Sums of Squares of Polynomials and Graphs

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Joint work with Luis Felipe Vargas (CWI)

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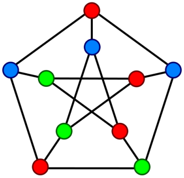
Computing the stability number $\alpha(G)$



$\alpha = 4 \quad \chi = 3 \quad \overline{\chi} = 5$

- **Stability number** $\alpha(G)$:
maximum cardinality of a set of pairwise non-adjacent vertices (**stable set**)
- **Clique cover number** $\overline{\chi}(G) := \chi(\overline{G})$:
minimum number of **cliques** covering V
- $\alpha(G) \leq \overline{\chi}(G)$

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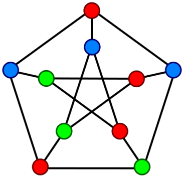

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Optimization over the **simplex** Δ_n Motzkin-Straus (1965)

$$\frac{1}{\alpha(G)} = \min \ x^{\mathsf{T}}(I + A_G)x \ \text{ s.t. } \ x \in \Delta_n = \{x : x \geq 0, \ \textstyle\sum_{i=1}^n x_i = 1\}$$

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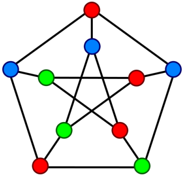

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S stable with size $\alpha(G) \rightsquigarrow x = \frac{\chi^S}{\alpha(G)} \in \Delta_n$ with value $\frac{1}{\alpha(G)}$

$$I + A_G = \begin{array}{c} \\ S \\ V \setminus S \end{array} \begin{array}{c} \begin{array}{cc} S & V \setminus S \end{array} \\ \begin{pmatrix} I & A_{G[S,V\setminus S]} \\ A_{G[V\setminus S,S]} & I + A_{G[V\setminus S]} \end{pmatrix} \end{array}$$

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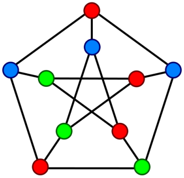

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Optimization over the **unit sphere** $\mathbb{S}^{n-1}$ — $x^{\circ 2} = (x_1^2, \dots, x_n^2)$

$$\frac{1}{\alpha(G)} = \min \ (x^{\circ 2})^T(I + A_G)x^{\circ 2} \ \text{ s.t. } x \in \mathbb{S}^{n-1} = \{x : \sum_{i=1}^n x_i^2 = 1\}$$

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Copositive cone

$$\mathrm{COP}_n = \{M \in \mathcal{S}^n : x^\mathsf{T} M x \geq 0 \quad \forall x \in \mathbb{R}^n_+\}$$

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$$\alpha(G) = \min_{\lambda \in \mathbb{R}} \ \lambda \ \text{ s.t. } \lambda(I + A_G) - J \in COP_n$$

$$J = ee^\mathsf{T}$$

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$\iff x^{\mathsf{T}}(I + A_G)x \geq 1/\lambda$ on Δ_n

SoS approximations for $\alpha(G)$

the cones $K_n^{(r)}$

the bounds $\vartheta^{(r)}(G)$

Tractable subcones of COP_n

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linear cone $C_n^{(r)} = \{M : (\sum_i x_i)^r x^\mathsf{T} M x \in \mathbb{R}_+[x]\}$ (nonnegative coefficients)

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Theorem (Pólya 1974; Powers-Reznick 2001)

If p is a form s.t. $p > 0$ on Δ_n, then there exists an integer $r \in \mathbb{N}$ s.t. $(\sum_i x_i)^r p$ has **nonnegative coefficients**

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in fact, for any $r \geq \binom{d}{2} \frac{L_p}{p_{\min}} - d$ $\qquad d = \deg(p),\ L_p = \max_\alpha |p_\alpha| \frac{\alpha!}{d!}$

Hierarchies of approximations for $\alpha(G)$

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  [de Klerk-Pasechnik 2002]

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- **Equality:** $\vartheta^{(r)}(G) = \alpha(G)$ for $r = \alpha(G) - 1$ if $\alpha(G) \leq 8$ [Gvozdenović-L 2007]

