+$ *are monotonic*

c) q_Λ^-, q_Λ^+, $\tilde{q}_\Lambda^-$, $\tilde{q}_\Lambda^+$ *are monotonic with respect to* Λ

d) q_Λ^-, q_Λ^+ *are m-quasiconcave,* $\tilde{q}_\Lambda^-$, $\tilde{q}_\Lambda^+$ *are m-quasiconvex.*

Recall that a functional $T\colon \mathcal{M} \to \mathbb{R}$ is m-quasiconvex if for each $F_1, F_2 \in \mathcal{M}$ and $\alpha \in (0,1)$ it holds that

$$T(\alpha F_1 + (1-\alpha)F_2) \leq \max(T(F_1), T(F_2)),$$

that is if the lower level sets of T are convex with respect to mixtures, and similarly T is m-quasiconcave if

$$T(\alpha F_1 + (1-\alpha)F_2) \geq \min(T(F_1), T(F_2)).$$

Nonincreasing case

Proposition (Bellini and P., 2020)

Let $\Lambda\colon \mathbb{R} \to [0,1]$ be nonincreasing. Then:

a) $\tilde{q}^-_\Lambda(F) = q^-_\Lambda(F)$ *and* $\tilde{q}^+_\Lambda(F) = q^+_\Lambda(F)$

b) q^-_Λ *and* q^+_Λ *are finite if and only if* $\Lambda \not\equiv 0$ *and* $\Lambda \not\equiv 1$

c) *If* $\Lambda \not\equiv 0$ *and* $\Lambda \not\equiv 1$, *then* q^-_Λ *is normalized if and only if* $\Lambda(x) > 0$ *for each* $x \in \mathbb{R}$, *and* q^+_Λ *is normalized if and only if* $\Lambda(x) < 1$ *for each* $x \in \mathbb{R}$.

d) *if* $\Lambda_1(x) = \Lambda_2(x)$ *on their common points of continuity, then* $q^-_{\Lambda_1}(F) = q^-_{\Lambda_2}(F)$ *and* $q^+_{\Lambda_1}(F) = q^+_{\Lambda_2}(F)$.

We say that the functional T is normalised if $T(\delta_x) = x$

Locality and axiomatization

Locality property: The locality property is satisfied by the usual quantiles, and is considered quite desirable since it expresses a very strong form of robustness with respect to outliers. It has been described by Koenker (2017): "[...]quantiles are inherently local and are nearly impervious to small perturbations of distributional mass. [...]"

Definition (Bellini and P., 2020)

We say that T is local if

$$T(F) \in (x, y) \Rightarrow T(F) = T(G), \text{ for each } G \in \mathcal{M} \text{ with } G = F \text{ on } (x, y).$$

Axiomatization of Λ-quantiles:

Theorem (Bellini and P., 2020)

Let $T \colon \mathcal{M} \to \mathbb{R}$ be normalized, monotonic and local. If T is weakly lower semicontinuous, then there exists a nonincreasing $\Lambda \colon \mathbb{R} \to [0, 1]$ such that $T(F) = q^-_\Lambda(F)$, while if T is weakly upper semicontinuous then $T(F) = q^+_\Lambda(F)$.

Robustness

Motivation: sensitivity of a risk estimator to small changes in the data set

Definition: A risk estimator $\hat{\rho}$ is robust if it is continuous with respect to the Lévy metric (Hampel et al., 1986; Huber, 1981; after considered by Cont et al. 2010 for general risk measures).

Robustness of the ΛVaR historical estimator

Theorem (Burzoni, P. and Ruffo, 2017)

If Λ is continuous and does not coincide with F on any interval, ΛVaR is continuous w.r.t the Lévy metric, hence the historical estimator of ΛVaR is robust.

Cont et al. (2010) showed that if a risk measure is continuous with respect to the Lévy metric, then its historical estimator is robust.

Convex level sets (CxLS)

Motivation: Necessary condition for elicitability (Osband, 1985) important for backtesting and applicability of supervised learning methods.

Definition: for each $F_1, F_2 \in \mathcal{M}$, $\alpha \in (0,1)$ and $\gamma \in \mathbb{R}$, a functional T satisfies the CxLS property if:

$$T(F_1) = T(F_2) = \gamma \Rightarrow T(\alpha F_1 + (1-\alpha)F_2) = \gamma,$$

Convex level sets of Λ-quantiles

If $\Lambda\colon \mathbb{R} \to [0,1]$ is nonincreasing, then $q_\Lambda^-(F)$ and $q_\Lambda^+(F)$ have the CxLS property.

If Λ is non-decreasing and piecewise constant with a finite number of jumps, then q_Λ has convex level sets on the set of increasing distribution functions (see Lemma 6 in Burzoni et al. 2017)

Elicitability

Definition: A statistical functional $T : \mathcal{M} \to \mathbb{R}$ is elicitable if there exists a scoring function $S : \mathbb{R} \times \mathbb{R} \to [0, +\infty)$ s.t.

$$T(F) = \arg\min_x \int S(x, y) dF(y) \quad \forall F \in \mathcal{M}.$$

Interpretation: $S(x, y)$ represents the magnitude of the forecasting error between the *ex-ante* risk prediction x and the *ex-post* realization $y \in \mathbb{R}$.

Theorem (Burzoni, P. and Ruffo, 2017, Bellini and Bignozzi, 2015)

If Λ is s.t. $0 < \lambda^m \leq \lambda^M < 1$, q_Λ^+ is elicitable on:

$$\mathcal{M}_\Lambda = \{F \in \mathcal{M}_1 \textit{ that overpass only once } \Lambda \textit{ at } \Lambda VaR \textit{ level}\}$$

with scoring function given by

$$S_\Lambda(x, y) = (y - x)^- - \int_y^x \Lambda(t) dt.$$

and $\mathcal{M}_1$ is set of distribution functions with finite first moment.

Example: two steps nonincreasing Lambda

Set

$$\Lambda(t) = \begin{cases} \lambda_1 \text{ if } t < \bar{t} \\ \lambda_2 \text{ if } t \geq \bar{t}, \end{cases}$$

with $\lambda_1 > \lambda_2$, in this case, the Λ-quantile is elicitable and:

$$S_\Lambda(x, y) = \begin{cases} \lambda_2(y - x)_+ + (1 - \lambda_2)(y - x)_- & \text{if } x, y \geq \bar{t} \\ \lambda_1(y - x)_+ + (1 - \lambda_1)(y - x)_- & \text{if } x, y \leq \bar{t} \\ \lambda_2(y - \bar{t}) + \lambda_1(\bar{t} - x) & \text{if } x < \bar{t} < y \\ (1 - \lambda_2)(x - \bar{t}) + (1 - \lambda_1)(\bar{t} - y) & \text{if } y < \bar{t} < x. \end{cases}$$

Notice that, the first two specifications correspond to the usual scoring functions for λ_1 and λ_2 quantiles.

Λ-quantile regression

In the same spirit of the quantile regression by Koenker and Bassett (1978), we assume that the Λ-quantile of Y depends linearly on a suitable vector $\boldsymbol{X}$ of covariates and some parameters β. The Λ-quantile forecasts are obtained by

$$\hat{q}_{\Lambda}(y|\boldsymbol{x}) = \hat{\beta}'\boldsymbol{x}.$$

and β estimated by

$$\hat{\beta} = \arg\min_{\beta} \sum_i S_{\Lambda}(\beta'\boldsymbol{x}_i, y_i),$$

where $S_{\Lambda}(\cdot,\cdot)$ is the loss function associated to q_{Λ}, $\boldsymbol{x}_i$ and y_i are the sample realizations of $\boldsymbol{X}$ and Y.
When $\Lambda = \lambda$ we get the classical linear λ-quantile regression.

Methods of computation in risk management

Building Lambda:

1. Λ **direction** - based on market trend
   - increasing: bearish market trend
   - decreasing: bullish or stable market trend
2. Λ **range of values**, $\left[\lambda^m, \lambda^M\right]$:
   - $\lambda^m \cong 0$ very close to zero. Critical choice.
   - best choice with high risk aversion: $\lambda^M = 0.01$
3. Λ **functional shape**, based on the risk aversion profile: linear, convex (power-law, exponential), concave: arc-tangent.
4. Λ **parameters**.

Simulation approach:

- Building F with Monte Carlo or historical simulation. Simulating the iid random variables and building the empirical distribution function.
- Applying the formula: the 'while' algorithm to compute the Λ-quantile.

Parametric approach: consist in applying a closed formula (if available).
Supervise learning approach: for instance, use the Lambda quantile regression.

Two-steps nonincreasing Λ parametric approach

Set

$$\Lambda(t) = \begin{cases} \lambda_1 \text{ if } t < \bar{t} \\ \lambda_2 \text{ if } t \geq \bar{t}, \end{cases}$$

with $0 < \lambda_1 \leq \lambda_2 < 1$ and $\bar{t} \in \mathbb{R}$. If $Y = \mu + \sigma Z$, where Z is a standard random variable, then:

$$q_\Lambda(Y) = \begin{cases} \mu + \sigma q_{\lambda_1}(Z) & \text{if } \sigma < \bar{\sigma}_1 \\ \bar{t} & \text{if } \bar{\sigma}_1 < \sigma < \bar{\sigma}_2 \\ \mu + \sigma q_{\lambda_2}(Z) & \text{if } \sigma > \bar{\sigma}_2 \end{cases},$$

where

$$\bar{\sigma}_1 := \frac{\bar{t} - \mu}{q_{\lambda_2}(Z)}, \quad \bar{\sigma}_2 := \frac{\bar{t} - \mu}{q_{\lambda_1}(Z)}.$$