De Klerk-Pasechnik Conjecture on $\vartheta^{(r)}(G)$

De Klerk-Pasechnik conjecture

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The polynomial $(\sum_i x_i^2)^r p_G$ is SoS for $r = \alpha(G) - 1$ (for some r)

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Recall: The polynomial p_G is **nonnegative** on $\mathbb{R}^n$ since $M_G \in \mathrm{COP}_n$

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If true:

- this would give a class of polynomials for which most known sufficient conditions for finite convergence do not apply

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Recall: The polynomial p_G is **nonnegative** on $\mathbb{R}^n$ since $M_G \in \mathrm{COP}_n$

If true:

- this would give a class of polynomials for which most known sufficient conditions for finite convergence do not apply

  [Reznick 1995] form $p > 0$ on $\mathbb{R}^n \setminus \{0\} \rightsquigarrow \exists r$ s.t. $(\sum_i x_i^2)^r p$ is SoS

De Klerk-Pasechnik conjecture

Conjecture

(1) $\vartheta^{(r)}(G) = \alpha(G)$ *for* $r = \alpha(G) - 1$ *[de Klerk-Pasechnik 2002]*

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- this shows that the *continuous* copositive-based hierarchy has the *same convergence behaviour* as the *discrete* SoS hierarchy for $\alpha(G)$

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Recall: Conjecture (1) holds for graphs with $\alpha(G) \le 8$

The cone $K_n^{(0)}$ and the bound $\vartheta^{(0)}(G)$

- $K_n^{(0)} = \mathrm{PSD}_n + \mathbb{R}_+^{n\times n}$ [Parrilo 2000]
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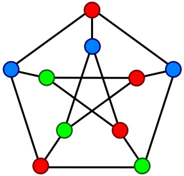
  This is not an equivalence!



$\vartheta\text{rank}(G) = 0$ for the Petersen graph

but $\alpha = 4 < \overline{\chi} = 5$

A link to matrix completion

For a graph G the following are equivalent:

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- $\chi(\overline{G}) = 3 \iff \vartheta(G) = 3$ with certificate P of **rank 2**
  $\rightsquigarrow$ hard instance of psd matrix completion **with rank constraint**

Examples with $\vartheta\mathrm{rank}(G) = 1$

- For the **5-cycle** C_5, $p_{C_5} = (x^{\circ 2})^{\mathsf{T}}(2(I + A_{C_5}) - J)x^{\circ 2}$

$$M_{C_5} = 2(I+A_{C_5})-J = \begin{pmatrix} 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 \end{pmatrix} \text{ is the } \textbf{Horn} \text{ matrix}$$

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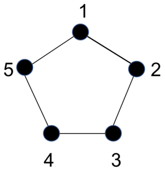

$$\left(\sum_i x_i^2\right) p_{C_5}$$
$$= \sum_{\text{circular}} x_1^2 (x_5^2 + x_1^2 + x_2^2 - x_3^2 - x_4^2)^2$$
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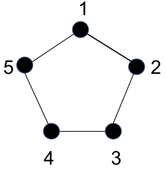

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- Odd cycles have rank 1: $\vartheta\text{rank}(C_{2n+1}) = 1$ [dK-P 2002]

Partial solution to the weak conjecture:

$\vartheta\mathrm{rank}(G) < \infty$ if G has no critical edges

Main steps

- Role of **critical** edges
- Link the bounds $\vartheta^{(r)}(G)$ to the Lasserre hierarchy for Motzkin-Straus (MS) formulation
- Characterize the minimizers of (MS)

Role of critical edges

Critical / a-critical graphs

- An edge e is **critical** if $\alpha(G \backslash e) = \alpha(G) + 1$
  G is **critical** if *all edges* are critical; G **a-critical** if *no edge* is critical

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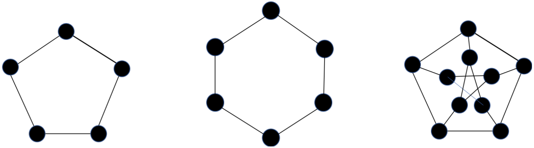