Methods for backtesting

Corbetta and P. (2018) framework

Violations:

$$I_t = \begin{cases} 1 & \text{if } x_t < q_{\Lambda_t} \\ 0 & \text{otherwise} \end{cases} \qquad I_t \sim B(\lambda_t)\,.$$

here, violations are **not** identically distributed, since:

$$\lambda_t^0 = \Lambda(q_{\Lambda_t})$$

but are assumed to be **independent**. Null and alternative hypothesis:

$$H_0\colon\ \lambda_t \leq \lambda_t^0 \text{ for every } t;$$
$$H_1\colon\ \lambda_t > \lambda_t^0 \text{ for some } t.$$

Poisson Binomial test:

$$Z_1 := \sum_{t=1}^{T} I_t \qquad \text{under } H_0 \quad Z_1 \sim Poiss.Bin(\{\lambda_t^0\}_t)$$

Lambda estimation: benchmark approach

Hitaj, Mateus, P., 2018

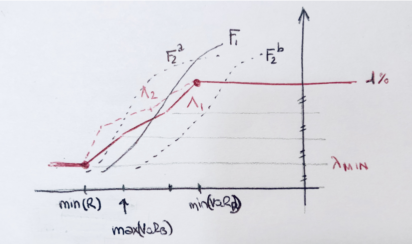
Estimation of Λ by **linear interpolation:**

1. Take a sample of **market benchmarks** (i.e. S&P500, FTSE100)
2. Compute min return, max, mean and min VaR_λ of the benchmarks;
3. Take an equipartition of the interval $(0; \lambda_M]$.

In-sample period and **holding period**: the same of the risk measure.

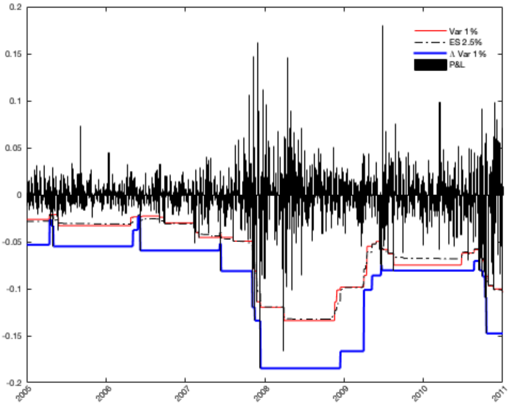
Interpretation: In the increasing case ΛVaR represents the tail of a market proxy distribution and captures:

- **different behaviour** of assets comparing with market (benchmark);
- **recent market trend** and specific asset reaction (dynamic approach).



Real data experiment: reactivity

Comparison between daily out-sample returns versus the risk measures.: In Hitaj, Mateus, P. (2018) and Corbetta and P. (2018) we compared historical simulation (in the figure below), normal, GARCH-t , EVT with Generalized Pareto approaches for a sample of equities severely affected by the financial crisis 2008.



Backtesting results: dynamic benchmark approach

Violations

	Historical						*Gaussian*					
	2006	2007	2008	2009	2010	2011	2006	2007	2008	2009	2010	2011
VaR 1%	3.42	5.33	11.58	0.75	3.08	6.83	4.58	7.08	14.92	1.75	4.17	9.42
ΛVaR 1% *(decr)*	2.21	3.00	6.38	0.67	1.79	4.13	4.33	6.29	13.67	1.50	3.58	8.88
ΛVaR 1% *(incr)*	1.17	1.04	3.92	0.42	0.96	2.75	3.33	4.92	11.25	1.04	2.88	6.83
	GARCH						*EVT*					
	2006	2007	2008	2009	2010	2011	2006	2007	2008	2009	2010	2011
VaR 1%	3.17	6.83	8.25	0.33	0.75	4.33	3.42	5.33	11.58	0.83	3.08	6.92
ΛVaR 1% *(decr)*	2.92	5.29	6.88	0.29	0.38	4.08	2.21	2.21	7.08	0.75	1.71	4.17
ΛVaR 1% *(incr)*	1.25	2.75	3.54	0.00	0.25	1.42	1.25	1.13	4.29	0.42	0.96	2.75