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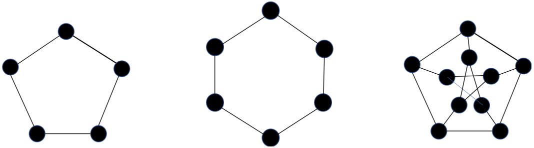

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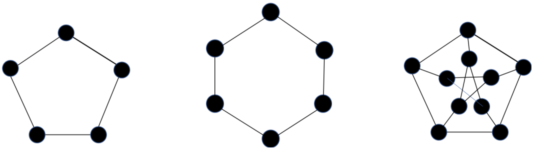

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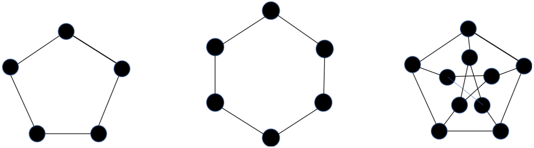

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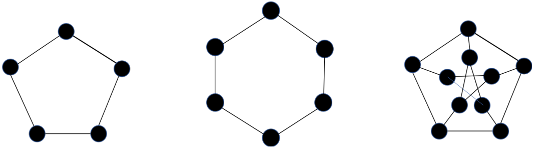

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But: We can prove the weak conjecture for **a-critical** graphs

Link to Lasserre hierarchy for other formulations of $\alpha(G)$

Motzkin-Straus formulation (MS)

$$\frac{1}{\alpha(G)} = \min \; x^{\mathsf{T}}(I + A_G)x \;\; \text{s.t.} \;\; x \in \Delta_n = \{x : x \geq 0, \; \sum_{i=1}^{n} x_i = 1\}$$

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Relation with $\vartheta^{(r)}(G)$: [L-V 2021]

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$\rightsquigarrow$ *It suffices to show finite convergence of Lasserre hierarchy for (MS)*

Motzkin-Straus formulation (MS)

$$\frac{1}{\alpha(G)} = \min \; x^{\mathsf{T}}(I + A_G)x \;\; \text{s.t.} \;\; x \in \Delta_n = \{x : x \geq 0, \; \textstyle\sum_{i=1}^n x_i = 1\}$$

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$$\text{las}_r^{\Delta}(G) = \frac{1}{\alpha(G)} \text{ for some } r \text{ ?}$$

Finite convergence for a-critical graphs

The weak conjecture holds for a-critical graphs

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Ex: For C_5, $x = (t, \frac{1}{2} - t, 0, \frac{1}{2}, 0)$ is a minimizer for any $t \in [0, \frac{1}{2}]$

Perturbed hierarchy

A_c: adjacency matrix of the **critical** edges of G, $\epsilon > 0$

$$\frac{1}{\alpha(G)} = \min \; x^{\mathsf{T}}(I + A_G + \epsilon A_c)x \;\; \text{s.t.} \;\; x \in \Delta_n$$

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Theorem (L-Vargas 2021)

If there is a polynomial time algorithm for deciding whether a standard quadratic program has finitely many minimizers then P=NP

Key: Reduce to the (hard) problem of testing whether an edge is critical

More about $\vartheta\mathrm{rank} = 0$

Critical graphs with $\vartheta\mathrm{rank} = 0$

- If G is **critical**, $\vartheta\mathrm{rank}(G) = 0 \iff G$ is a disjoint union of cliques [L-V'21]

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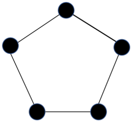
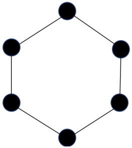
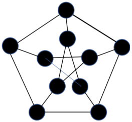
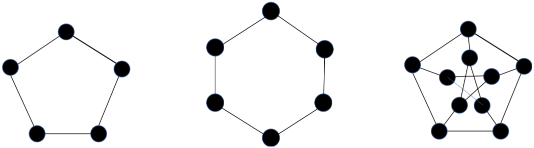