Poisson Binomial test

	Historical						*Gaussian*					
	2006	2007	2008	2009	2010	2011	2006	2007	2008	2009	2010	2011
VaR 1%	100%	58%	0%	100%	75%	25%	58%	33%	0%	92%	50%	8%
ΛVaR 1% *(decr)*	96%	79%	21%	100%	96%	71%	38%	17%	0%	88%	46%	8%
ΛVaR 1% *(incr)*	75 %	83%	0%	100%	79%	21%	4%	4%	0%	42%	38%	8%
	2006	2007	2008	2009	2010	2011	2006	2007	2008	2009	2010	2011
	GARCH						*EVT*					
VaR 1%	75%	50%	33%	100%	100%	67%	100%	58%	0%	100%	75%	25%
ΛVaR 1% *(decr)*	71%	58%	33%	100%	100%	67%	100%	92%	8%	100%	96%	54%
ΛVaR 1% *(incr)*	71%	54%	25%	100%	92%	58%	67%	75%	0%	100%	79%	21%

Testing Lambda Quantile regression

Forecasting two-steps decreasing Λ-quantiles

As a second example, we compare the forecasting of the Λ-quantile using a two-step non decreasing Lambda with $\lambda_1 = 0.98$, $\lambda_2 = 0.99$, $\bar{x} = 0.015$ with two methods:

- estimation of an **AR(1)-Garch(1,1) model** with t innovations on the logreturns of the S&P500 with rolling windows of 250 days
- **Lambda quantile regression** of the S&P500 with rolling windows of 250 days with two covariates: the lagged logreturn of S&P500 and the lagged logreturn of the VIX Index.

The dataset is composed of 2360 daily logreturns of the S&P500 and of the VIX from 4/1/2010 to 21/5/2019, that originated 2110 forecasted values of the Λ-quantiles.

Real data experiment: reactivity

Forecasting two-steps decreasing Λ-quantiles

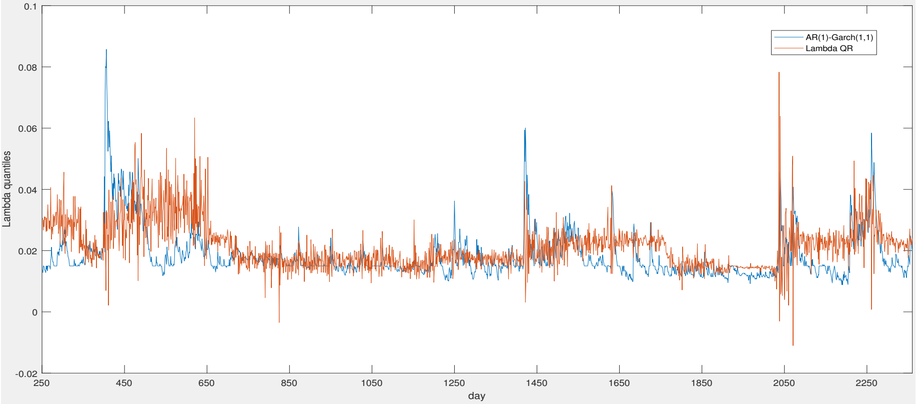


Figure: Comparison between the Λ-quantile forecasts by means of AR(1)-Garch(1,1) and Λ-quantile regression.

Backtesting results: Lambda quantile regression

Window	LQR $\bar{t} = 0.015$	AG $\bar{t} = 0.015$	LQR $\bar{t} = 0.02$	AG $\bar{t} = 0.02$	LQR $\bar{t} = 0.025$	AG $\bar{t} = 0.025$
1	0.00354	0.00883	0.00217	0.01305	0.00100	0.01621
2	0.92466	0.11819	0.92021	0.29092	0.90534	0.35728
3	0.18572	0.24224	0.06367	0.47086	0.02861	0.19778
4	0.07903	0.28899	0.04225	0.22259	0.02373	0.16812
5	0.15965	0.01676	0.37083	0.05461	0.17979	0.03529
6	0.00274	0.01722	0.00227	0.03748	0.00067	0.04928
7	0.24388	0.50511	0.38102	0.34824	0.14627	0.26996
8	0.40906	0.12261	0.52231	0.06931	0.41508	0.06417
9	0.00157	0.09371	0.00093	0.06027	0.00425	0.04092
10	0.24224	0.05835	0.37313	0.03803	0.34373	0.02852

Table: Backtesting results of the Λ-quantile forecasts based on the AR(1)-GARCH(1,1) approach (AG) and the Λ-quantile regression (LQR). The table displays the p-values of the Poisson Binomial test.

Work in progress and future research

- Improve on the estimation of the Lambda function used as proxy of the market tail distribution. In particular, we want that ΛVaR releases capital in case of good market conditions. Possible answer: adding a variable that detect the change of the market condition.
- Risk contribution and sensitivity analysis of Lambda quantiles (in collaboration with A. Ince and S. Pesenti)
- Theoretical properties of the Lambda quantile regression (with F. Bellini) and exploring application of statistical learning techniques such as:
  - Lambda quantile regression trees
  - Lambda quantile regression forests
- Extending this backtesting framework to other risk measures.

Main references

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