C_5 **critical**, ϑrank=1 Even cycle, Petersen: **a-critical**, ϑrank=0

This characterization does not hold for general graphs

Critical graphs with $\vartheta\text{rank} = 0$

- If G is **critical**, $\vartheta\text{rank}(G) = 0 \iff G$ is a disjoint union of cliques [L-V'21]



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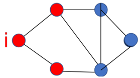
- One can reduce algorithmically deciding if $\vartheta\text{rank}(G) = 0$ to the same question for **a-critical graphs** (in poly-time for *fixed* α) [L-V'21]

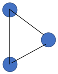
- Complexity of deciding whether $\vartheta\text{rank} = 0$?

What makes the analysis of $\vartheta\mathrm{rank}(G)$ so difficult?

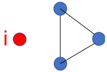
Tentative induction proof



$G, i \in V$



$G_i := G \backslash i^{\perp}$



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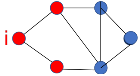

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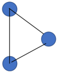

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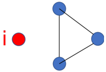

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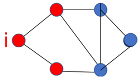

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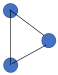

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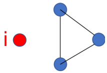

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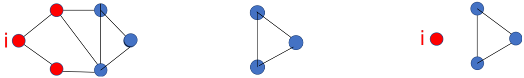

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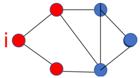

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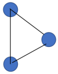

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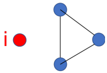

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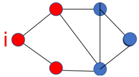

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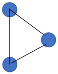

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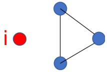

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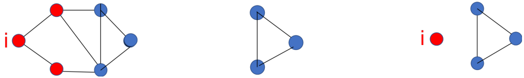

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$\implies \vartheta\mathrm{rank}(G) \leq \alpha - 1$ $\qquad$ (by Lemma)

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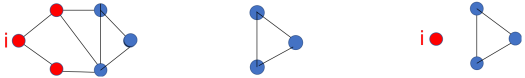

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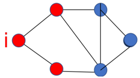

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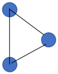

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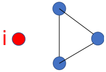

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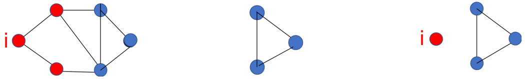

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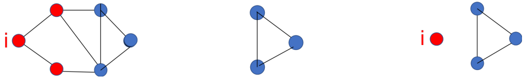

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But: (*) does **not** hold in general

because $K^{(r)}$ $(r \geq 1)$ is **not closed** under adding a zero row/column

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Open question: For $n = 5$, $\bigcup_{r\geq 0} K_5^{(r)} = \mathrm{COP}_5$?

Adding isolated nodes to graphs with $\vartheta\mathrm{rank} = 1$

Theorem (L-Vargas 2021)

Let $H = G \oplus p$ isolated nodes, where $\vartheta\mathrm{rank}(G) = 1$. If the subgraph $G_c = (V, E_c)$ of **critical** *edges in G is connected, then*

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Tools: Use the characterization of the cone $K^{(1)}$ and knowledge about the zeros of $x^{\mathsf{T}}(\alpha(I + A_G) - J)x$ (via the minimizers of (MS))

$M \in K^{(1)} \iff$ there exist matrices $P(i) \succeq 0$ $(i \in [n])$ such that
(1) $P(i)_{ii} = M_{ii}$ for $i \in [n]$
(2) $P(j)_{ii} + 2P(i)_{ij} = M_{ii} + 2M_{ij}$ for $i \neq j \in [n]$
(3) $P(i)_{jk} + P(j)_{ik} + P(k)_{ij} \leq M_{ij} + M_{ik} + M_{jk}$ for $i \neq j \neq k \in [n]$

[Parrilo 2000]

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only critical graphs on 8 nodes with $\vartheta\mathrm{rank} = 2$

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- The de Klerk-Pasechnik Conjecture offers a rich playground where real algebra (sums of squares), optimization and graph theory meet

Some references

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